Exercise 4D

1 a $3\cos\theta = 2\sin(\theta + 60°)$

$\Rightarrow 3\cos\theta = 2(\sin\theta\cos 60° + \cos\theta\sin 60°)$

$\Rightarrow 3\cos\theta = 2\left(\frac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta\right) = \sin\theta + \sqrt{3}\cos\theta$

$\Rightarrow \left(3-\sqrt{3}\right)\cos\theta = \sin\theta$

$\Rightarrow \tan\theta = 3-\sqrt{3} = 1.2679... \quad \left(\text{as } \tan\theta = \frac{\sin\theta}{\cos\theta}\right)$

As $\tan\theta$ is positive, θ is in the first and third quadrants

$\theta = \tan^{-1}(1.2679),\ 180° + \tan^{-1}(1.2679)$

$\theta = 51.7°,\ 231.7°$

b $\sin(\theta + 30°) + 2\sin\theta = 0$

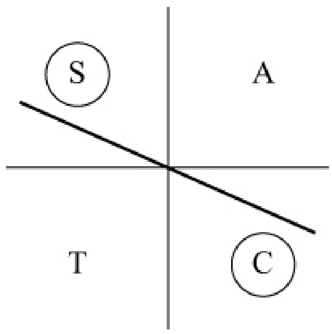
$\Rightarrow \sin\theta\cos 30° + \cos\theta\sin 30° + 2\sin\theta = 0$

$\Rightarrow \frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta + 2\sin\theta = 0$

$\Rightarrow \left(4+\sqrt{3}\right)\sin\theta = -\cos\theta$

$\Rightarrow \tan\theta = -\frac{1}{4+\sqrt{3}}$

As $\tan\theta$ is negative, θ is in the second and fourth quadrants



$\theta = \tan^{-1}\left(-\frac{1}{4+\sqrt{3}}\right) + 180°,\ \tan^{-1}\left(-\frac{1}{4+\sqrt{3}}\right) + 360°$

$\theta = 170.1°,\ 350.1°$

1 c $\cos(\theta+25°)+\sin(\theta+65°)=1$

$\Rightarrow \cos\theta\cos 25° - \sin\theta\sin 25° + \sin\theta\cos 65° + \cos\theta\sin 65° = 1$

As $\sin(90-x)° = \cos x°$ and $\cos(90-x)° = \sin x°$

$\cos 25° = \sin 65°$ and $\sin 25° = \cos 65°$

So $\cos\theta\sin 65° - \sin\theta\cos 65° + \sin\theta\cos 65° + \cos\theta\sin 65° = 1$

$\Rightarrow 2\cos\theta\sin 65° = 1$

$\Rightarrow \cos\theta = \dfrac{1}{2\sin 65°} = 0.55168...$

$\theta = \cos^{-1}(0.55168),\ 360° - \cos^{-1}(0.55168)$

$\theta = 56.5°,\ 303.5°$

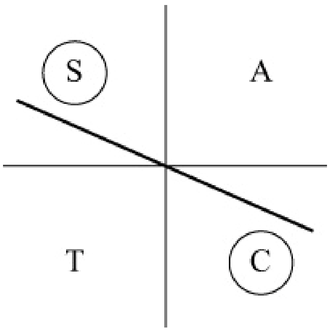
d $\cos\theta = \cos(\theta+60°)$

$\Rightarrow \cos\theta = \cos\theta\cos 60° - \sin\theta\sin 60° = \dfrac{1}{2}\cos\theta - \dfrac{\sqrt{3}}{2}\sin\theta$

$\Rightarrow \cos\theta = -\sqrt{3}\sin\theta$

$\Rightarrow \tan\theta = -\dfrac{1}{\sqrt{3}}$ $\quad\left(\text{as } \tan\theta = \dfrac{\sin\theta}{\cos\theta}\right)$

As $\tan\theta$ is negative, θ is in the second and fourth quadrants



$\theta = \tan^{-1}\left(-\dfrac{1}{\sqrt{3}}\right)+180°,\ \tan^{-1}\left(-\dfrac{1}{\sqrt{3}}\right)+360°$

$\theta = 150.0°,\ 330.0°$

2 a $\sin\left(\theta+\dfrac{\pi}{4}\right) \equiv \sin\theta\cos\dfrac{\pi}{4} + \cos\theta\sin\dfrac{\pi}{4}$

$\equiv \dfrac{1}{\sqrt{2}}\sin\theta + \dfrac{1}{\sqrt{2}}\cos\theta \equiv \dfrac{1}{\sqrt{2}}(\sin\theta+\cos\theta)$

2 b $\frac{1}{\sqrt{2}}(\sin\theta+\cos\theta)=\frac{1}{\sqrt{2}}$

$\Rightarrow \sin\left(\theta+\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$

Find all answers for $\theta+\frac{\pi}{4}$. As $0\leqslant\theta\leqslant 2\pi$ so $\frac{\pi}{4}\leqslant\theta+\frac{\pi}{4}\leqslant 2\pi+\frac{\pi}{4}$

$\theta+\frac{\pi}{4}=\frac{\pi}{4},\frac{3\pi}{4},\frac{9\pi}{4}$, so $\theta=0,\frac{\pi}{2},2\pi$

c As $\sin\left(\theta+\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}(\sin\theta+\cos\theta)$

When $\sin\theta+\cos\theta=1$

$\sin\left(\theta+\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$

So $\theta=0,\frac{\pi}{2},2\pi$

3 a $\cos\theta\cos 30°-\sin\theta\sin 30°=0.5$

$\Rightarrow \cos(\theta+30°)=0.5$

$\Rightarrow \theta+30°=60°,\ 300°$

$\Rightarrow \theta=30°,\ 270°$

b $\cos\theta\cos 30°-\sin\theta\sin 30°\equiv\cos\theta\times\frac{\sqrt{3}}{2}-\sin\theta\times\frac{1}{2}$

So $\cos\theta\cos 30°-\sin\theta\sin 30°=\frac{1}{2}$ is identical to $\sqrt{3}\cos\theta-\sin\theta=1$

Solutions are same as (a), i.e. $30°,\ 270°$

4 a $3\sin(x-y)-\sin(x+y)=0$

$\Rightarrow 3\sin x\cos y-3\cos x\sin y-\sin x\cos y-\cos x\sin y=0$

$\Rightarrow 2\sin x\cos y=4\cos x\sin y$

$\Rightarrow \frac{2\sin x\cos y}{\cos x\cos y}=\frac{4\cos x\sin y}{\cos x\cos y}$

$\Rightarrow \frac{2\sin x}{\cos x}=\frac{4\sin y}{\cos y}$

$\Rightarrow 2\tan x=4\tan y$

b Put $y=45°\ \Rightarrow\ \tan x=2$

So $x=\tan^{-1}2,\ 180°+\tan^{-1}2$

$x=63.4°,\ 243.4°$ (1 d.p.)

5 a $\sin 2\theta = \sin\theta,\ 0 \leqslant \theta \leqslant 2\pi$

$\Rightarrow\ 2\sin\theta\cos\theta = \sin\theta$

$\Rightarrow\ 2\sin\theta\cos\theta - \sin\theta = 0$

$\Rightarrow\ \sin\theta(2\cos\theta - 1) = 0$

$\Rightarrow\ \sin\theta = 0 \text{ or } \cos\theta = \tfrac{1}{2}$

Solution set: $0,\ \frac{\pi}{3},\ \pi,\ \frac{5\pi}{3},\ 2\pi$

b $\cos 2\theta = 1 - \cos\theta,\ -180° < \theta \leqslant 180°$

$\Rightarrow\ 2\cos^2\theta - 1 = 1 - \cos\theta$

$\Rightarrow\ 2\cos^2\theta + \cos\theta - 2 = 0$

$\Rightarrow\ \cos\theta = \frac{-1 \pm \sqrt{17}}{4}$ (using the quadratic formula)

As $\frac{-1-\sqrt{17}}{4} \leqslant -1$, this gives the only solution as $\cos\theta = \frac{-1+\sqrt{17}}{4} = 0.78077...$

As $\cos\theta$ is positive, θ is in the first and fourth quadrants

Using a calculator $\cos^{-1} 0.78077 = 38.7°$ (1 d.p.)

Solutions are $\pm 38.7°$ (1 d.p.)

c $3\cos 2\theta = 2\cos^2\theta,\quad 0 \leqslant \theta < 360°$

$\Rightarrow\ 3(2\cos^2\theta - 1) = 2\cos^2\theta$

$\Rightarrow\ 6\cos^2\theta - 3 = 2\cos^2\theta$

$\Rightarrow\ 4\cos^2\theta = 3$

$\Rightarrow\ \cos^2\theta = \frac{3}{4}\ \Rightarrow\ \cos\theta = \pm\frac{\sqrt{3}}{2}$

θ will be in all four quadrants.

Solution set: 30°, 150°, 210°, 330°

d $\sin 4\theta = \cos 2\theta,\quad 0 \leqslant \theta \leqslant \pi$

$\Rightarrow\ 2\sin 2\theta\cos 2\theta = \cos 2\theta$

$\Rightarrow\ \cos 2\theta(2\sin 2\theta - 1) = 0$

$\Rightarrow\ \cos 2\theta = 0 \text{ or } \sin 2\theta = \tfrac{1}{2}$

$\cos 2\theta = 0$ in $0 \leq 2\theta \leq 2\pi$

$\Rightarrow\ 2\theta = \frac{\pi}{2}, \frac{3\pi}{2}\ \Rightarrow\ \theta = \frac{\pi}{4}, \frac{3\pi}{4}$

$\sin 2\theta = \tfrac{1}{2}$ in $0 \leqslant 2\theta \leqslant 2\pi$

$\Rightarrow\ 2\theta = \frac{\pi}{6}, \frac{5\pi}{6}\ \Rightarrow\ \theta = \frac{\pi}{12}, \frac{5\pi}{12}$

Solution set: $\frac{\pi}{12},\ \frac{\pi}{4},\ \frac{5\pi}{12},\ \frac{3\pi}{4}$

5 e $3\cos\theta - \sin\frac{\theta}{2} - 1 = 0, \quad 0 \leqslant \theta \leqslant 720°$

$\Rightarrow 3\left(1 - 2\sin^2\frac{\theta}{2}\right) - \sin\frac{\theta}{2} - 1 = 0$

$\Rightarrow 6\sin^2\frac{\theta}{2} + \sin\frac{\theta}{2} - 2 = 0$

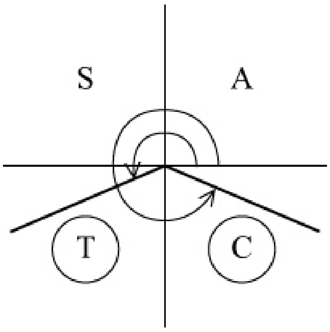
$\Rightarrow \left(3\sin\frac{\theta}{2} + 2\right)\left(2\sin\frac{\theta}{2} - 1\right) = 0$

$\Rightarrow \sin\frac{\theta}{2} = -\frac{2}{3}$ or $\sin\frac{\theta}{2} = \frac{1}{2}$

$\sin\frac{\theta}{2} = \frac{1}{2}$ in $0 \leqslant \frac{\theta}{2} \leqslant 360°$

$\Rightarrow \frac{\theta}{2} = 30°, 150° \Rightarrow \theta = 60°, 300°$

$\sin\frac{\theta}{2} = -\frac{2}{3}$ in $0 \leqslant \frac{\theta}{2} \leqslant 360°$



$\Rightarrow \frac{\theta}{2} = 180° - \sin^{-1}\left(-\frac{2}{3}\right), 360° + \sin^{-1}\left(-\frac{2}{3}\right) = 221.8°, 318.2°$ (1 d.p.)

$\Rightarrow \theta = 443.6°, 636.4°$ (1 d.p.)

Solution set: $60°, 300°, 443.6°, 636.4°$

f $\cos^2\theta - \sin 2\theta = \sin^2\theta, \qquad 0 \leqslant \theta \leqslant \pi$

$\Rightarrow \cos^2\theta - \sin^2\theta = \sin 2\theta$

$\Rightarrow \cos 2\theta = \sin 2\theta$

$\Rightarrow \tan 2\theta = 1$ (divide both sides by $\cos 2\theta$)

$\tan 2\theta = 1$ in $0 \leqslant 2\theta \leqslant 2\pi$

$\Rightarrow 2\theta = \frac{\pi}{4}, \frac{5\pi}{4}$

$\Rightarrow \theta = \frac{\pi}{8}, \frac{5\pi}{8}$

5 **g** $2\sin\theta = \sec\theta, \quad 0 \leqslant \theta \leqslant 2\pi$

$\Rightarrow \ 2\sin\theta = \dfrac{1}{\cos\theta}$

$\Rightarrow \ 2\sin\theta\cos\theta = 1$

$\Rightarrow \ \sin 2\theta = 1$

$\sin 2\theta = 1$ in $0 \leqslant 2\theta \leqslant 4\pi$

$\Rightarrow \ 2\theta = \dfrac{\pi}{2}, \ \dfrac{5\pi}{2}$

$\Rightarrow \ \theta = \dfrac{\pi}{4}, \ \dfrac{5\pi}{4}$

h $2\sin 2\theta = 3\tan\theta, \quad 0 \leqslant \theta < 360°$

$\Rightarrow \ 4\sin\theta\cos\theta = \dfrac{3\sin\theta}{\cos\theta}$

$\Rightarrow \ 4\sin\theta\cos^2\theta = 3\sin\theta$

$\Rightarrow \ \sin\theta(4\cos^2\theta - 3) = 0$

$\Rightarrow \ \sin\theta = 0$ or $\cos^2\theta = \frac{3}{4}$

$\sin\theta = 0 \ \Rightarrow \ \theta = 0°, \ 180°$

$\cos^2\theta = \dfrac{3}{4} \ \Rightarrow \cos\theta = \pm\dfrac{\sqrt{3}}{2} \ \Rightarrow \ \theta = \ 30°, \ 150°, \ 210°, \ 330°$

Solution set: $0°, \ 30°, \ 150°, \ 180°, \ 210°, \ 330°$

5 i $2\tan\theta = \sqrt{3}(1-\tan\theta)(1+\tan\theta), \quad 0 \leqslant \theta \leqslant 2\pi$

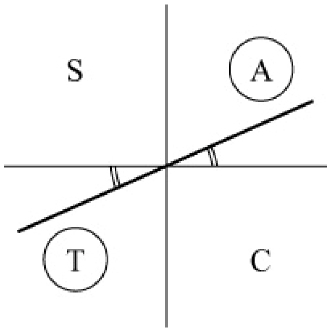
$\Rightarrow 2\tan\theta = \sqrt{3}(1-\tan^2\theta)$

$\Rightarrow \sqrt{3}\tan^2\theta + 2\tan\theta - \sqrt{3} = 0$

$\Rightarrow (\sqrt{3}\tan\theta - 1)(\tan\theta + \sqrt{3}) = 0$

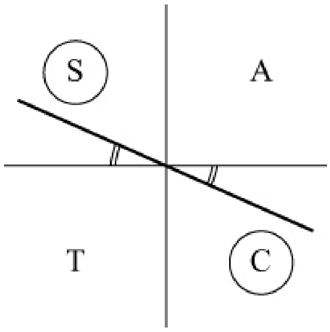
$\Rightarrow \tan\theta = \frac{1}{\sqrt{3}}$ or $\tan\theta = -\sqrt{3}$

$\tan\theta = \frac{1}{\sqrt{3}}, \; 0 \leqslant \theta \leqslant 2\pi$



$\Rightarrow \theta = \tan^{-1}\frac{1}{\sqrt{3}}, \; \pi + \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}, \; \frac{7\pi}{6}$

$\tan\theta = -\sqrt{3}, \quad 0 \leqslant \theta \leqslant 2\pi$



$\Rightarrow \theta = \pi + \tan^{-1}\left(-\sqrt{3}\right), \; 2\pi + \tan^{-1}\left(-\sqrt{3}\right) = \frac{2\pi}{3}, \; \frac{5\pi}{3}$

Solution set: $\frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$

5 j $\sin^2\theta = 2\sin 2\theta, \quad -180° < \theta \leqslant 180°$

$\Rightarrow \sin^2\theta = 4\sin\theta\cos\theta$

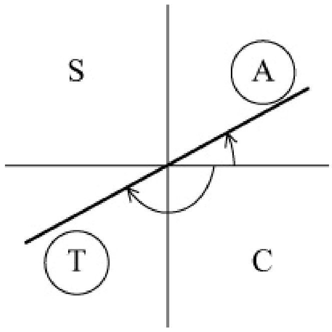
$\Rightarrow \sin\theta(\sin\theta - 4\cos\theta) = 0$

$\Rightarrow \sin\theta = 0 \text{ or } \sin\theta = 4\cos\theta$

$\Rightarrow \sin\theta = 0 \text{ or } \tan\theta = 4$

$\sin\theta = 0 \Rightarrow \theta = 0°, 180°$

$\tan\theta = 4 \Rightarrow \theta = \tan^{-1}4, -180° + \tan^{-1}4 = 76.0°, -104.0°$ (1 d.p.)



Solution set: $-104.0°, 0°, 76.0°, 180°$

k $4\tan\theta = \tan 2\theta, \quad 0 \leqslant \theta \leqslant 360°$

$\Rightarrow 4\tan\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

$\Rightarrow 2\tan\theta(1-\tan^2\theta) = \tan\theta$

$\Rightarrow \tan\theta(2 - 2\tan^2\theta - 1) = 0$

$\Rightarrow \tan\theta(1 - 2\tan^2\theta) = 0$

$\Rightarrow \tan\theta = 0 \text{ or } \tan\theta = \pm\sqrt{\dfrac{1}{2}}$

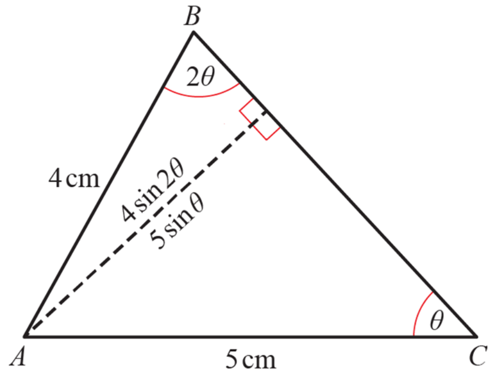
$\tan\theta = 0 \Rightarrow \theta = 0°, 180°, 360°$

$\tan\theta = \pm\sqrt{\dfrac{1}{2}}$

$\Rightarrow \theta = 35.3°, 144.7°, 215.3°, 324.7°$

Solution set: $0°, 35.3°, 144.7°, 180°, 215.3°, 324.7°, 360°$

6 Sketch ΔABC



$4\sin 2\theta = 5\sin\theta$

$\Rightarrow 8\sin\theta\cos\theta = 5\sin\theta$

$\Rightarrow 8\sin\theta\cos\theta - 5\sin\theta = 0$

$\Rightarrow \sin\theta(8\cos\theta - 5) = 0$

$\Rightarrow \sin\theta = 0$ or $\cos\theta = \frac{5}{8}$

As ABC is a triangle, $0 < \theta < 90°$, so $\theta = 0°$ or $180°$ are not possible solutions.

So $\theta = \cos^{-1}\frac{5}{8} = 51.3°$

7 a As $5\sin 2\theta = 10\sin\theta\cos\theta$

$5\sin 2\theta + 4\sin\theta = 10\sin\theta\cos\theta + 4\sin\theta = 0$

$2\sin\theta(5\cos\theta + 2) = 0$

So $a = 2$, $b = 5$ and $c = 2$

b $2\sin\theta(5\cos\theta + 2) = 0, \quad 0 \leqslant \theta < 360°$

$\Rightarrow \sin\theta = 0$ or $\cos\theta = -\frac{2}{5}$

$\sin\theta = 0 \ \Rightarrow \theta = 0°, \ 180°$

$\cos\theta = -\frac{2}{5} \ \Rightarrow \theta = \cos^{-1}\left(-\frac{2}{5}\right), \ 360° - \cos^{-1}\left(-\frac{2}{5}\right) = 113.6°, \ 246.4°$ (1 d.p.)

Solution set: $\theta = 0°, \ 113.6°, \ 180°, \ 246.4°$

8 a As $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = 1 - 2\sin^2\theta$

$\sin 2\theta + \cos 2\theta = 1$

$\Rightarrow 2\sin\theta\cos\theta + (1 - 2\sin^2\theta) = 1$

$\Rightarrow 2\sin\theta\cos\theta - 2\sin^2\theta = 0$

$\Rightarrow 2\sin\theta(\cos\theta - \sin\theta) = 0$

b $2\sin\theta(\cos\theta - \sin\theta) = 0, \quad 0 \leqslant \theta < 360°$

$\Rightarrow \sin\theta = 0$ or $\cos\theta = \sin\theta$

$\sin\theta = 0 \Rightarrow \theta = 0°, \ 180°$

$\cos\theta = \sin\theta \Rightarrow \tan\theta = 1 \Rightarrow \ \theta = 45°, \ 225°$

Solution set: $\theta = 0°, \ 45°, \ 180°, \ 225°$

9 a LHS $\equiv (\cos 2\theta - \sin 2\theta)^2$

$\equiv \cos^2 2\theta - 2\sin 2\theta \cos 2\theta + \sin^2 2\theta$

$\equiv (\cos^2 2\theta + \sin^2 2\theta) - (2\sin 2\theta \cos 2\theta)$

$\equiv 1 - \sin 4\theta \qquad (\sin^2 A + \cos^2 A \equiv 1,\ \sin 2A \equiv 2\sin A\cos A)$

$\equiv$ RHS

b You can use $(\cos 2\theta - \sin 2\theta)^2 = \frac{1}{2}$ but this also solves the equation

$\cos 2\theta - \sin 2\theta = -\dfrac{1}{\sqrt{2}}$

so you need to check your final answers.

As $(\cos 2\theta - \sin 2\theta)^2 \equiv 1 - \sin 4\theta$

$\Rightarrow\ \frac{1}{2} = 1 - \sin 4\theta$

$\Rightarrow\ \sin 4\theta = \frac{1}{2}$

$0 \leqslant \theta < \pi$, so $0 \leqslant 4\theta < 4\pi$

$\Rightarrow\ 4\theta = \dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{13\pi}{6}, \dfrac{17\pi}{6}$

$\Rightarrow\ \theta = \dfrac{\pi}{24}, \dfrac{5\pi}{24}, \dfrac{13\pi}{24}, \dfrac{17\pi}{24}$

Checking these values in $\cos 2\theta - \sin 2\theta = \dfrac{1}{\sqrt{2}}$ eliminates $\dfrac{5\pi}{24}, \dfrac{13\pi}{24}$

which apply to $\cos 2\theta - \sin 2\theta = -\dfrac{1}{\sqrt{2}}$

Solutions are $\dfrac{\pi}{24}, \dfrac{17\pi}{24}$

10 a i RHS $\equiv \dfrac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$

$\equiv \dfrac{2\tan\frac{\theta}{2}}{\sec^2\frac{\theta}{2}}$

$\equiv \dfrac{2\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} \times \cos^2\frac{\theta}{2}$

$\equiv 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$\equiv \sin\theta \qquad (\sin 2A = 2\sin A\cos A)$

$\equiv$ LHS

10 a ii RHS $\equiv \dfrac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$

$\equiv \dfrac{1-\tan^2\frac{\theta}{2}}{\sec^2\frac{\theta}{2}}$

$\equiv \cos^2\frac{\theta}{2}\left(1-\tan^2\frac{\theta}{2}\right)$

$\equiv \cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2} \qquad \left(\tan^2\frac{\theta}{2}=\dfrac{\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}}\right)$

$\equiv \cos\theta \qquad (\cos 2A=\cos^2 A-\sin^2 A)$

$\equiv$ LHS

b Let $\tan\frac{\theta}{2}=t$

i $\sin\theta+2\cos\theta=1$

$\Rightarrow \dfrac{2t}{1+t^2}+\dfrac{2(1-t^2)}{1+t^2}=1$

$\Rightarrow 2t+2-2t^2=1+t^2$

$\Rightarrow 3t^2-2t-1=0$

$\Rightarrow (3t+1)(t-1)=0$

$\Rightarrow \tan\frac{\theta}{2}=-\frac{1}{3}$ or $\tan\frac{\theta}{2}=1,\quad 0\leqslant\frac{\theta}{2}\leqslant 180°$

$\tan\frac{\theta}{2}=1 \Rightarrow \frac{\theta}{2}=45° \Rightarrow \theta=90°$

$\tan\frac{\theta}{2}=-\frac{1}{3} \Rightarrow \frac{\theta}{2}=161.56° \Rightarrow \theta=323.1°$ (1 d.p.)

Solution set: $90°,\ 323.1°$

10 b ii $3\cos\theta - 4\sin\theta = 2$

$$\Rightarrow \frac{3(1-t^2)}{1+t^2} - \frac{4\times 2t}{1+t^2} = 2$$

$$\Rightarrow 3(1-t^2) - 8t = 2(1+t^2)$$

$$\Rightarrow 5t^2 + 8t - 1 = 0$$

$$\Rightarrow t = \frac{-8\pm\sqrt{84}}{10}$$

For $\tan\frac{\theta}{2} = \frac{-8+\sqrt{84}}{10} \qquad 0 \leqslant \frac{\theta}{2} \leqslant 180°$

$\frac{\theta}{2} = 6.65° \Rightarrow \theta = 13.3°$ (1 d.p.)

For $\tan\frac{\theta}{2} = \frac{-8-\sqrt{84}}{10} \qquad 0 \leqslant \frac{\theta}{2} \leqslant 180°$

$\frac{\theta}{2} = 120.2° \Rightarrow \theta = 240.4°$ (1 d.p.)

Solution set: 13.3°, 240.4°

11 a RHS $\equiv 1 + 2\cos 2x$

$\equiv 1 + 2(\cos^2 x - \sin^2 x)$

$\equiv 1 + 2\cos^2 x - 2\sin^2 x$

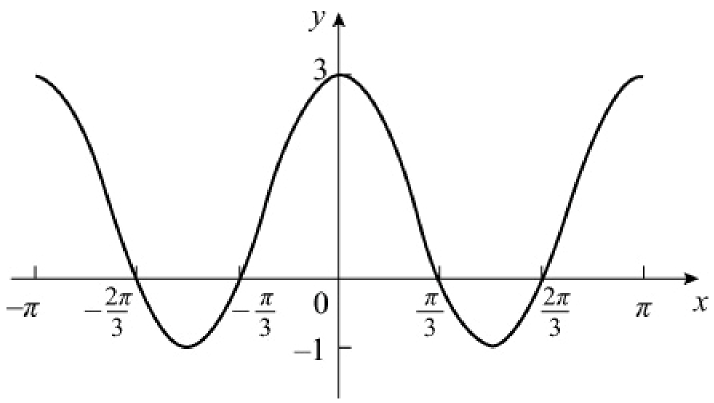
$\equiv \cos^2 x + \sin^2 x + 2\cos^2 x - 2\sin^2 x \qquad$ (using $\sin^2 x + \cos^2 x \equiv 1$)

$\equiv 3\cos^2 x - \sin^2 x$

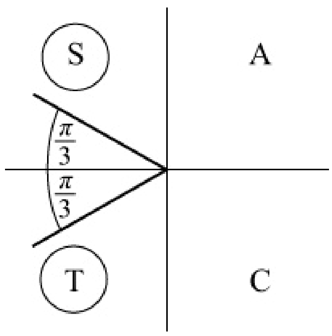
$\equiv$ LHS

11 b $y = 3\cos^2 x - \sin^2 x$ is the same as $y = 1 + 2\cos 2x$

Using your work on transformations, this curve is the result of
(i) stretching $y = \cos x$ by scale factor $\frac{1}{2}$ in the x direction, then
(ii) stretching the result by scale factor 2 in the y direction, then
(iii) translating by 1 in the positive y direction.



The curve crosses y-axis at (0, 3). It crosses x-axis where $y = 0$
i.e. where $1 + 2\cos 2x = 0$ $\quad -\pi \leqslant x \leqslant \pi$
$\Rightarrow \cos 2x = -\frac{1}{2} \quad -2\pi \leqslant 2x \leqslant 2\pi$



So $2x = -\frac{4\pi}{3}, -\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$

$\Rightarrow x = -\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}$

The curve meets the x-axis at $\left(-\frac{2\pi}{3}, 0\right), \left(-\frac{\pi}{3}, 0\right), \left(\frac{\pi}{3}, 0\right), \left(\frac{2\pi}{3}, 0\right)$

12 a $\cos^2 \frac{\theta}{2} \equiv \frac{1+\cos\theta}{2}, \ \sin^2 \frac{\theta}{2} \equiv \frac{1-\cos\theta}{2}$

So $2\cos^2 \frac{\theta}{2} - 4\sin^2 \frac{\theta}{2} \equiv (1+\cos\theta) - 2(1-\cos\theta) \equiv 3\cos\theta - 1$

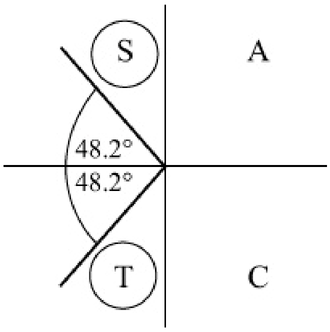
12 b $3\cos\theta - 1 = -3, \quad 0 \leqslant \theta < 360°$

$\Rightarrow \; 3\cos\theta = -2$

$\Rightarrow \; \cos\theta = -\frac{2}{3}$

As $\cos\theta$ is negative, θ is in second and third quadrants.

Calculator value is $\cos^{-1}\left(-\frac{2}{3}\right) = 131.8°$ (1 d.p.)



Solutions are $131.8°,\; 360° - 131.8° = 131.8°,\; 228.2°$ (1 d.p.)

13 a As $\sin^2 A + \cos^2 A \equiv 1$ so $(\sin^2 A + \cos^2 A)^2 \equiv 1$

$\Rightarrow \; \sin^4 A + \cos^4 A + 2\sin^2 A\cos^2 A \equiv 1$

$\Rightarrow \; \sin^4 A + \cos^4 A \equiv 1 - 2\sin^2 A\cos^2 A$

$\equiv 1 - \frac{1}{2}(4\sin^2 A\cos^2 A)$

$\equiv 1 - \frac{1}{2}\left((2\sin A\cos A)^2\right)$

$\equiv 1 - \frac{1}{2}\sin^2 2A$

$\equiv \frac{1}{2}(2 - \sin^2 2A)$

b As $\cos 2A \equiv 1 - 2\sin^2 A$ so $\cos 4A \equiv 1 - 2\sin^2 2A$ so $\sin^2 2A \equiv \dfrac{1-\cos 4A}{2}$

$\Rightarrow$ from (a) $\sin^4 A + \cos^4 A \equiv \dfrac{1}{2}\left(2 - \dfrac{1-\cos 4A}{2}\right) \equiv \dfrac{1}{2}\left(\dfrac{4-1+\cos 4A}{2}\right) \equiv \dfrac{1}{4}(3+\cos 4A)$

13 c Using part (b)

$8\sin^4\theta + 8\cos^4\theta = 7$

$\Rightarrow \; 8 \times \frac{1}{4}(3 + \cos 4\theta) = 7$

$\Rightarrow \; 3 + \cos 4\theta = \frac{7}{2}$

$\Rightarrow \; \cos 4\theta = \frac{1}{2}$

Solve $\cos 4\theta = \frac{1}{2}$ in $0 < 4\theta < 4\pi$

$\Rightarrow \; 4\theta = \dfrac{\pi}{3}, \dfrac{5\pi}{3}, \dfrac{7\pi}{3}, \dfrac{11\pi}{3}$

$\Rightarrow \; \theta = \dfrac{\pi}{12}, \dfrac{5\pi}{12}, \dfrac{7\pi}{12}, \dfrac{11\pi}{12}$

14 a $\cos 3\theta \equiv \cos(2\theta+\theta) \equiv \cos 2\theta \cos\theta - \sin 2\theta \sin\theta$

$\equiv (\cos^2\theta - \sin^2\theta)\cos\theta - (2\sin\theta\cos\theta)\sin\theta$

$\equiv \cos^3\theta - \sin^2\theta\cos\theta - 2\sin^2\theta\cos\theta$

$\equiv \cos^3\theta - 3\sin^2\theta\cos\theta$

$\equiv \cos^3\theta - 3(1-\cos^2\theta)\cos\theta$

$\equiv 4\cos^3\theta - 3\cos\theta$

b $6\cos\theta - 8\cos^3\theta + 1 = 0$, $0 < \theta < \pi$

$\Rightarrow 1 = 8\cos^3\theta - 6\cos\theta$

$\Rightarrow 4\cos^3\theta - 3\cos\theta = \frac{1}{2}$

$\Rightarrow \cos 3\theta = \frac{1}{2}$, $0 < 3\theta < 3\pi$ using the result from part (a)

So $3\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$

$\Rightarrow \theta = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}$