## Exercise 4C

**1** $\sin 2A \equiv \sin(A+A) \equiv \sin A\cos A + \cos A\sin A$
$\equiv 2\sin A\cos A$

**2 a** $\cos 2A \equiv \cos(A+A)$
$\equiv \cos A\cos A - \sin A\sin A$
$\equiv \cos^2 A - \sin^2 A$

**b i** $\cos 2A \equiv \cos^2 A - \sin^2 A$
Use $\cos^2 A + \sin^2 A \equiv 1$ to simplify, so
$\cos 2A \equiv \cos^2 A - (1-\cos^2 A)$
$\equiv 2\cos^2 A - 1$

**ii** $\cos 2A \equiv \cos^2 A - \sin^2 A$
$\equiv (1-\sin^2 A) - \sin^2 A$
$\equiv 1 - 2\sin^2 A$

**3** $\tan 2A \equiv \tan(A+A)$
$$\equiv \frac{\tan A + \tan A}{1-\tan A\tan A} \equiv \frac{2\tan A}{1-\tan^2 A}$$

**4 a** $2\sin 10°\cos 10° = \sin 20°$
(using $2\sin A\cos A \equiv \sin 2A$)

**b** $1 - 2\sin^2 25° = \cos 50°$
using $\cos 2A \equiv 1 - 2\sin^2 A$

**c** $\cos^2 40° - \sin^2 40° = \cos 80°$
using $\cos 2A \equiv \cos^2 A - \sin^2 A$

**d** $$\frac{2\tan 5°}{1-\tan^2 5°} = \tan 10°$$
using $\tan 2A \equiv \dfrac{2\tan A}{1-\tan^2 A}$

**e** $$\frac{1}{2\sin 24.5°\cos 24.5°} = \frac{1}{\sin 49°}$$
$= \operatorname{cosec} 49°$

**f** $6\cos^2 30° - 3 = 3(2\cos^2 30° - 1)$
$= 3\cos 60°$

**g** $$\frac{\sin 8°}{\sec 8°} = \sin 8°\cos 8°$$
$= \frac{1}{2}(2\sin 8°\cos 8°) = \frac{1}{2}\sin 16°$

**4 h** $$\cos^2\frac{\pi}{16} - \sin^2\frac{\pi}{16} = \cos\frac{2\pi}{16} = \cos\frac{\pi}{8}$$

**5 a** $2\sin 22.5°\cos 22.5° = \sin 2\times 22.5°$
$= \sin 45° = \dfrac{\sqrt{2}}{2}$

**b** $2\cos^2 15° - 1 = \cos(2\times 15°)$
$= \cos 30° = \dfrac{\sqrt{3}}{2}$

**c** $(\sin 75° - \cos 75°)^2$
$= \sin^2 75° + \cos^2 75° - 2\sin 75°\cos 75°$
$= 1 - \sin(2\times 75°)$
as $\sin^2 75° + \cos^2 75° = 1$, and this gives
$(\sin 75° - \cos 75°)^2 = 1 - \sin 150°$
$= 1 - \frac{1}{2} = \frac{1}{2}$

**d** $$\frac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}} = \tan\left(2\times\frac{\pi}{8}\right) = \tan\frac{\pi}{8} = 1$$

**6 a** $(\sin A + \cos A)^2$
$\equiv \sin^2 A + 2\sin A\cos A + \cos^2 A$
$\equiv 1 + \sin 2A$

**b** $$\left(\sin\frac{\pi}{8} + \cos\frac{\pi}{8}\right)^2$$
$$= 1 + \sin\frac{\pi}{4} = 1 + \frac{\sqrt{2}}{2} = \frac{2+\sqrt{2}}{4}$$

**7 a** $\cos^2 3\theta - \sin^2 3\theta \equiv \cos(2\times 3\theta) \equiv \cos 6\theta$

**b** $6\sin 2\theta\cos 2\theta \equiv 3(2\sin 2\theta\cos 2\theta)$
$\equiv 3\sin(2\times 2\theta)$
$\equiv 3\sin 4\theta$

**c** $$\frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}} \equiv \tan\left(2\times\frac{\theta}{2}\right) \equiv \tan\theta$$

**7 d** $2-4\sin^2\frac{\theta}{2}\equiv 2\left(1-2\sin^2\left(\frac{\theta}{2}\right)\right)$

$\equiv 2\cos\left(2\times\frac{\theta}{2}\right)\equiv 2\cos\theta$

**e** $\sqrt{1+\cos 2\theta}\equiv\sqrt{1+(2\cos^2\theta-1)}$

$\equiv\sqrt{2\cos^2\theta}\equiv\sqrt{2}\cos\theta$

**f** $\sin^2\theta\cos^2\theta\equiv\frac{1}{4}(4\sin^2\theta\cos^2\theta)$

$\equiv\frac{1}{4}(2\sin\theta\cos\theta)^2$

$\equiv\frac{1}{4}\sin^2 2\theta$

**g** $4\sin\theta\cos\theta\cos 2\theta\equiv 2(2\sin\theta\cos\theta)\cos 2\theta$

$\equiv 2\sin 2\theta\cos 2\theta$

$\equiv\sin 4\theta$

As $\sin 2A\equiv 2\sin A\cos A$ with $A\equiv 2\theta$

**h** $\frac{\tan\theta}{\sec^2\theta-2}\equiv\frac{\tan\theta}{(1+\tan^2\theta)-2}$

$\equiv\frac{\tan\theta}{\tan^2\theta-1}$

$\equiv-\frac{\tan\theta}{1-\tan^2\theta}$

$\equiv-\frac{1}{2}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$

$\equiv-\frac{1}{2}\tan 2\theta$

**i** $\cos^4\theta-2\sin^2\theta\cos^2\theta+\sin^4\theta$

$\equiv(\cos^2\theta-\sin^2\theta)^2\equiv\cos^2 2\theta$

**8** $p=2\cos\theta\ \Rightarrow\ \cos\theta=\frac{p}{2}$

$\cos 2\theta=q$

Using $\cos 2\theta=2\cos^2\theta-1$

$\Rightarrow q=2\left(\frac{p}{2}\right)^2-1$

$\Rightarrow q=\frac{p^2}{2}-1$

**9 a** $\cos^2\theta=x,\ \cos 2\theta=1-y$

Using $\cos 2\theta\equiv 2\cos^2\theta-1$

$\Rightarrow\ 1-y=2x-1$

$\Rightarrow y=2-2x=2(1-x)$

Any form of this equation is correct

**b** $y=\cot 2\theta\ \Rightarrow\ \tan 2\theta=\frac{1}{y}$

$x=\tan\theta$

Using $\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}$

$\Rightarrow\ \frac{1}{y}=\frac{2x}{1-x^2}$

$\Rightarrow\ 2xy=1-x^2$

Any form of this equation is correct

**c** $x=\sin\theta,\ y=2\sin\theta\cos\theta$

$\Rightarrow\ y=2x\cos\theta$

$\Rightarrow\ \cos\theta=\frac{y}{2x}$

Using $\sin^2\theta+\cos^2\theta\equiv 1$

$\Rightarrow\ x^2+\frac{y^2}{4x^2}=1$

$\Rightarrow\ 4x^4+y^2=4x^2$

or $y^2=4x^2(1-x^2)$

Any form of this equation is correct

**d** $x=3\cos 2\theta+1\ \Rightarrow\ \cos 2\theta=\frac{x-1}{3}$

$y=2\sin\theta\ \Rightarrow\ \sin\theta=\frac{y}{2}$

Using $\cos 2\theta=1-2\sin^2\theta$

$\Rightarrow\ \frac{x-1}{3}=1-\frac{2y^2}{4}=1-\frac{y^2}{2}$

Multiplying both sides by 6 gives

$2(x-1)=6-3y^2$

$\Rightarrow\ 3y^2=6-2(x-1)=8-2x$

$\Rightarrow\ y^2=\frac{2(4-x)}{3}$

Any form of this equation is correct

**10** $\cos 2x=2\cos^2 x-1$

$\cos 2x=2\left(\frac{1}{4}\right)^2-1=\frac{1}{8}-1=-\frac{7}{8}$

**11** $\cos 2\theta=1-2\sin^2\theta$

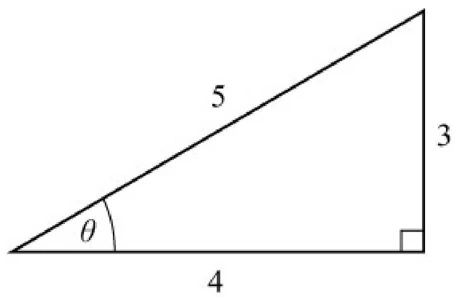
So $\frac{23}{25}=1-2\sin^2\theta$

$\Rightarrow\ 2\sin^2\theta=1-\frac{23}{25}=\frac{2}{25}$

$\Rightarrow\ \sin^2\theta=\frac{1}{25}$

$\Rightarrow\ \sin\theta=\pm\frac{1}{5}$

**12** Draw a right-angled triangle with $\theta$ as one of the angles. The hypotenuse is 5




So $\sin\theta = \frac{3}{5}$, $\cos\theta = \frac{4}{5}$, $\tan\theta = \frac{3}{4}$

**a i** $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta} = \dfrac{\frac{3}{2}}{1-\frac{9}{16}}$

$= \dfrac{\frac{3}{2}}{\frac{7}{16}} = \dfrac{3}{2}\times\dfrac{16}{7} = \dfrac{24}{7}$

**ii** $\sin 2\theta = 2\sin\theta\cos\theta = 2\times\frac{3}{5}\times\frac{4}{5} = \frac{24}{25}$

**iii** $\cos 2\theta = \cos^2\theta - \sin^2\theta = \frac{16}{25} - \frac{9}{25} = \frac{7}{25}$

**b** $\sin 4\theta = 2\sin 2\theta\cos 2\theta$

$= 2\times\frac{24}{25}\times\frac{7}{25} = \frac{336}{625}$

**13 a i** $\cos 2A = 2\cos^2 A - 1$

$= 2\left(-\frac{1}{3}\right)^2 - 1 = \frac{2}{9} - 1 = -\frac{7}{9}$

**ii** $\cos 2A = 1 - 2\sin^2 A$

$\Rightarrow -\frac{7}{9} = 1 - 2\sin^2 A$

$\Rightarrow 2\sin^2 A = 1 + \frac{7}{9} = \frac{16}{9}$

$\Rightarrow \sin^2 A = \frac{8}{9}$

$\Rightarrow \sin A = \pm\dfrac{\sqrt{8}}{3} = \pm\dfrac{2\sqrt{2}}{3}$

But $A$ is in the second quarter so $\sin A$ is positive, and the solution is

$\sin A = \dfrac{2\sqrt{2}}{3}$

**iii** $\operatorname{cosec} 2A = \dfrac{1}{\sin 2A} = \dfrac{1}{2\sin A\cos A}$

$= \dfrac{1}{2\times\frac{2\sqrt{2}}{3}\times\left(-\frac{1}{3}\right)}$

$= -\dfrac{9}{4\sqrt{2}} = -\dfrac{9\sqrt{2}}{8}$

**13 b** $\tan 2A = \dfrac{\sin 2A}{\cos 2A} = \dfrac{-\frac{4\sqrt{2}}{9}}{-\frac{7}{9}}$

$= -\dfrac{4\sqrt{2}}{9}\times -\dfrac{9}{7} = \dfrac{4\sqrt{2}}{7}$

**14** Using $\tan\theta = \dfrac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$

$\Rightarrow \dfrac{3}{4} = \dfrac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$

$\Rightarrow 3 - 3\tan^2\dfrac{\theta}{2} = 8\tan\dfrac{\theta}{2}$

$\Rightarrow 3\tan^2\dfrac{\theta}{2} + 8\tan\dfrac{\theta}{2} - 3 = 0$

$\Rightarrow \left(3\tan\dfrac{\theta}{2} - 1\right)\left(\tan\dfrac{\theta}{2} + 3\right) = 0$

so $\tan\dfrac{\theta}{2} = \frac{1}{3}$ or $\tan\dfrac{\theta}{2} = -3$

But $\pi < \theta < \dfrac{3\pi}{2}$ so $\dfrac{\pi}{2} < \dfrac{\theta}{2} < \dfrac{3\pi}{4}$

As $\dfrac{\theta}{2}$ is in the second quadrant, so $\tan\dfrac{\theta}{2}$ is negative, and the solution is $\tan\dfrac{\theta}{2} = -3$

**15** $\cos x + \sin x = m$

$\cos x - \sin x = n$

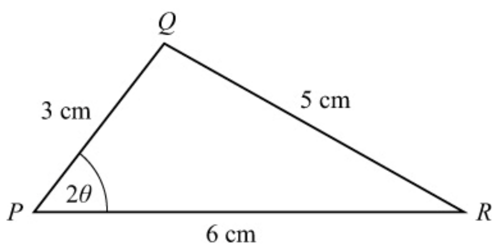
Multiply the equations

$(\cos x + \sin x)(\cos x - \sin x) = mn$

$\Rightarrow \cos^2 x - \sin^2 x = mn$

$\Rightarrow \cos 2x = mn$

**16**




**a** Using cosine rule with

$\cos P = \dfrac{q^2 + r^2 - p^2}{2qr}$

$\cos 2\theta = \dfrac{36 + 9 - 25}{2\times 6\times 3} = \dfrac{20}{36} = \dfrac{5}{9}$

**b** Using $\cos 2\theta = 1 - 2\sin^2\theta$

$\frac{5}{9} = 1 - 2\sin^2\theta$

$\Rightarrow\ 2\sin^2\theta = 1 - \frac{5}{9} = \frac{4}{9}$

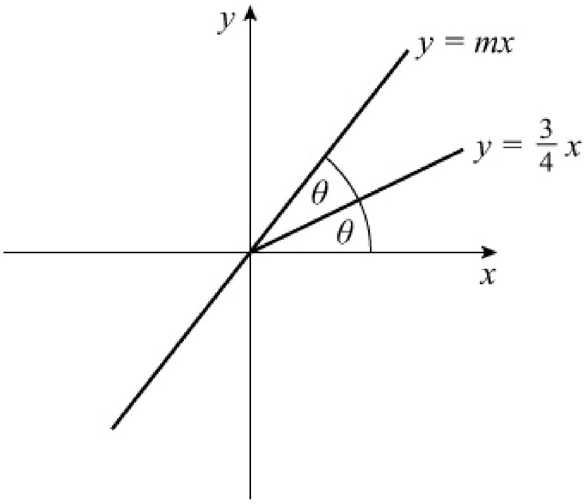
$\Rightarrow\ \sin^2\theta = \frac{2}{9}$

$\Rightarrow\ \sin\theta = \pm\frac{\sqrt{2}}{3}$

As $2\theta$ is acute, $\theta$ must be in the first quadrant so $\sin\theta$ is positive, so

$\sin\theta = \frac{\sqrt{2}}{3}$

**17** Sketch the problem,




**a** The gradient of line $l$ is $\frac{3}{4}$, which is $\tan\theta$.

So $\tan\theta = \frac{3}{4}$

**b** The gradient of $y = mx$ is m and as $y = \frac{3}{4}x$ bisects the angle between $y = mx$ and the $x$-axis

$m = \tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$

$= \frac{2\times\frac{3}{4}}{1 - \left(\frac{3}{4}\right)^2} = \frac{\frac{3}{2}}{\frac{7}{16}} = \frac{3}{2}\times\frac{16}{7} = \frac{24}{7}$

**18 a** $\cos 2A \equiv \cos(A + A)$

$\equiv \cos A\cos A - \sin A\sin A$

$\equiv \cos^2 A - \sin^2 A$

$\equiv \cos^2 A - (1 - \cos^2 A)$

$\equiv 2\cos^2 A - 1$

**b** The lines intersect when

$4\cos 2x = 6\cos^2 x - 3\sin 2x$

This equation can be written as

$\cos 2x + 3\cos 2x = 6\cos^2 x + 3\sin 2x$

Use the fact that $3\cos 2x = 6\cos^2 x - 3$, so the equation becomes

$\cos 2x + 6\cos^2 x - 3$

$= 6\cos x^2 - 3\sin 2x$

$\Rightarrow \cos 2x - 3 = 3\sin 2x$

$\Rightarrow \cos 2x + 3\sin 2x - 3 = 0$

**19** $\tan 2A \equiv \frac{\sin 2A}{\cos 2A} \equiv \frac{2\sin A\cos A}{\cos^2 A - \sin^2 A}$

$\equiv \dfrac{\dfrac{2\sin A\cos A}{\cos^2 A}}{\dfrac{\cos^2 A - \sin^2 A}{\cos^2 A}}$

$\equiv \dfrac{\dfrac{2\sin A}{\cos A}}{1 - \dfrac{\sin^2 A}{\cos^2 A}}$

$\equiv \frac{2\tan A}{1 - \tan^2 A}$