

Exercise 4B

$$\begin{aligned}
 1 \quad \mathbf{a} \quad \cos 15^\circ &= \cos(45^\circ - 30^\circ) \\
 &= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ \\
 &= \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \times \frac{1}{2} \\
 &= \frac{\sqrt{2}\sqrt{3} + \sqrt{2}}{4} = \frac{\sqrt{2}(\sqrt{3} + 1)}{4}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \sin 75^\circ &= \sin(45^\circ + 30^\circ) \\
 &= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ \\
 &= \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \times \frac{1}{2} \\
 &= \frac{\sqrt{2}\sqrt{3} + \sqrt{2}}{4} = \frac{\sqrt{2}(\sqrt{3} + 1)}{4}
 \end{aligned}$$

Note $\sin 75^\circ = \cos(90^\circ - 75^\circ) = \cos 15^\circ$

$$\begin{aligned}
 \mathbf{c} \quad \sin(120^\circ + 45^\circ) &= \sin 120^\circ \cos 45^\circ + \cos 120^\circ \sin 45^\circ \\
 &= \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} + \left(-\frac{1}{2}\right) \times \frac{\sqrt{2}}{2} \\
 &= \frac{\sqrt{2}\sqrt{3} - \sqrt{2}}{4} = \frac{\sqrt{2}(\sqrt{3} - 1)}{4}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad \tan 165^\circ &= \tan(120^\circ + 45^\circ) \\
 &= \frac{\tan 120^\circ + \tan 45^\circ}{1 - \tan 120^\circ \tan 45^\circ} \\
 \tan 120^\circ &= \frac{\sin 120^\circ}{\cos 120^\circ} = \frac{\sin 60^\circ}{-\cos 60^\circ} \\
 &= -\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3} \\
 \text{So } \tan 120^\circ &= \frac{-\sqrt{3} + 1}{1 + \sqrt{3}} \\
 &= \frac{(1 - \sqrt{3} + 1)(\sqrt{3} - 1)}{(\sqrt{3} + 1)(\sqrt{3} - 1)} \\
 &= \frac{-4 + 2\sqrt{3}}{2} \\
 &= -2 + \sqrt{3}
 \end{aligned}$$

$$\begin{aligned}
 2 \quad \mathbf{a} \quad &\text{Using } \sin(A + B) \text{ expansion} \\
 \sin 30^\circ \cos 60^\circ + \cos 30^\circ \sin 60^\circ &= \sin(30^\circ + 60^\circ) = \sin 90^\circ = 1
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \cos 110^\circ \cos 20^\circ + \sin 110^\circ \sin 20^\circ &= \cos(110^\circ - 20^\circ) = \cos 90^\circ = 0
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad \sin 33^\circ \cos 27^\circ + \cos 33^\circ \sin 27^\circ &= \sin(33^\circ + 27^\circ) = \sin 60^\circ = \frac{\sqrt{3}}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad \cos \frac{\pi}{8} \cos \frac{\pi}{8} - \sin \frac{\pi}{8} \sin \frac{\pi}{8} &= \cos\left(\frac{\pi}{8} + \frac{\pi}{8}\right) = \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad \sin 60^\circ \cos 15^\circ - \cos 60^\circ \sin 15^\circ &= \sin(60^\circ - 15^\circ) = \sin 45^\circ = \frac{\sqrt{2}}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad \cos 70^\circ \cos 50^\circ - \cos 70^\circ \tan 70^\circ \sin 50^\circ &= \cos 70^\circ \cos 50^\circ - \sin 70^\circ \sin 50^\circ
 \end{aligned}$$

Simplifying as

$$\left(\cos \theta \times \tan \theta = \cancel{\cos \theta} \times \frac{\sin \theta}{\cancel{\cos \theta}} = \sin \theta \right)$$

$$\begin{aligned}
 \text{So } \cos 70^\circ (\cos 50^\circ - \tan 70^\circ \sin 50^\circ) &= \cos(70^\circ + 50^\circ) \\
 &= \cos 120^\circ = -\cos 60^\circ = -\frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{g} \quad \frac{\tan 45^\circ + \tan 15^\circ}{1 - \tan 45^\circ \tan 15^\circ} &= \tan(45^\circ + 15^\circ) = \tan 60^\circ = \sqrt{3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{h} \quad \text{Use the fact that } \tan 45^\circ = 1 \text{ to rewrite as} \\
 \frac{1 - \tan 15^\circ}{1 + \tan 15^\circ} &= \frac{\tan 45^\circ - \tan 15^\circ}{1 + \tan 45^\circ \tan 15^\circ} \\
 &= \tan(45^\circ - 15^\circ) = \tan 30^\circ = \frac{\sqrt{3}}{3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{i} \quad \frac{\tan \frac{7\pi}{12} - \tan \frac{\pi}{3}}{1 + \tan \frac{7\pi}{12} \tan \frac{\pi}{3}} &= \tan\left(\frac{7\pi}{12} - \frac{\pi}{3}\right) \\
 &= \tan \frac{3\pi}{12} = \tan \frac{\pi}{4} = 1
 \end{aligned}$$

- 2 j This is very similar to part (e) but to appreciate this you need to rewrite the equation as

$$\begin{aligned} & \sqrt{3}\cos 15^\circ - \sin 15^\circ \\ &= 2\left(\frac{\sqrt{3}}{2}\cos 15^\circ - \frac{1}{2}\sin 15^\circ\right) \\ &= 2(\sin 60^\circ \cos 15^\circ - \cos 60^\circ \sin 15^\circ) \\ &= 2\sin(60 - 15)^\circ \\ &= 2\sin 45^\circ \\ &= \sqrt{2} \end{aligned}$$

3 a $\tan(45^\circ + 30^\circ) = \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ}$

b $\tan 75^\circ = \frac{1 + \frac{\sqrt{3}}{3}}{1 - \frac{\sqrt{3}}{3}}$

$$\begin{aligned} &= \frac{3 + \sqrt{3}}{3 - \sqrt{3}} = \frac{(3 + \sqrt{3})(3 + \sqrt{3})}{(3 - \sqrt{3})(3 + \sqrt{3})} \\ &= \frac{12 + 6\sqrt{3}}{6} = 2 + \sqrt{3} \end{aligned}$$

4 $\cot(A + B) = 2$

$$\Rightarrow \tan(A + B) = \frac{1}{2}$$

$$\Rightarrow \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{1}{2}$$

But as $\cot A = \frac{1}{4}$, then $\tan A = 4$.

$$\text{So } \frac{4 + \tan B}{1 - 4 \tan B} = \frac{1}{2}$$

$$\Rightarrow 8 + 2 \tan B = 1 - 4 \tan B$$

$$\Rightarrow 6 \tan B = -7$$

$$\Rightarrow \tan B = -\frac{7}{6}$$

$$\text{So } \cot B = \frac{1}{\tan B} = -\frac{6}{7}$$

5 a $\cos 105^\circ = \cos(45^\circ + 60^\circ)$

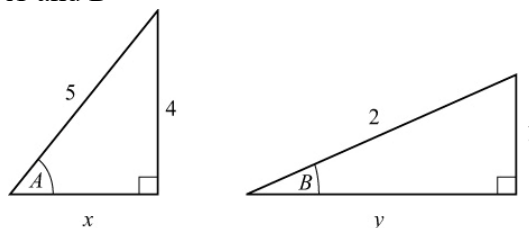
$$\begin{aligned} &= \cos 45^\circ \cos 60^\circ - \sin 45^\circ \sin 60^\circ \\ &= \frac{1}{\sqrt{2}} \times \frac{1}{2} - \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} \\ &= \frac{1 - \sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{2} - \sqrt{6}}{4} \end{aligned}$$

b $\sec 105^\circ = \frac{1}{\cos 105^\circ}$

$$\begin{aligned} &= \frac{1}{\frac{\sqrt{2} - \sqrt{6}}{4}} = \frac{4}{\sqrt{2} - \sqrt{6}} \\ &= \frac{4}{\sqrt{2} - \sqrt{6}} \times \frac{\sqrt{2} + \sqrt{6}}{\sqrt{2} + \sqrt{6}} \\ &= \frac{4(\sqrt{2} + \sqrt{6})}{-4} = -\sqrt{2}(1 + \sqrt{3}) \end{aligned}$$

So $a = 2$ and $b = 3$

- 6 Draw the right-angled triangles containing A and B



Using Pythagoras' theorem gives

$$x = 3 \text{ and } y = \sqrt{3}$$

a $\sin(A + B) = \sin A \cos B + \cos A \sin B$

$$= \frac{4}{5} \times \frac{\sqrt{3}}{2} + \frac{3}{5} \times \frac{1}{2} = \frac{4\sqrt{3} + 3}{10}$$

b $\cos(A - B) = \cos A \cos B + \sin A \sin B$

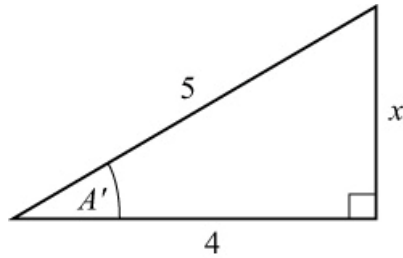
$$= \frac{3}{5} \times \frac{\sqrt{3}}{2} + \frac{4}{5} \times \frac{1}{2} = \frac{3\sqrt{3} + 4}{10}$$

c $\sec(A - B) = \frac{1}{\cos(A - B)} = \frac{10}{3\sqrt{3} + 4}$

$$\begin{aligned} &= \frac{10(3\sqrt{3} - 4)}{(3\sqrt{3} + 4)(3\sqrt{3} - 4)} \\ &= \frac{10(3\sqrt{3} - 4)}{27 - 16} \\ &= \frac{10(3\sqrt{3} - 4)}{11} \end{aligned}$$

- 7 Let $A' = 180^\circ - A$. As A is the second quadrant $\cos A' = -\cos A$

Draw a right-angled triangle where $\cos A' = \frac{4}{5}$



Using Pythagoras' theorem $x = 3$

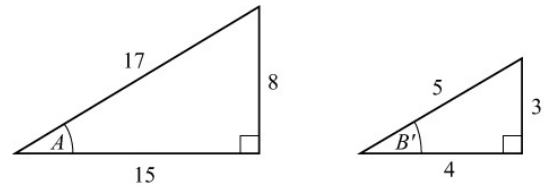
So $\sin A' = \frac{3}{5}$, $\tan A' = \frac{3}{4}$

- a As A is in the second quadrant,
 $\sin A = \sin A'$, $\sin A = \frac{3}{5}$
- b $\cos(\pi + A) = \cos \pi \cos A - \sin \pi \sin A$
 $= -\cos A$
As $\cos \pi = -1$, $\sin \pi = 0$
So $\cos(\pi + A) = \frac{4}{5}$
- c $\sin\left(\frac{\pi}{3} + A\right) = \sin \frac{\pi}{3} \cos A + \cos \frac{\pi}{3} \sin A$
 $= \left(\frac{\sqrt{3}}{2}\right)\left(-\frac{4}{5}\right) + \left(\frac{1}{2}\right)\left(\frac{3}{5}\right)$
 $= \frac{3 - 4\sqrt{3}}{10}$

- d As A is in the second quadrant,
 $\tan A = -\tan A' = -\frac{3}{4}$
- $$\tan\left(\frac{\pi}{4} + A\right) = \frac{\tan \frac{\pi}{4} + \tan A}{1 - \tan \frac{\pi}{4} \tan A}$$
- $$= \frac{1 + \tan A}{1 - \tan A} = \frac{\frac{1}{4}}{\frac{7}{4}} = \frac{1}{7}$$

- 8 Let $B' = 180^\circ - B$. As B is in the second quadrant $\cos B' = -\cos B$, $\sin B' = \sin B$ and $\tan B' = -\tan B$.

Drawing right-angled triangles for A and B' , use Pythagoras' theorem to find the missing sides, which are 15 and 3.

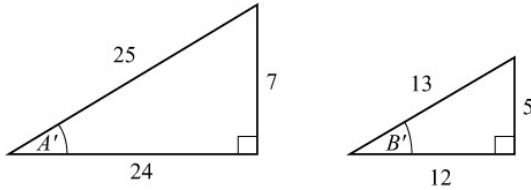


So $\sin A = \frac{8}{17}$, $\cos A = \frac{15}{17}$, $\tan A = \frac{8}{15}$
and $\sin B = \frac{3}{5}$, $\cos B = -\frac{4}{5}$, $\tan B = -\frac{3}{4}$

- a $\sin(A - B) = \sin A \cos B - \cos A \sin B$
 $= \left(\frac{8}{17}\right)\left(-\frac{4}{5}\right) - \left(\frac{15}{17}\right)\left(\frac{3}{5}\right)$
 $= \frac{-32 - 45}{85} = -\frac{77}{85}$
- b $\cos(A - B) = \cos A \cos B + \sin A \sin B$
 $= \left(\frac{15}{17}\right)\left(-\frac{4}{5}\right) + \left(\frac{8}{17}\right)\left(\frac{3}{5}\right)$
 $= \frac{-60 + 24}{85} = -\frac{36}{85}$
- c $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$
 $= \frac{\frac{8}{15} - \frac{3}{4}}{1 - \frac{24}{60}} = \frac{\frac{77}{60}}{\frac{36}{60}} = \frac{77}{36}$
So $\cot(A - B) = \frac{1}{\tan(A - B)} = \frac{36}{77}$

- 9 Angle A is in the third quadrant as it is reflex and $\tan A$ is positive. Let $A' = A - 180^\circ$, so $\sin A = -\sin A'$, $\cos A = -\cos A'$, $\tan A = \tan A'$. Let $B' = 180^\circ - B$. As B is in the second quadrant $\cos B' = -\cos B$, $\sin B' = \sin B$ and $\tan B' = -\tan B$.

Drawing right-angled triangles for A' and B' use Pythagoras' theorem to find the missing sides, which are 25 and 12.



So $\sin A = -\frac{7}{25}$, $\cos A = -\frac{24}{25}$, $\tan A = \frac{7}{24}$
and $\sin B = \frac{5}{13}$, $\cos B = -\frac{12}{13}$, $\tan B = -\frac{5}{12}$

$$\begin{aligned} \mathbf{a} \quad \sin(A+B) &= \sin A \cos B + \cos A \sin B \\ &= \left(-\frac{7}{25}\right)\left(-\frac{12}{13}\right) + \left(-\frac{24}{25}\right)\left(\frac{5}{13}\right) \\ &= \frac{84-120}{325} = -\frac{36}{325} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \tan(A-B) &= \frac{\tan A - \tan B}{1 + \tan A \tan B} \\ &= \frac{\frac{7}{24} + \frac{5}{12}}{1 - \left(\frac{7}{24}\right)\left(\frac{5}{12}\right)} = \frac{\frac{17}{24}}{\frac{253}{288}} = \frac{204}{253} \end{aligned}$$

$$\mathbf{c} \quad \operatorname{cosec}(A+B) = \frac{1}{\sin(A+B)} = -\frac{325}{36}$$

$$\begin{aligned} \mathbf{10 a} \quad \tan(A+B) &= \frac{\tan A + \tan B}{1 - \tan A \tan B} \\ &= \frac{\frac{1}{5} + \frac{2}{3}}{1 - \frac{1}{5} \times \frac{2}{3}} = \frac{\frac{13}{15}}{\frac{15-2}{15}} = \frac{13}{15} = 1 \end{aligned}$$

As $\tan(A+B)$ is positive, $A+B$ is in the first or third quadrants, but as A and B are both acute $A+B$ cannot be in the third quadrant, so $A+B = \tan^{-1} 1 = 45^\circ$

- b** A is reflex but $\tan A$ is positive, so A is in the third quadrant, i.e. $180^\circ < A < 270^\circ$ and $0^\circ < B < 90^\circ$. As $\tan(A+B)$ is positive, $A+B$ is in the first or third quadrants.

As $180^\circ < A+B < 360^\circ$, it must be in the third quadrant, so $A+B = \tan^{-1} 1 = 225^\circ$