Exercise 4A

1 a i $\angle FAB = \angle CAF + \angle BAC$
$= (\alpha - \beta) + \beta = \alpha$
So $\angle FAB = \alpha$

ii $\angle FAB$ and $\angle ABD$ are alternate angles
so $\angle FAB = \angle ABD$
so $\angle ABD = \alpha$
$\angle ECB = 90 - \alpha$, so $\angle ECB = 90 - (90 - \alpha) = \alpha$

iii $\cos\beta = \dfrac{AB}{1}$
So $AB = \cos\beta$

iv $\sin\beta = \dfrac{BC}{1}$
So $BC = \sin\beta$

b i $\angle ABD = \alpha$, so $\sin\alpha = \dfrac{AD}{AB}$

As $AB = \cos\beta$, this gives $\sin\alpha = \dfrac{AD}{\cos\beta}$

So $AD = \sin\alpha\cos\beta$

ii $\cos\alpha = \dfrac{BD}{AB} = \dfrac{BD}{\cos\beta}$
So $BD = \cos\alpha\cos\beta$

c i $\angle BEC = \alpha$, so $\cos\alpha = \dfrac{CE}{BC}$

As $BC = \sin\beta$, this gives $\cos\alpha = \dfrac{CE}{\sin\beta}$
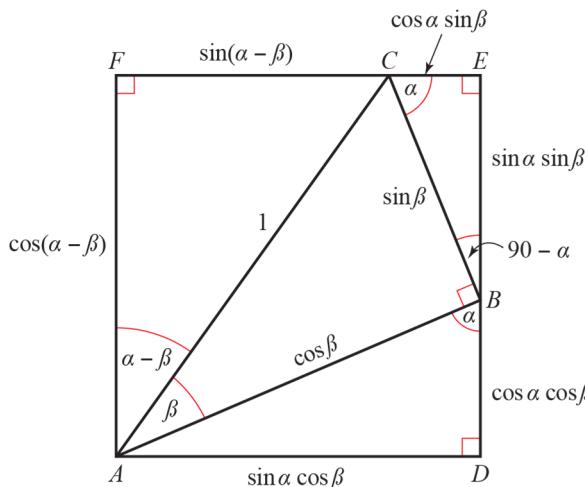
So $CE = \cos\alpha\sin\beta$

ii $\sin\alpha = \dfrac{BE}{BC} = \dfrac{BE}{\sin\beta}$
So $BE = \sin\alpha\sin\beta$

d i $\sin(\alpha - \beta) = \dfrac{FC}{1}$
So $FC = \sin(\alpha - \beta)$

1 d ii $\cos(\alpha - \beta) = \dfrac{FA}{1}$

So $FA = \cos(\alpha - \beta)$

e i The completed diagram should look like this:

$FC + CE = AD$, so $FC = AD - CE$
$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$

ii $AF = DB + BE$
$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$

2 $\tan(A-B) = \dfrac{\sin(A-B)}{\cos(A-B)}$

$= \dfrac{\sin A \cos B - \cos A \sin B}{\cos A \cos B + \sin A \sin B}$

Divide the numerator and denominator by $\cos A \cos B$

$$\tan(A-B) = \dfrac{\dfrac{\sin A \cos B}{\cos A \cos B} - \dfrac{\cos A \sin B}{\cos A \cos B}}{\dfrac{\cos A \cos B}{\cos A \cos B} + \dfrac{\sin A \sin B}{\cos A \cos B}} = \dfrac{\dfrac{\sin A}{\cos A} - \dfrac{\sin B}{\cos B}}{1 + \dfrac{\sin A \sin B}{\cos A \cos B}}$$

$$= \dfrac{\tan A - \tan B}{1 + \tan A \tan B} \text{ as required}$$

3 $\sin(A + B) = \sin A \cos B + \cos A \sin B$
$\sin(P + (-Q)) = \sin P \cos(-Q) + \cos P \sin(-Q)$
As $\cos(-P) = \cos P$ and $\sin(-P) = -\sin P$, this gives
$\sin(P - Q) \equiv \sin P \cos Q - \cos P \sin Q$

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Pearson

4 Example: $A = 60°$, $B = 30°$

$$\sin A = \frac{\sqrt{3}}{2}, \ \sin B = \frac{1}{2}$$

$$\sin(A+B) = 1; \ \sin A + \sin B = \frac{\sqrt{3}}{2} + \frac{1}{2} \neq 1$$

This proves $\sin(A+B) = \sin A + \sin B$ is not true for all values.

There will be many values of A and B for which the statement is true, e.g. $A = -30°$ and $B = +30°$, and this is the danger of trying to prove a statement by taking particular examples. To prove a statement requires a sound argument; to disprove it only requires one counterexample.

5 $\cos(A - B) \equiv \cos A \cos B + \sin A \sin B$
Set $A = \theta$, $B = \theta$

$$\Rightarrow \ \cos(\theta - \theta) \equiv \cos\theta\cos\theta + \sin\theta\sin\theta$$

$$\Rightarrow \ \cos 0 \equiv \cos^2\theta + \sin^2\theta$$

So $\cos^2\theta + \sin^2\theta \equiv 1$ (since $\cos 0 = 1$)

6 a $\sin(A - B) \equiv \sin A \cos B - \cos A \sin B$

Set $A = \frac{\pi}{2}$, $B = \theta$

$$\Rightarrow \ \sin\left(\frac{\pi}{2} - \theta\right) \equiv \sin\frac{\pi}{2}\cos\theta - \cos\frac{\pi}{2}\sin\theta$$

$$\Rightarrow \ \sin\left(\frac{\pi}{2} - \theta\right) \equiv \cos\theta$$

since $\sin\frac{\pi}{2} = 1, \cos\frac{\pi}{2} = 0$

b $\cos(A - B) \equiv \cos A \cos B + \sin A \sin B$

Set $A = \frac{\pi}{2}$, $B = \theta$

$$\Rightarrow \ \cos\left(\frac{\pi}{2} - \theta\right) \equiv \cos\frac{\pi}{2}\cos\theta + \sin\frac{\pi}{2}\sin\theta$$

$$\Rightarrow \ \cos\left(\frac{\pi}{2} - \theta\right) \equiv \sin\theta$$

since $\cos\frac{\pi}{2} = 0, \sin\frac{\pi}{2} = 1$

7 $\sin\left(x + \frac{\pi}{6}\right) = \sin x \cos\frac{\pi}{6} + \cos x \sin\frac{\pi}{6}$

$$= \frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x$$

8 $\cos\left(x+\dfrac{\pi}{3}\right)=\cos x\cos\dfrac{\pi}{3}-\sin x\sin\dfrac{\pi}{3}$

$=\dfrac{1}{2}\cos x-\dfrac{\sqrt{3}}{2}\sin x$

9 a Using $\sin(A+B)\equiv\sin A\cos B+\cos A\sin B$ gives
   $\sin 15°\cos 20°+\cos 15°\sin 20°\equiv\sin(15°+20°)\equiv\sin 35°$

b Using $\sin(A-B)\equiv\sin A\cos B-\cos A\sin B$ gives
   $\sin 58°\cos 23°-\cos 58°\sin 23°\equiv\sin(58°-23°)\equiv\sin 35°$

c Using $\cos(A+B)\equiv\cos A\cos B-\sin A\sin B$ gives
   $\cos 130°\cos 80°-\sin 130°\sin 80°\equiv\cos(130°+80°)\equiv\cos 210°$

d Using $\tan(A-B)\equiv\dfrac{\tan A-\tan B}{1+\tan A\tan B}$ gives

   $\dfrac{\tan 76°-\tan 45°}{1+\tan 76°\tan 45°}\equiv\tan(76°-45°)\equiv\tan 31°$

e Using $\cos(A-B)\equiv\cos A\cos B+\sin A\sin B$ gives
   $\cos 2\theta\cos\theta+\sin 2\theta\sin\theta\equiv\cos(2\theta-\theta)\equiv\cos\theta$

f Using $\cos(A+B)\equiv\cos A\cos B-\sin A\sin B$ gives
   $\cos 4\theta\cos 3\theta-\sin 4\theta\sin 3\theta\equiv\cos(4\theta+3\theta)\equiv\cos 7\theta$

g Using $\sin(A+B)\equiv\sin A\cos B+\cos A\sin B$ gives
   $\sin\tfrac{1}{2}\theta\cos\tfrac{5}{2}\theta+\cos\tfrac{1}{2}\theta\sin\tfrac{5}{2}\theta\equiv\sin\left(\tfrac{1}{2}\theta+\tfrac{5}{2}\theta\right)\equiv\sin 3\theta$

h Using $\tan(A+B)\equiv\dfrac{\tan A+\tan B}{1-\tan A\tan B}$ gives

   $\dfrac{\tan 2\theta+\tan 3\theta}{1-\tan 2\theta\tan 3\theta}\equiv\tan(2\theta+3\theta)\equiv\tan 5\theta$

i Using $\sin(P-Q)\equiv\sin P\cos Q-\cos P\sin Q$ gives
   $\sin(A+B)\cos B-\cos(A+B)\sin B\equiv\sin\left((A+B)-B\right)\equiv\sin A$

j Using $\cos(A+B)\equiv\cos A\cos B-\sin A\sin B$ gives
   $\cos\left(\dfrac{3x+2y}{2}\right)\cos\left(\dfrac{3x-2y}{2}\right)-\sin\left(\dfrac{3x+2y}{2}\right)\sin\left(\dfrac{3x-2y}{2}\right)\equiv\cos\left(\left(\dfrac{3x+2y}{2}\right)+\left(\dfrac{3x-2y}{2}\right)\right)$

   $\equiv\cos\left(\dfrac{6x}{2}\right)\equiv\cos 3x$

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10 a Use the fact that $\dfrac{1}{\sqrt{2}} = \cos\dfrac{\pi}{4} = \sin\dfrac{\pi}{4}$ to write

$$\frac{1}{\sqrt{2}}(\sin x + \cos x) = \frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = \sin x \cos\frac{\pi}{4} + \cos x \sin\frac{\pi}{4} = \sin\left(x + \frac{\pi}{4}\right)$$

or

$$\frac{1}{\sqrt{2}}(\sin x + \cos x) = \frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x = \cos x \cos\frac{\pi}{4} + \sin x \sin\frac{\pi}{4} = \cos\left(x - \frac{\pi}{4}\right)$$

b Use the fact that $\dfrac{1}{\sqrt{2}} = \cos\dfrac{\pi}{4} = \sin\dfrac{\pi}{4}$ to write

$$\frac{1}{\sqrt{2}}(\cos x - \sin x) = \frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x = \cos x \cos\frac{\pi}{4} - \sin x \sin\frac{\pi}{4} = \cos\left(x + \frac{\pi}{4}\right)$$

c Use the fact that $\dfrac{1}{2} = \cos\dfrac{\pi}{3} = \sin\dfrac{\pi}{6}$ and $\dfrac{\sqrt{3}}{2} = \cos\dfrac{\pi}{6} = \sin\dfrac{\pi}{3}$ to write

$$\frac{1}{2}(\sin x + \sqrt{3}\cos x) = \frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x = \sin x \cos\frac{\pi}{3} + \cos x \sin\frac{\pi}{3} = \sin\left(x + \frac{\pi}{3}\right)$$

or

$$\frac{1}{2}(\sin x + \sqrt{3}\cos x) = \frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x = \cos x \cos\frac{\pi}{6} + \sin x \sin\frac{\pi}{6} = \cos\left(x - \frac{\pi}{6}\right)$$

d Use the fact that $\dfrac{1}{\sqrt{2}} = \cos\dfrac{\pi}{4} = \sin\dfrac{\pi}{4}$ to write

$$\frac{1}{\sqrt{2}}(\sin x - \cos x) = \frac{1}{\sqrt{2}}\sin x - \frac{1}{\sqrt{2}}\cos x = \sin x \cos\frac{\pi}{4} - \cos x \sin\frac{\pi}{4} = \sin\left(x - \frac{\pi}{4}\right)$$

11 $\cos y = \sin(x + y)$

$\Rightarrow \quad \cos y = \sin x \cos y + \cos x \sin y$

Divide throughout by $\cos x \cos y$

$$\frac{\cancel{\cos y}^{1}}{\cos x \cancel{\cos y}} = \frac{\sin x \cancel{\cos y}}{\cos x \cancel{\cos y}} + \frac{\cancel{\cos x} \sin y}{\cancel{\cos x} \cos y}$$

$\Rightarrow \quad \sec x = \tan x + \tan y$

$\Rightarrow \quad \tan y = \sec x - \tan x$

12 As $\tan(x - y) = 3$

so $\dfrac{\tan x - \tan y}{1 + \tan x \tan y} = 3$

$\Rightarrow \quad \tan x - \tan y = 3 + 3\tan x \tan y$

$\Rightarrow \quad 3\tan x \tan y + \tan y = \tan x - 3$

$\Rightarrow \quad \tan y(3\tan x + 1) = \tan x - 3$

$\Rightarrow \quad \tan y = \dfrac{\tan x - 3}{3\tan x + 1}$

13 $\sin x(\cos y + 2\sin y) = \cos x(2\cos y - \sin y)$

$\Rightarrow \ \sin x\cos y + 2\sin x\sin y = 2\cos x\cos y - \cos x\sin y$

$\Rightarrow \ \sin x\cos y + \cos x\sin y = 2(\cos x\cos y - \sin x\sin y)$

$\Rightarrow \ \sin(x+y) = 2\cos(x+y)$

$\Rightarrow \ \dfrac{\sin(x+y)}{\cos(x+y)} = 2$

$\Rightarrow \ \tan(x+y) = 2$

14 a $\tan(x - 45°) = \frac{1}{4}$

$\Rightarrow \ \dfrac{\tan x - \tan 45°}{1 + \tan x\tan 45°} = \dfrac{1}{4}$

$\Rightarrow \ 4\tan x - 4 = 1 + \tan x \ \ (\text{as } \tan 45° = 1)$

$\Rightarrow \ 3\tan x = 5$

$\Rightarrow \ \tan x = \frac{5}{3}$

b $\sin(x - 60°) = 3\cos(x + 30°)$

$\Rightarrow \ \sin x\cos 60° - \cos x\sin 60° = 3\cos x\cos 30° - 3\sin x\sin 30°$

$\Rightarrow \ \dfrac{1}{2}\sin x - \dfrac{\sqrt{3}}{2}\cos x = \dfrac{3\sqrt{3}}{2}\cos x - \dfrac{3}{2}\sin x$

$\Rightarrow \ 4\sin x = 4\sqrt{3}\cos x$

$\Rightarrow \ \dfrac{\sin x}{\cos x} = \dfrac{4\sqrt{3}}{4}$

$\Rightarrow \ \tan x = \sqrt{3}$

c $\tan(x - 60°) = 2$

$\Rightarrow \ \dfrac{\tan x - \tan 60°}{1 + \tan x\tan 60°} = 2$

$\Rightarrow \ \dfrac{\tan x - \sqrt{3}}{1 + \sqrt{3}\tan x} = 2 \ \ (\text{as } \tan 60° = \sqrt{3})$

$\Rightarrow \ \tan x - \sqrt{3} = 2 + 2\sqrt{3}\tan x$

$\Rightarrow \ (2\sqrt{3} - 1)\tan x = -(2 + \sqrt{3})$

$\Rightarrow \ \tan x = -\dfrac{(2 + \sqrt{3})}{2\sqrt{3} - 1} = -\dfrac{(2 + \sqrt{3})(2\sqrt{3} + 1)}{(2\sqrt{3} - 1)(2\sqrt{3} + 1)}$

$\qquad\quad = -\dfrac{8 + 5\sqrt{3}}{11}$

15 $\tan\left(x + \dfrac{\pi}{3}\right) = \dfrac{1}{2}$

$\Rightarrow \dfrac{\tan x + \tan\frac{\pi}{3}}{1 - \tan x \tan\frac{\pi}{3}} = \dfrac{1}{2}$

$\Rightarrow \dfrac{\tan x + \sqrt{3}}{1 - \sqrt{3}\tan x} = \dfrac{1}{2} \quad \left(\text{since } \tan\dfrac{\pi}{3} = \sqrt{3}\right)$

$\Rightarrow 2\tan x + 2\sqrt{3} = 1 - \sqrt{3}\tan x$

$\Rightarrow (2 + \sqrt{3})\tan x = 1 - 2\sqrt{3}$

$\Rightarrow \tan x = \dfrac{1 - 2\sqrt{3}}{2 + \sqrt{3}} = \dfrac{(1 - 2\sqrt{3})(2 - \sqrt{3})}{(2 + \sqrt{3})(2 - \sqrt{3})}$

$\qquad\quad = \dfrac{2 - 4\sqrt{3} - \sqrt{3} + 6}{1} = 8 - 5\sqrt{3}$

16 Write $\theta = \left(\theta + \dfrac{2\pi}{3}\right) - \dfrac{2\pi}{3}$ and $\theta + \dfrac{4\pi}{3} = \left(\theta + \dfrac{2\pi}{3}\right) + \dfrac{2\pi}{3}$

Now use the appropriate addition formulae for cos

$\cos\left(\left(\theta + \dfrac{2\pi}{3}\right) - \dfrac{2\pi}{3}\right) = \cos\left(\theta + \dfrac{2\pi}{3}\right)\cos\dfrac{2\pi}{3} + \sin\left(\theta + \dfrac{2\pi}{3}\right)\sin\dfrac{2\pi}{3}$

$\cos\left(\left(\theta + \dfrac{2\pi}{3}\right) + \dfrac{2\pi}{3}\right) = \cos\left(\theta + \dfrac{2\pi}{3}\right)\cos\dfrac{2\pi}{3} - \sin\left(\theta + \dfrac{2\pi}{3}\right)\sin\dfrac{2\pi}{3}$

Now add up all terms

$\cos\theta + \cos\left(\theta + \dfrac{2\pi}{3}\right) + \cos\left(\theta + \dfrac{4\pi}{3}\right)$

$\equiv \cos\left(\left(\theta + \dfrac{2\pi}{3}\right) - \dfrac{2\pi}{3}\right) + \cos\left(\theta + \dfrac{2\pi}{3}\right) + \cos\left(\left(\theta + \dfrac{2\pi}{3}\right) + \dfrac{2\pi}{3}\right)$

$\equiv \cos\left(\theta + \dfrac{2\pi}{3}\right)\cos\dfrac{2\pi}{3} + \sin\left(\theta + \dfrac{2\pi}{3}\right)\sin\dfrac{2\pi}{3} + \cos\left(\theta + \dfrac{2\pi}{3}\right) + \cos\left(\theta + \dfrac{2\pi}{3}\right)\cos\dfrac{2\pi}{3}$

$\qquad - \sin\left(\theta + \dfrac{2\pi}{3}\right)\sin\dfrac{2\pi}{3}$

$\equiv 2\cos\left(\theta + \dfrac{2\pi}{3}\right)\cos\dfrac{2\pi}{3} + \cos\left(\theta + \dfrac{2\pi}{3}\right)$

$\equiv 0$ as $\cos\dfrac{2\pi}{3} = -\dfrac{1}{2}$

Challenge

a i Area $= \frac{1}{2} ab \sin \theta = \frac{1}{2} x (y \cos B)(\sin A)$

$\qquad = \frac{1}{2} xy \sin A \cos B$

$\quad$ **ii** Area $= \frac{1}{2} ab \sin \theta = \frac{1}{2} y (x \cos A)(\sin B)$

$\qquad = \frac{1}{2} xy \cos A \sin B$

$\quad$ **iii** Area $= \frac{1}{2} ab \sin \theta = \frac{1}{2} xy \sin(A + B)$

b Area $T_1 + T_2 = $ Area $T_1 + $ Area T_2

$\Rightarrow \frac{1}{2} xy \sin(A + B) = \frac{1}{2} xy \sin A \cos B + \frac{1}{2} xy \cos A \sin B$

$\Rightarrow \sin(A + B) = \sin A \cos B + \cos A \sin B$