Chapter review

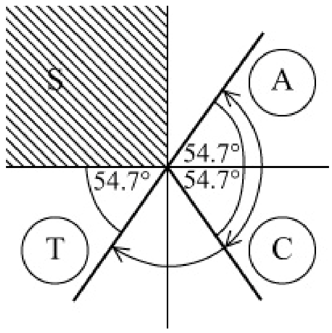
1 $\tan x = 2\cot x, \; -180° \le x \le 90°$

$\Rightarrow \tan x = \frac{2}{\tan x}$

$\Rightarrow \tan^2 x = 2$

$\Rightarrow \tan x = \pm\sqrt{2}$

Calculator value for $\tan x = +\sqrt{2}$ is 54.7° (1 d.p.)



Solutions are required in the 1st, 3rd and 4th quadrants.

Solution set is:

$-125.3°, \; -54.7°, \; 54.7°$ (1 d.p.)

2 $p = 2\sec\theta \;\Rightarrow\; \sec\theta = \frac{p}{2}$

$q = 4\cos\theta \;\Rightarrow\; \cos\theta = \frac{q}{4}$

$\sec\theta = \frac{1}{\cos\theta} \;\Rightarrow\; \frac{p}{2} = \frac{1}{\frac{q}{4}} = \frac{4}{q} \;\Rightarrow\; p = \frac{8}{q}$

3 $p = \sin\theta \;\Rightarrow\; \frac{1}{p} = \frac{1}{\sin\theta} = \operatorname{cosec}\theta$

$q = 4\cot\theta \;\Rightarrow\; \cot\theta = \frac{q}{4}$

Using $1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta$

$\Rightarrow 1 + \frac{q^2}{16} = \frac{1}{p^2}$ (multiply by $16p^2$)

$\Rightarrow 16p^2 + p^2q^2 = 16$

$\Rightarrow p^2q^2 = 16 - 16p^2 = 16(1 - p^2)$

4 a i $\operatorname{cosec}\theta = 2\cot\theta, \; 0 < \theta < 180°$

$\Rightarrow \frac{1}{\sin\theta} = \frac{2\cos\theta}{\sin\theta}$

$\Rightarrow 2\cos\theta = 1$

$\Rightarrow \cos\theta = \frac{1}{2}$

$\Rightarrow \theta = 60°$

ii $2\cot^2\theta = 7\operatorname{cosec}\theta - 8, \; 0 < \theta < 180°$

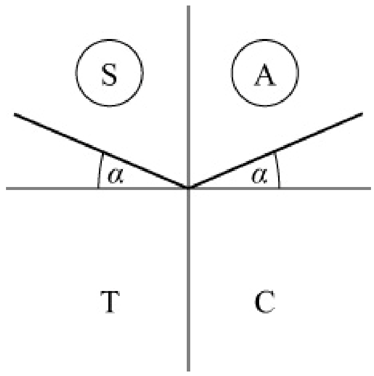
$\Rightarrow 2(\operatorname{cosec}^2\theta - 1) = 7\operatorname{cosec}\theta - 8$

$\Rightarrow 2\operatorname{cosec}^2\theta - 7\operatorname{cosec}\theta + 6 = 0$

$\Rightarrow (2\operatorname{cosec}\theta - 3)(\operatorname{cosec}\theta - 2) = 0$

$\Rightarrow \operatorname{cosec}\theta = \frac{3}{2}$ or $\operatorname{cosec}\theta = 2$

So $\sin\theta = \frac{2}{3}$ or $\sin\theta = \frac{1}{2}$



Solutions are $\alpha°$ and $(180 - \alpha)°$ where α is the calculator value.

$\sin\theta = \frac{2}{3}$

Calculator value is 41.8° (1 d.p.)

Solutions are 41.8°, 138.2°

$\sin\theta = \frac{1}{2}$

Calculator value is 30° (1 d.p.)

Solutions are 30°, 150°

Solution set is:

30°, 41.8°, 138.2°, 150°

4 b i $\sec(2\theta - 15°) = \text{cosec}\,135°$,
$0 \le \theta \le 360°$

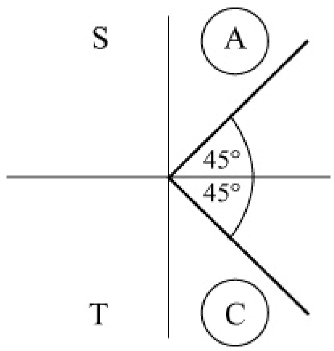
$\Rightarrow \cos(2\theta - 15°) = \dfrac{1}{\text{cosec}\,135°}$

$= \sin 135° = \dfrac{\sqrt{2}}{2}$

Solve $\cos(2\theta - 15°) = \dfrac{\sqrt{2}}{2}$, in the interval $-15° \le 2\theta - 15° \le 705°$

The calculator value is 45°

cos is positive, so $(2\theta - 15°)$ is in the 1st and 4th quadrants.



So $(2\theta - 15°) = 45°, 315°, 405°, 675°$

$\Rightarrow 2\theta = 60°, 330°, 420°, 690°$

$\Rightarrow \theta = 30°, 165°, 210°, 345°$

ii $\sec^2\theta + \tan\theta = 3,\ 0 \le \theta \le 360°$

$\Rightarrow 1 + \tan^2\theta + \tan\theta = 3$

$\Rightarrow \tan^2\theta + \tan\theta - 2 = 0$

$\Rightarrow (\tan\theta - 1)(\tan\theta + 2) = 0$

$\Rightarrow \tan\theta = 1$ *or* $\tan\theta = -2$

$\tan\theta = 1 \Rightarrow \theta = 45°, 180° + 45°$,

i.e. $45°, 225°$

$\tan\theta = -2$,

calculator value is $-63.4°$ (1 d.p.)

$\Rightarrow \theta = 180° + (-63.4°) = 116.6°$

$\theta = 360° + (-63.4°) = 296.6°$

Solution set is:

$45°, 116.6°, 225°, 296.6°$

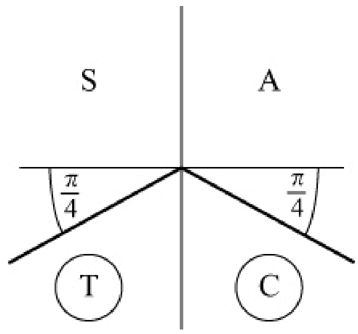
c i $\text{cosec}\left(x + \dfrac{\pi}{15}\right) = -\sqrt{2},\ 0 \le x \le 2\pi$

$\Rightarrow \sin\left(x + \dfrac{\pi}{15}\right) = -\dfrac{1}{\sqrt{2}}$

Calculator value is $-\dfrac{\pi}{4}$

$\sin\left(x + \dfrac{\pi}{15}\right)$ is negative,

so $x + \dfrac{\pi}{15}$ is in 3rd and 4th quadrants.



So $x + \dfrac{\pi}{15} = \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

$\Rightarrow x = \dfrac{5\pi}{4} - \dfrac{\pi}{15}, \dfrac{7\pi}{4} - \dfrac{\pi}{15}$

$= \dfrac{75\pi - 4\pi}{60}, \dfrac{105\pi - 4\pi}{60}$

$= \dfrac{71\pi}{60}, \dfrac{101\pi}{60}$

ii $\sec^2 x = \dfrac{4}{3},\ 0 \le x \le 2\pi$

$\Rightarrow \cos^2 x = \dfrac{3}{4}$

$\Rightarrow \cos x = \pm\dfrac{\sqrt{3}}{2}$

Calculator value for $\cos x = +\dfrac{\sqrt{3}}{2}$ is $\dfrac{\pi}{6}$

As $\cos x$ is $\pm$, x is in all four quadrants. Solution set is:

$x = \dfrac{\pi}{6}, \pi - \dfrac{\pi}{6}, \pi + \dfrac{\pi}{6}, 2\pi - \dfrac{\pi}{6}$

$= \dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$

5 $5\sin x\cos y + 4\cos x\sin y = 0$

$\Rightarrow \dfrac{5\sin x\cos y}{\sin x\sin y} + \dfrac{4\cos x\sin y}{\sin x\sin y} = 0$

(divide by $\sin x\sin y$)

$\Rightarrow \dfrac{5\cos y}{\sin y} + \dfrac{4\cos x}{\sin x} = 0$

So $5\cot y + 4\cot x = 0$

As $\cot x = 2$

$5\cot y + 8 = 0$

$5\cot y = -8$

$\cot y = -\dfrac{8}{5}$

6 a LHS $\equiv (\tan\theta + \cot\theta)(\sin\theta + \cos\theta)$

$\equiv \left(\dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta}\right)(\sin\theta + \cos\theta)$

$\equiv \left(\dfrac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta}\right)(\sin\theta + \cos\theta)$

$\equiv \left(\dfrac{1}{\cos\theta\sin\theta}\right)(\sin\theta + \cos\theta)$

$\equiv \dfrac{\sin\theta}{\cos\theta\sin\theta} + \dfrac{\cos\theta}{\cos\theta\sin\theta}$

$\equiv \dfrac{1}{\cos\theta} + \dfrac{1}{\sin\theta}$

$\equiv \sec\theta + \operatorname{cosec}\theta \equiv$ RHS

b LHS $\equiv \dfrac{\operatorname{cosec} x}{\operatorname{cosec} x - \sin x}$

$\equiv \dfrac{\dfrac{1}{\sin x}}{\dfrac{1}{\sin x} - \sin x}$

$\equiv \dfrac{\dfrac{1}{\sin x}}{\dfrac{1 - \sin^2 x}{\sin x}}$

$\equiv \dfrac{1}{\sin x} \times \dfrac{\sin x}{1 - \sin^2 x}$

$\equiv \dfrac{1}{1 - \sin^2 x}$

$\equiv \dfrac{1}{\cos^2 x}$

(using $\sin^2 x + \cos^2 x \equiv 1$)

$\equiv \sec^2 x \equiv$ RHS

c LHS $\equiv (1 - \sin x)(1 + \operatorname{cosec} x)$

$\equiv 1 - \sin x + \operatorname{cosec} x - \sin x\operatorname{cosec} x$

$\equiv 1 - \sin x + \operatorname{cosec} x - 1$

$\left(\text{as } \operatorname{cosec} x = \dfrac{1}{\sin x}\right)$

$\equiv \operatorname{cosec} x - \sin x$

$\equiv \dfrac{1}{\sin x} - \sin x$

$\equiv \dfrac{1 - \sin^2 x}{\sin x}$

$\equiv \dfrac{\cos^2 x}{\sin x}$

$\equiv \dfrac{\cos x}{\sin x} \times \cos x$

$\equiv \cos x\cot x \equiv$ RHS

d LHS $\equiv \dfrac{\cot x}{\operatorname{cosec} x - 1} - \dfrac{\cos x}{1 + \sin x}$

$\equiv \dfrac{\dfrac{\cos x}{\sin x}}{\dfrac{1}{\sin x} - 1} - \dfrac{\cos x}{1 + \sin x}$

$\equiv \dfrac{\dfrac{\cos x}{\sin x}}{\dfrac{1 - \sin x}{\sin x}} - \dfrac{\cos x}{1 + \sin x}$

$\equiv \dfrac{\cos x}{1 - \sin x} - \dfrac{\cos x}{1 + \sin x}$

$\equiv \dfrac{\cos x(1 + \sin x) - \cos x(1 - \sin x)}{(1 - \sin x)(1 + \sin x)}$

$\equiv \dfrac{2\cos x\sin x}{1 - \sin^2 x}$

$\equiv \dfrac{2\cos x\sin x}{\cos^2 x}$

$\equiv 2\dfrac{\sin x}{\cos x}$

$\equiv 2\tan x \equiv$ RHS

6 e LHS $\equiv \dfrac{1}{\operatorname{cosec}\theta - 1} + \dfrac{1}{\operatorname{cosec}\theta + 1}$

$\equiv \dfrac{(\operatorname{cosec}\theta + 1) + (\operatorname{cosec}\theta - 1)}{(\operatorname{cosec}\theta - 1)(\operatorname{cosec}\theta + 1)}$

$\equiv \dfrac{2\operatorname{cosec}\theta}{\operatorname{cosec}^2\theta - 1}$

$\equiv \dfrac{2\operatorname{cosec}\theta}{\cot^2\theta}$

$(1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta)$

$\equiv \dfrac{2}{\sin\theta} \times \dfrac{\sin^2\theta}{\cos^2\theta}$

$\equiv 2 \times \dfrac{1}{\cos\theta} \times \dfrac{\sin\theta}{\cos\theta}$

$\equiv 2\sec\theta\tan\theta \equiv$ RHS

f LHS $\equiv \dfrac{(\sec\theta - \tan\theta)(\sec\theta + \tan\theta)}{1 + \tan^2\theta}$

$\equiv \dfrac{\sec^2\theta - \tan^2\theta}{\sec^2\theta}$

$\equiv \dfrac{(1 + \tan^2\theta) - \tan^2\theta}{\sec^2\theta}$

$\equiv \dfrac{1}{\sec^2\theta}$

$\equiv \cos^2\theta \equiv$ RHS

7 a LHS $\equiv \dfrac{\sin x}{1 + \cos x} + \dfrac{1 + \cos x}{\sin x}$

$\equiv \dfrac{\sin^2 x + (1 + \cos x)^2}{(1 + \cos x)\sin x}$

$\equiv \dfrac{\sin^2 x + 1 + 2\cos x + \cos^2 x}{(1 + \cos x)\sin x}$

$\equiv \dfrac{2 + 2\cos x}{(1 + \cos x)\sin x}$

$\left(\sin^2 x + \cos^2 x \equiv 1\right)$

$\equiv \dfrac{2(1 + \cos x)}{(1 + \cos x)\sin x}$

$\equiv \dfrac{2}{\sin x}$

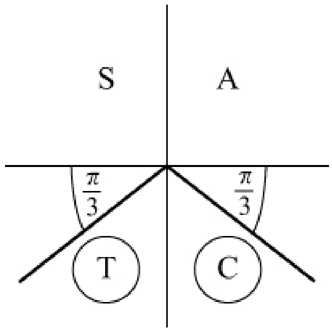
$\equiv 2\operatorname{cosec} x \equiv$ RHS

b Solve $2\operatorname{cosec} x = -\dfrac{4}{\sqrt{3}}, \quad -2\pi \le x \le 2\pi$

$\Rightarrow \operatorname{cosec} x = -\dfrac{2}{\sqrt{3}}$

$\Rightarrow \sin x = -\dfrac{\sqrt{3}}{2}$

Calculator value is $-\dfrac{\pi}{3}$



Solutions in $-2\pi \le x \le 2\pi$ are

$-\dfrac{\pi}{3}, -\pi + \dfrac{\pi}{3}, \pi + \dfrac{\pi}{3}, 2\pi - \dfrac{\pi}{3}$

i.e. $-\dfrac{\pi}{3}, -\dfrac{2\pi}{3}, \dfrac{4\pi}{3}, \dfrac{5\pi}{3}$

8 RHS $\equiv (\operatorname{cosec}\theta + \cot\theta)^2$

$\equiv \left(\dfrac{1}{\sin\theta} + \dfrac{\cos\theta}{\sin\theta}\right)^2$

$\equiv \dfrac{(1 + \cos\theta)^2}{\sin^2\theta}$

$\equiv \dfrac{(1 + \cos\theta)^2}{1 - \cos^2\theta}$

$\equiv \dfrac{(1 + \cos\theta)(1 + \cos\theta)}{(1 + \cos\theta)(1 - \cos\theta)}$

$\equiv \dfrac{1 + \cos\theta}{1 - \cos\theta} \equiv$ LHS

9 a $\sec A = -3,\ \frac{\pi}{2} < A < \pi,$

i.e. A is in 2nd quadrant.

As $1 + \tan^2 A = \sec^2 A$

$1 + \tan^2 A = 9$

$\tan^2 A = 8$

$\tan A = \pm\sqrt{8} = \pm 2\sqrt{2}$

As A is in 2nd quadrant, $\tan A$ is negative.

So $\tan A = -2\sqrt{2}$

b $\sec A = -3$, so $\cos A = -\frac{1}{3}$

As $\tan A = \frac{\sin A}{\cos A}$

$\sin A = \cos A \times \tan A = -\frac{1}{3} \times -2\sqrt{2} = \frac{2\sqrt{2}}{3}$

So $\operatorname{cosec} A = \frac{3}{2\sqrt{2}} = \frac{3\sqrt{2}}{2\times 2} = \frac{3\sqrt{2}}{4}$

An alternative approach is to use the fact that $\operatorname{cosec}^2 \theta \equiv 1 + \cot^2 \theta$

$\operatorname{cosec}^2 A = 1 + \cot^2 A = 1 + \frac{1}{8} = \frac{9}{8}$

$\Rightarrow \operatorname{cosec} A = \pm\frac{3}{2\sqrt{2}} = \pm\frac{3\sqrt{2}}{4}$

As $\frac{\pi}{2} < A < \pi$, $\operatorname{cosec} A$ is positive, so

$\operatorname{cosec} A = \frac{3\sqrt{2}}{4}$

10 $\sec\theta = k,\ |k| \geq 1$

θ is in the 2nd quadrant $\Rightarrow k$ is negative

a $\cos\theta = \frac{1}{\sec\theta} = \frac{1}{k}$

b Using $1 + \tan^2\theta = \sec^2\theta$

$\tan^2\theta = k^2 - 1$

c $\tan\theta = \pm\sqrt{k^2 - 1}$

In the 2nd quadrant, $\tan\theta$ is negative

So $\tan\theta = -\sqrt{k^2 - 1}$

$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{-\sqrt{k^2-1}} = -\frac{1}{\sqrt{k^2-1}}$

d Using $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

$\operatorname{cosec}^2\theta = 1 + \frac{1}{k^2-1} = \frac{k^2-1+1}{k^2-1} = \frac{k^2}{k^2-1}$

So $\operatorname{cosec}\theta = \pm\frac{k}{\sqrt{k^2-1}}$

In the 2nd quadrant, $\operatorname{cosec}\theta$ is positive

As k is negative, $\operatorname{cosec}\theta = -\frac{k}{\sqrt{k^2-1}}$

11 $\sec\left(x + \frac{\pi}{4}\right) = 2,\ 0 \leq x \leq 2\pi$

$\Rightarrow \cos\left(x + \frac{\pi}{4}\right) = \frac{1}{2},\ 0 \leq x \leq 2\pi$

$\Rightarrow x + \frac{\pi}{4} = \cos^{-1}\frac{1}{2},\ 2\pi - \cos^{-1}\frac{1}{2}$

$= \frac{\pi}{3},\ 2\pi - \frac{\pi}{3}$

So $x = \frac{\pi}{3} - \frac{\pi}{4},\ \frac{5\pi}{3} - \frac{\pi}{4}$

$= \frac{4\pi - 3\pi}{12},\ \frac{20\pi - 3\pi}{12}$

$= \frac{\pi}{12},\ \frac{17\pi}{12}$

12 $\arcsin\left(\frac{1}{2}\right)$ is the angle α in the interval

$-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$ whose sine is $\frac{1}{2}$

So $\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$

Similarly, $\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

So $\arcsin\left(\frac{1}{2}\right) - \arcsin\left(-\frac{1}{2}\right) = \frac{\pi}{6} - \left(-\frac{\pi}{6}\right) = \frac{\pi}{3}$

13 $\sec^2 x - \frac{2\sqrt{3}}{3}\tan x - 2 = 0,\ 0 \le x \le 2\pi$

$\Rightarrow\ (1+\tan^2 x) - \frac{2\sqrt{3}}{3}\tan x - 2 = 0$

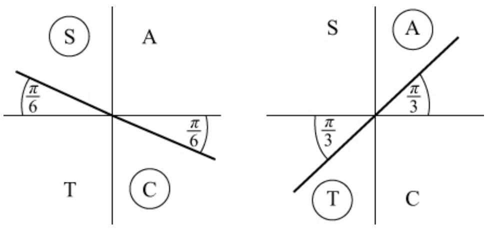
$\Rightarrow\ \tan^2 x - \frac{2\sqrt{3}}{3}\tan x - 1 = 0$

(This does factorise!)

$\left(\tan x + \frac{\sqrt{3}}{3}\right)\left(\tan x - \sqrt{3}\right) = 0$

$\Rightarrow\ \tan x = -\frac{\sqrt{3}}{3}$ or $\tan x = \sqrt{3}$

Calculator values are $-\frac{\pi}{6}$ and $\frac{\pi}{3}$



Solution set: $\frac{\pi}{3}, \frac{5\pi}{6}, \frac{4\pi}{3}, \frac{11\pi}{6}$

14 a $\sec x \operatorname{cosec} x - 2\sec x - \operatorname{cosec} x + 2$

$= \sec x(\operatorname{cosec} x - 2) - (\operatorname{cosec} x - 2)$

$= (\operatorname{cosec} x - 2)(\sec x - 1)$

b So $\sec x \operatorname{cosec} x - 2\sec x - \operatorname{cosec} x + 2 = 0$

$\Rightarrow\ (\operatorname{cosec} x - 2)(\sec x - 1) = 0$

$\Rightarrow\ \operatorname{cosec} x = 2$ or $\sec x = 1$

$\Rightarrow\ \sin x = \frac{1}{2}$ or $\cos x = 1$

$\sin x = \frac{1}{2},\quad 0 \le x \le 360°$

$\Rightarrow\ x = 30°, (180 - 30)°$

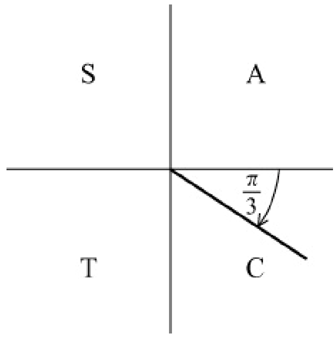
$\cos x = 1,\quad 0 \le x \le 360°,$

$\Rightarrow\ x = 0°, 360°$ (from the graph)

As $\operatorname{cosec} x$ is not defined for $x = 0°, 360°$, the equation is not defined for these values, so $x = 0°, 360°$ are not solutions

So the solution set is: 30°, 150°

15 $\arctan(x-2) = -\frac{\pi}{3}$

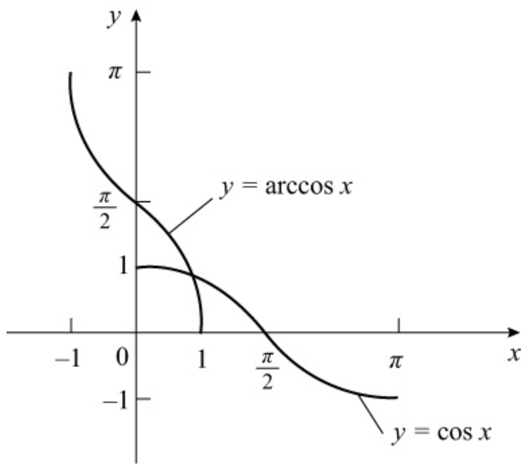


$\Rightarrow\ x - 2 = \tan\left(-\frac{\pi}{3}\right)$

$\Rightarrow\ x - 2 = -\sqrt{3}$

$\Rightarrow\ x = 2 - \sqrt{3}$

16



17 a As $1 + \tan^2 x \equiv \sec^2 x$

$\sec^2 x - \tan^2 x \equiv 1$

$\Rightarrow\ (\sec x - \tan x)(\sec x + \tan x) \equiv 1$

(difference of two squares)

As $\tan x + \sec x = -3$ is given,

so $-3(\sec x - \tan x) = 1$

$\Rightarrow\ \sec x - \tan x = -\frac{1}{3}$

17 b $\sec x + \tan x = -3$

and $\sec x - \tan x = -\frac{1}{3}$

i Add the equations

$\Rightarrow 2\sec x = -\frac{10}{3} \Rightarrow \sec x = -\frac{5}{3}$

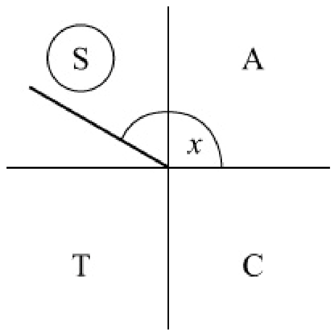
ii Subtract the equations

$\Rightarrow 2\tan x = -3 - \left(-\frac{1}{3}\right) = -\frac{8}{3}$

$\Rightarrow \tan x = -\frac{4}{3}$

c As $\sec x$ and $\tan x$ are both negative, $\cos x$ and $\tan x$ are both negative.

So x must be in the 2nd quadrant.



Solving $\tan x = -\frac{4}{3}$, where x is in the 2nd quadrant, gives

$x = 180° + (-53.1°) = 126.9°$ (1 d.p.)

18 $p = \sec\theta - \tan\theta,\ q = \sec\theta + \tan\theta$

Multiply together:

$pq = (\sec\theta - \tan\theta)(\sec\theta + \tan\theta)$

$= \sec^2\theta - \tan^2\theta = 1$

(since $1 + \tan^2\theta = \sec^2\theta$)

$\Rightarrow p = \frac{1}{q}$

(There are several ways of solving this problem.)

19 a LHS $\equiv \sec^4\theta - \tan^4\theta$

$\equiv (\sec^2\theta - \tan^2\theta)(\sec^2\theta + \tan^2\theta)$

$\equiv 1 \times (\sec^2\theta + \tan^2\theta)$

(as $\sec^2\theta \equiv 1 + \tan^2\theta$

$\Rightarrow \sec^2\theta - \tan^2\theta \equiv 1$)

$\equiv \sec^2\theta + \tan^2\theta \equiv$ RHS

b $\sec^4\theta = \tan^4\theta + 3\tan\theta$

$\Rightarrow \sec^4\theta - \tan^4\theta = 3\tan\theta$

$\Rightarrow \sec^2\theta + \tan^2\theta = 3\tan\theta$

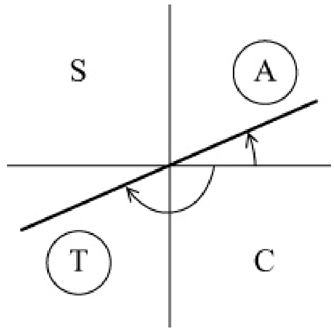
(using part (a))

$\Rightarrow (1 + \tan^2\theta) + \tan^2\theta = 3\tan\theta$

$\Rightarrow 2\tan^2\theta - 3\tan\theta + 1 = 0$

$\Rightarrow (2\tan\theta - 1)(\tan\theta - 1) = 0$

$\Rightarrow \tan\theta = \frac{1}{2}$ or $\tan\theta = 1$



In the interval $-180° \le \theta \le 180°$

$\tan\theta = \frac{1}{2} \Rightarrow \theta = \tan^{-1}\frac{1}{2}, -180° + \tan^{-1}\frac{1}{2}$

$= 26.6°, -153.4°$ (1 d.p.)

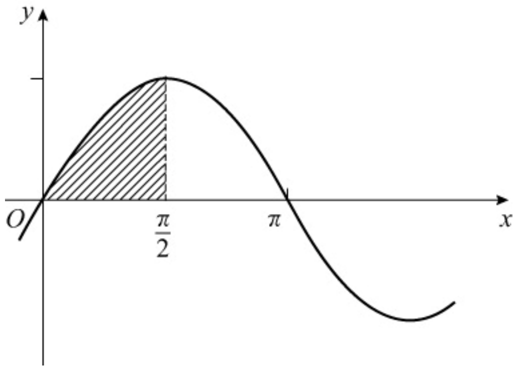
$\tan\theta = 1 \Rightarrow \theta = \tan^{-1}1, -180° + \tan^{-1}1$

$= 45°,\ -135°$

Solution set is:

$-153.4°,\ -135°,\ 26.6°,\ 45°$

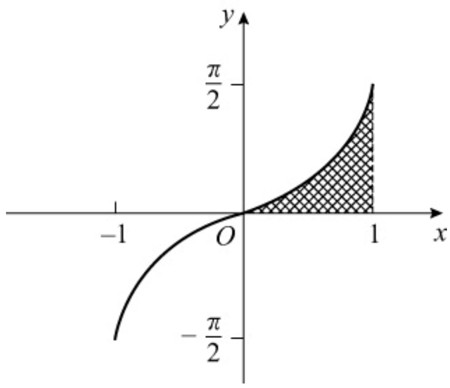
20 a $y = \sin x$



$\int_0^{\frac{\pi}{2}} \sin x \, dx$ represents the area between

$y = \sin x$, the x-axis and $x = \frac{\pi}{2}$

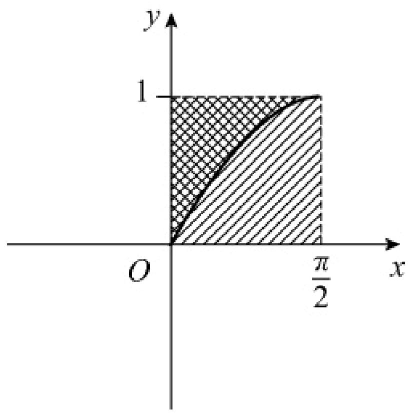
b $y = \arcsin x, \quad -1 \leq x \leq 1$



$\int_0^1 \arcsin x \, dx$ represents the area between

$y = \arcsin x$, the x-axis and $x = 1$

c The curves are the same with the axes interchanged.
The shaded area in (b) could be added to the graph in (a) to form a rectangle with sides 1 and $\frac{\pi}{2}$, as in the diagram.



Area of rectangle $= 1 \times \frac{\pi}{2} = \frac{\pi}{2}$

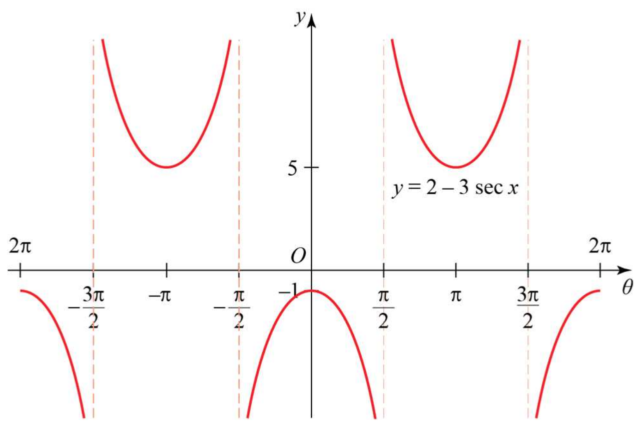
So $\int_0^{\frac{\pi}{2}} \sin x \, dx + \int_0^1 \arcsin x \, dx = \frac{\pi}{2}$

21 $\cot 60° \sec 60° = \frac{1}{\tan 60°} \times \frac{1}{\cos 60°}$

$= \frac{1}{\sqrt{3}} \times \frac{1}{\frac{1}{2}} = \frac{2}{\sqrt{3}}$

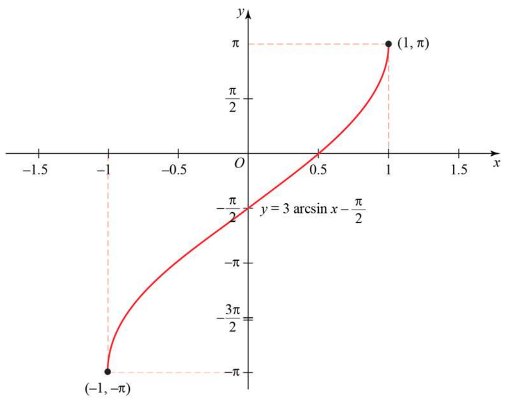
$= \frac{2\sqrt{3}}{3}$

22 a The graph of $y = 2 - 3\sec x$ is $y = \sec x$ stretched by a scale factor 3 in the y direction, then reflected in the x-axis and then translated by the vector $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$.



b $-1 < k < 5$

23 a The graph of $y = 3\arcsin x - \frac{\pi}{2}$ is $y = \arcsin x$ stretched by a scale factor 3 in the y direction and then translated by the vector $\begin{pmatrix} 0 \\ -\frac{\pi}{2} \end{pmatrix}$



b Curve meets the x-axis when $y = 0$

$\Rightarrow 3\arcsin x - \frac{\pi}{2} = 0$

$\Rightarrow \arcsin x = \frac{\pi}{6}$

$\Rightarrow \sin\frac{\pi}{6} = x$

$\Rightarrow x = \frac{1}{2}$

Curve meets the x-axis at $\left(\frac{1}{2}, 0\right)$

24 a Let $y = \arccos x,\ 0 < x \le 1$

$\Rightarrow \cos y = x$

$\Rightarrow \sin y = \sqrt{1-x^2}$

Note that as $0 < x \le 1,\ 0 \le y < \frac{\pi}{2}$,

so $\sin y$ is positive

Thus $\tan y = \frac{\sqrt{1-x^2}}{x}$,

which is valid for $0 < x \le 1$

$\Rightarrow y = \arctan\frac{\sqrt{1-x^2}}{x}$

So $\arccos x = \arctan\frac{\sqrt{1-x^2}}{x}$ for $0 < x \le 1$

b Let $y = \arccos x,\ -1 \le x < 0$

$\Rightarrow \cos y = x$

$\Rightarrow \sin y = \sqrt{1-x^2}$

As $-1 \le x < 0,\ \frac{\pi}{2} < y \le \pi$,

so $\sin y$ is positive

$\Rightarrow \tan y = \frac{\sin y}{\cos y} = \frac{\sqrt{1-x^2}}{x}$

for $-1 \le x < 0,\ \frac{\pi}{2} < y \le \pi$

Note that as $y > \frac{\pi}{2}$, it is not in the range of $y = \arccos x$

However, from the tan curve, we know that $\tan(y - \pi) = \tan y$

So $\tan(y-\pi) = \frac{\sqrt{1-x^2}}{x}$

for $-1 \le x < 0,\ -\frac{\pi}{2} < y - \pi \le 0$

We can now use the inverse function

$y - \pi = \arctan\frac{\sqrt{1-x^2}}{x}$

So $y = \pi + \arctan\frac{\sqrt{1-x^2}}{x}$

for $-1 \le x < 0$

Thus $\arccos x = \pi + \arctan\frac{\sqrt{1-x^2}}{x}$

for $-1 \le x < 0$