

Exercise 3E

- 1 a $\arccos(0)$ is the angle α in $0 \leq \alpha \leq \pi$
for which $\cos \alpha = 0$

Refer to graph of $y = \cos \theta \Rightarrow \alpha = \frac{\pi}{2}$

So $\arccos(0) = \frac{\pi}{2}$

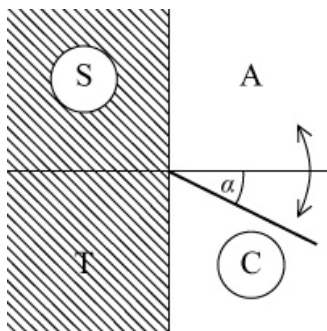
- b $\arcsin(1)$ is the angle α in $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$
for which $\sin \alpha = 1$

Refer to graph of $y = \sin \theta \Rightarrow \alpha = \frac{\pi}{2}$

So $\arcsin(1) = \frac{\pi}{2}$

- c $\arctan(-1)$ is the angle α in $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$
for which $\tan \alpha = -1$

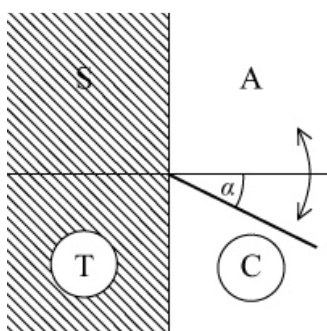
So $\arctan(-1) = -\frac{\pi}{4}$



- d $\arcsin\left(-\frac{1}{2}\right)$ is the angle α

in $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$ for which $\sin \alpha = -\frac{1}{2}$

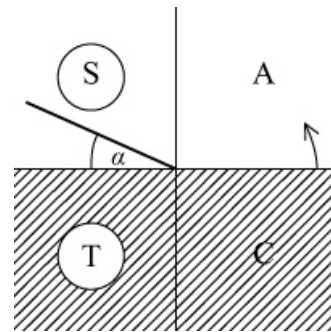
So $\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$



- e $\arccos\left(-\frac{1}{\sqrt{2}}\right)$ is the angle α in $0 \leq \alpha \leq \pi$

for which $\cos \alpha = -\frac{1}{\sqrt{2}}$

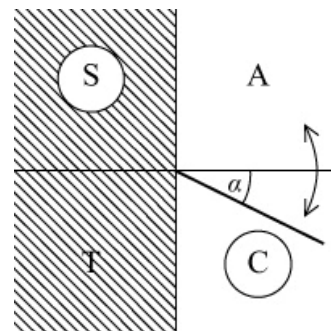
So $\arccos\left(-\frac{1}{\sqrt{2}}\right) = \frac{3\pi}{4}$



- f $\arctan\left(-\frac{1}{\sqrt{3}}\right)$ is the angle α

in $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$ for which $\tan \alpha = -\frac{1}{\sqrt{3}}$

So $\arctan\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$

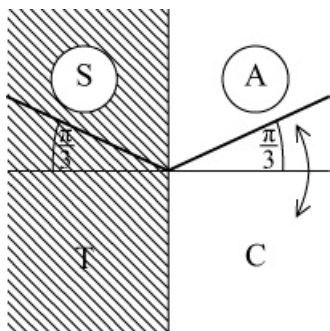


- g $\arcsin\left(\sin \frac{\pi}{3}\right)$ is the angle α

in $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$ for which $\sin \alpha = \sin \frac{\pi}{3}$

So $\arcsin\left(\sin \frac{\pi}{3}\right) = \frac{\pi}{3}$

- 1 h $\arcsin\left(\sin\frac{2\pi}{3}\right)$ is the angle α in
 $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$ for which $\sin\alpha = \sin\frac{2\pi}{3}$
 So $\arcsin\left(\sin\frac{2\pi}{3}\right) = \frac{\pi}{3}$



2 a $\arcsin\left(\frac{1}{2}\right) + \arcsin\left(-\frac{1}{2}\right) = \frac{\pi}{6} + \left(-\frac{\pi}{6}\right) = 0$

b $\arccos\left(\frac{1}{2}\right) - \arccos\left(-\frac{1}{2}\right) = \frac{\pi}{3} - \frac{2\pi}{3} = -\frac{\pi}{3}$

c $\arctan(1) - \arctan(-1) = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$

3 a $\sin\left(\arcsin\frac{1}{2}\right)$

$\arcsin\frac{1}{2} = \alpha$ where $\sin\alpha = \frac{1}{2}$,

and $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$

So $\arcsin\frac{1}{2} = \frac{\pi}{6}$

$\Rightarrow \sin\left(\arcsin\frac{1}{2}\right) = \sin\frac{\pi}{6} = \frac{1}{2}$

b $\sin\left(\arcsin\left(-\frac{1}{2}\right)\right)$

$\arcsin\left(-\frac{1}{2}\right) = \alpha$

where $\sin\alpha = -\frac{1}{2}$, $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$

So $\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

$\Rightarrow \sin\left(\arcsin\left(-\frac{1}{2}\right)\right) = \sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$

c $\tan(\arctan(-1))$

$\arctan(-1) = \alpha$

where $\tan\alpha = -1$, $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$

So $\arctan(-1) = -\frac{\pi}{4}$

$\Rightarrow \tan(\arctan(-1)) = \tan\left(-\frac{\pi}{4}\right) = -1$

d $\cos(\arccos 0)$

$\arccos 0 = \alpha$ where $\cos\alpha = 0$, $0 \leq \alpha \leq \pi$

So $\arccos 0 = \frac{\pi}{2}$

$\Rightarrow \cos(\arccos 0) = \cos\frac{\pi}{2} = 0$

4 a $\sin\left(\arccos\frac{1}{2}\right)$

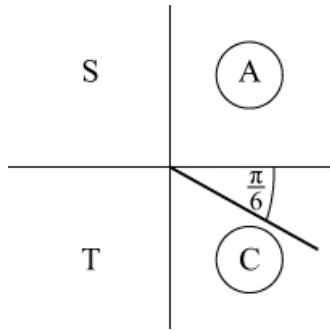
$\arccos\frac{1}{2} = \frac{\pi}{3}$

$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$$4 \text{ b } \cos\left(\arcsin\left(-\frac{1}{2}\right)\right)$$

$$\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

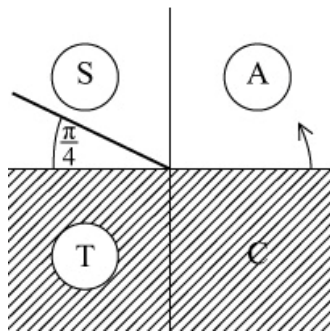
$$\cos\left(-\frac{\pi}{6}\right) = +\frac{\sqrt{3}}{2}$$



$$c \text{ } \tan\left(\arccos\left(-\frac{\sqrt{2}}{2}\right)\right)$$

$$\arccos\left(-\frac{\sqrt{2}}{2}\right) = \alpha$$

$$\text{where } \cos \alpha = -\frac{\sqrt{2}}{2}, 0 \leq \alpha \leq \pi$$



$$\text{So } \arccos\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$$

$$\tan\frac{3\pi}{4} = -1$$

$$d \text{ } \sec(\arctan\sqrt{3})$$

$$\arctan\sqrt{3} = \frac{\pi}{3}$$

(the angle whose \tan is $\sqrt{3}$)

$$\sec\frac{\pi}{3} = \frac{1}{\cos\frac{\pi}{3}} = \frac{1}{\frac{1}{2}} = 2$$

$$e \text{ } \operatorname{cosec}(\arcsin(-1))$$

$$\arcsin(-1) = \alpha$$

$$\text{where } \sin \alpha = -1, -\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$$

$$\text{So } \arcsin(-1) = -\frac{\pi}{2}$$

$$\begin{aligned} \Rightarrow \operatorname{cosec}(\arcsin(-1)) &= \frac{1}{\sin\left(-\frac{\pi}{2}\right)} \\ &= \frac{1}{-1} = -1 \end{aligned}$$

$$f \text{ } \sin\left(2\arcsin\left(\frac{\sqrt{2}}{2}\right)\right)$$

$$\arcsin\frac{\sqrt{2}}{2} = \frac{\pi}{4}$$

$$\text{So } \sin\left(2\arcsin\left(\frac{\sqrt{2}}{2}\right)\right) = \sin\frac{\pi}{2} = 1$$

- 5 As k is positive, the first two positive solutions of $\sin x = k$ are $\arcsin k$ and $\pi - \arcsin k$ i.e. α and $\pi - \alpha$
(Try a few examples, taking specific values for k).

- 6 a $\arcsin x$ is the angle α in $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$
such that $\sin \alpha = x$

$$\text{In this case } x = \sin k \text{ where } 0 < k < \frac{\pi}{2}$$

As \sin is an increasing function

$$\sin 0 < x < \sin\frac{\pi}{2}$$

$$\Rightarrow 0 < x < 1$$

$$b \text{ i } \cos k = \pm\sqrt{1 - \sin^2 k} = \pm\sqrt{1 - x^2}$$

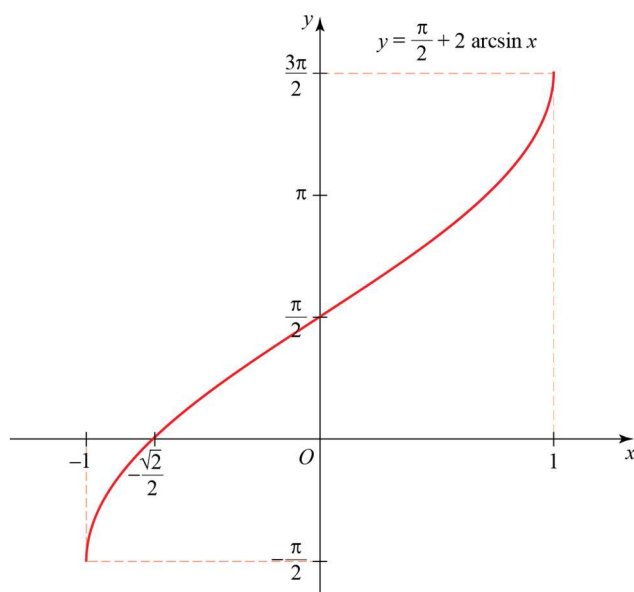
k is in the 1st quadrant $\Rightarrow \cos k > 0$

$$\text{So } \cos k = \sqrt{1 - x^2}$$

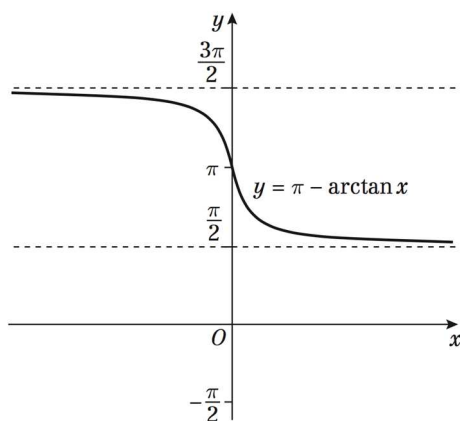
$$b \text{ ii } \tan k = \frac{\sin k}{\cos k} = \frac{x}{\sqrt{1 - x^2}}$$

6 c k is now in the 4th quadrant, where $\cos k$ is positive. So the value of $\cos k$ remains the same and there is no change to $\tan k$.

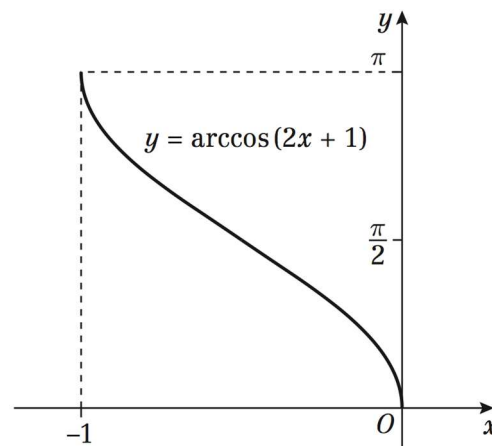
7 a The graph of $y = \frac{\pi}{2} + 2 \arcsin x$ is $y = \arcsin x$ stretched by a scale factor 2 in the y direction and then translated by the vector $\begin{pmatrix} 0 \\ \frac{\pi}{2} \end{pmatrix}$



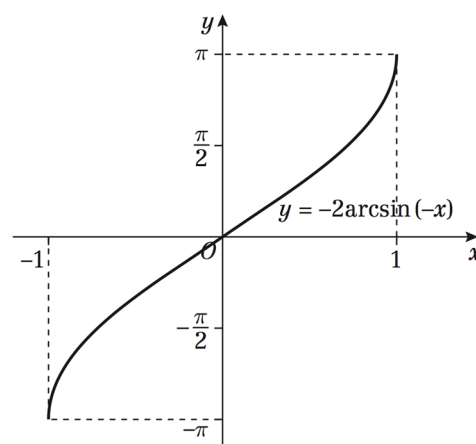
b The graph of $y = \pi - \arctan x$ is $y = \arctan x$ reflected in the x -axis and then translated by the vector $\begin{pmatrix} 0 \\ \pi \end{pmatrix}$



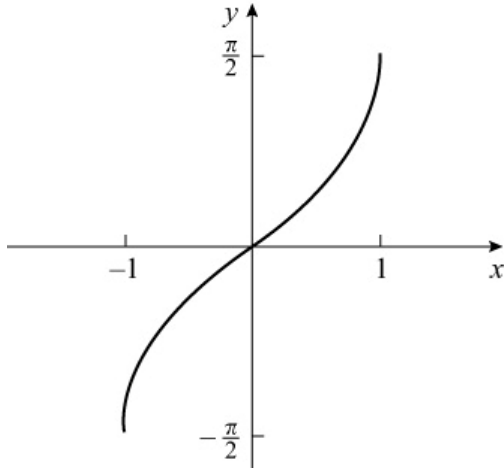
7 c The graph of $y = \arccos(2x + 1)$ is $y = \arccos x$ translated by the vector $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$ and then stretched by scale factor $\frac{1}{2}$ in the x direction



d The graph of $y = -2 \arcsin(-x)$ is $y = \arcsin x$ reflected in the y -axis, then reflected in the x -axis and then stretched by a scale factor 2 in the y direction



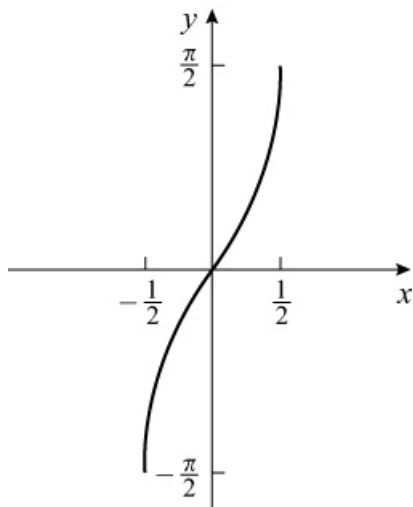
8 a $y = \arcsin x$



Range is $-\frac{\pi}{2} \leq f(x) \leq \frac{\pi}{2}$

- b The graph of $y = f(2x)$ is the graph of $y = f(x)$ stretched in the x direction by scale factor $\frac{1}{2}$

$y = g(x)$



c $g : x \mapsto \arcsin 2x$

The domain is $-\frac{1}{2} \leq x \leq \frac{1}{2}$

d Let $y = \arcsin 2x$

$$\Rightarrow 2x = \sin y$$

$$\Rightarrow x = \frac{1}{2} \sin y$$

So $g^{-1} : x \mapsto \frac{1}{2} \sin x, -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

9 a Let $y = \arccos x$

$$\text{As } 0 \leq x \leq 1 \Rightarrow 0 \leq y \leq \frac{\pi}{2}$$

$$\cos y = x, \text{ and using } \cos^2 y + \sin^2 y = 1$$

$$\Rightarrow \sin^2 y = 1 - \cos^2 y$$

$$\Rightarrow \sin y = \sqrt{1 - \cos^2 y} = \sqrt{1 - x^2}$$

Note, $\sin y \geq 0$ since $0 \leq y \leq \frac{\pi}{2}$

$$\text{so } \sin y = \sqrt{1 - x^2}$$

$$\sin y = \sqrt{1 - x^2}$$

$$\Rightarrow y = \arcsin \sqrt{1 - x^2}$$

Therefore, $\arccos x = \arcsin \sqrt{1 - x^2}$
for $0 \leq x \leq 1$

b For $-1 \leq x \leq 0, \frac{\pi}{2} \leq \arccos x \leq \pi$

But \arcsin has a range of $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$\text{So } \arccos x \neq \arcsin \sqrt{1 - x^2},$$

for $-1 \leq x \leq 0$

An alternative approach is to provide a counterexample.

$$\text{Let } x = -\frac{1}{\sqrt{2}}$$

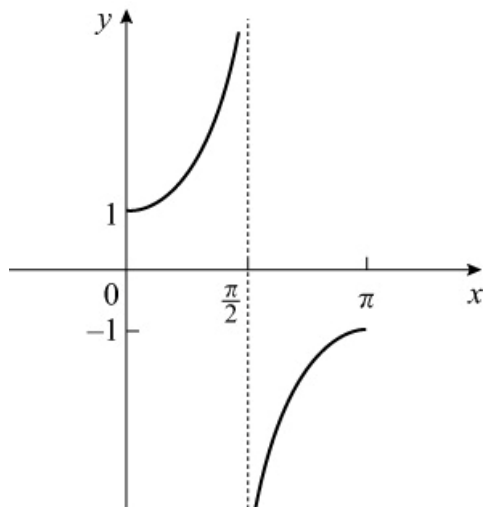
$$\arccos x = \frac{3\pi}{4}$$

$$\arcsin \sqrt{1 - x^2} = \arcsin \frac{1}{\sqrt{2}} = \frac{\pi}{4}$$

$$\text{So } \arccos x \neq \arcsin \sqrt{1 - x^2}$$

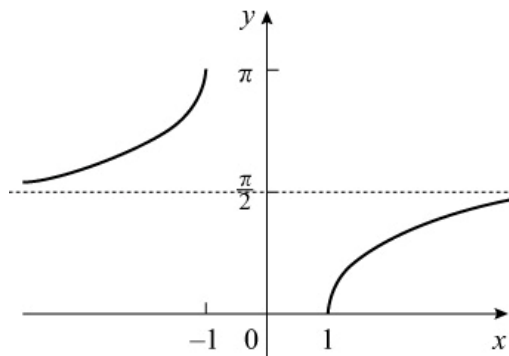
Challenge

a $y = \sec x$



- b Reflect the graph drawn for part (a) in the line $y = x$

$$y = \arcsin x, x \leq -1, x \geq 1$$



Range is $0 \leq \arcsin x \leq \pi$, for $\arcsin x \neq \frac{\pi}{2}$