Exercise 3D

1 a Use $1+\tan^2\theta=\sec^2\theta$
with θ replaced with $\frac{1}{2}\theta$
$1+\tan^2\left(\frac{1}{2}\theta\right)=\sec^2\left(\frac{1}{2}\theta\right)$

b $(\sec\theta-1)(\sec\theta+1)$ (multiply out)
$=\sec^2\theta-1$
$=(1+\tan^2\theta)-1$
$=\tan^2\theta$

c $\tan^2\theta(\text{cosec}^2\theta-1)$
$=\tan^2\theta\left((1+\cot^2\theta)-1\right)$
$=\tan^2\theta\cot^2\theta$
$=\tan^2\theta\times\dfrac{1}{\tan^2\theta}$
$=1$

d $(\sec^2\theta-1)\cot\theta$
$=\tan^2\theta\cot\theta$
$=\tan^2\theta\times\dfrac{1}{\tan\theta}$
$=\tan\theta$

e $(\text{cosec}^2\theta-\cot^2\theta)^2$
$=\left((1+\cot^2\theta)-\cot^2\theta\right)^2$
$=1^2=1$

f $2-\tan^2\theta+\sec^2\theta$
$=2-\tan^2\theta+(1+\tan^2\theta)$
$=2-\tan^2\theta+1+\tan^2\theta$
$=3$

g $\dfrac{\tan\theta\sec\theta}{1+\tan^2\theta}$
$=\dfrac{\tan\theta\sec\theta}{\sec^2\theta}$
$=\dfrac{\tan\theta}{\sec\theta}$
$=\tan\theta\cos\theta$
$=\dfrac{\sin\theta}{\cos\theta}\times\cos\theta$
$=\sin\theta$

h $(1-\sin^2\theta)(1+\tan^2\theta)$
$=\cos^2\theta\times\sec^2\theta$
$=\cos^2\theta\times\dfrac{1}{\cos^2\theta}$
$=1$

i $\dfrac{\text{cosec}\,\theta\cot\theta}{1+\cot^2\theta}$
$=\dfrac{\text{cosec}\,\theta\cot\theta}{\text{cosec}^2\theta}$
$=\dfrac{1}{\text{cosec}\,\theta}\times\cot\theta$
$=\dfrac{\sin\theta}{1}\times\dfrac{\cos\theta}{\sin\theta}$
$=\cos\theta$

j $\sec^4\theta-2\sec^2\theta\tan^2\theta+\tan^4\theta$
$=(\sec^2\theta-\tan^2\theta)^2$ (factorise)
$=\left((1+\tan^2\theta)-\tan^2\theta\right)^2$
$=1^2=1$

k $4\,\text{cosec}^2 2\theta+4\,\text{cosec}^2 2\theta\cot^2 2\theta$
$=4\,\text{cosec}^2 2\theta(1+\cot^2 2\theta)$
$=4\,\text{cosec}^2 2\theta\,\text{cosec}^2 2\theta$
$=4\,\text{cosec}^4 2\theta$

2 $\text{cosec}\,x=\dfrac{k}{\text{cosec}\,x}$
$\Rightarrow\ \text{cosec}^2 x=k$
$\Rightarrow\ 1+\cot^2 x=k$
$\Rightarrow\ \cot^2 x=k-1$
$\Rightarrow\ \cot x=\pm\sqrt{k-1}$

3 a $\cot\theta = \sqrt{3}$ $\quad 90° < \theta < 180°$

$\Rightarrow \cot^2\theta = 3$

$\Rightarrow 1 + \cot^2\theta = 1 + 3 = 4$

$\Rightarrow \operatorname{cosec}^2\theta = 4$

$\Rightarrow \sin^2\theta = \frac{1}{4}$

$\Rightarrow \sin\theta = \frac{1}{2}$

(as θ is in 2nd quadrant, $\sin\theta$ is positive)

b Using $\sin^2\theta + \cos^2\theta = 1$

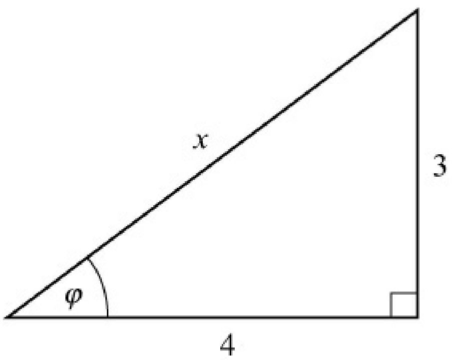
$\Rightarrow \cos^2\theta = 1 - \sin^2\theta = 1 - \frac{1}{4} = \frac{3}{4}$

$\Rightarrow \cos\theta = -\frac{\sqrt{3}}{4}$

(as θ is in 2nd quadrant, $\cos\theta$ is negative)

4 $\tan\theta = \frac{3}{4}$ $\quad 180° < \theta < 270°$

Draw a right-angled triangle where $\tan\varphi = \frac{3}{4}$



Using Pythagoras' theorem, $x = 5$

So $\cos\varphi = \frac{4}{5}$ and $\sin\varphi = \frac{3}{5}$

As θ is in the 3rd quadrant, both $\sin\theta$ and $\cos\theta$ are negative.

a $\sec\theta = \frac{1}{\cos\theta} = -\frac{1}{\cos\varphi} = -\frac{5}{4}$

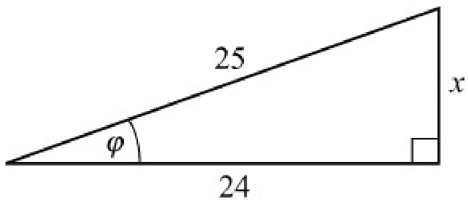
b $\cos\theta = -\cos\varphi = -\frac{4}{5}$

c $\sin\theta = -\sin\varphi = -\frac{3}{5}$

5 $\cos\theta = \frac{24}{25}$, θ reflex

As $\cos\theta$ is positive and θ reflex, θ is in the 4th quadrant.

Use right-angled triangle where $\cos\varphi = \frac{24}{25}$



Using Pythagoras' theorem,

$25^2 = x^2 + 24^2$

$\Rightarrow x^2 = 25^2 - 24^2 = 49$

$\Rightarrow x = 7$

So $\tan\varphi = \frac{7}{24}$ and $\sin\varphi = \frac{7}{25}$

As θ is in the 4th quadrant, both $\tan\theta$ and $\sin\theta$ are negative

a $\tan\theta = -\frac{7}{24}$

b $\operatorname{cosec}\theta = \frac{1}{\sin\theta} = -\frac{1}{\frac{7}{25}} = -\frac{25}{7}$

6 a LHS $\equiv \sec^4\theta - \tan^4\theta$

$\equiv (\sec^2\theta - \tan^2\theta)(\sec^2\theta + \tan^2\theta)$

(difference of two squares)

$\equiv (1)(\sec^2\theta + \tan^2\theta)$

(as $1 + \tan^2\theta \equiv \sec^2\theta$

$\Rightarrow \sec^2\theta - \tan^2\theta \equiv 1$)

$\equiv \sec^2\theta + \tan^2\theta \equiv$ RHS

b LHS $\equiv \operatorname{cosec}^2 x - \sin^2 x$

$\equiv (1 + \cot^2 x) - (1 - \cos^2 x)$

$\equiv 1 + \cot^2 x - 1 + \cos^2 x$

$\equiv \cot^2 x + \cos^2 x \equiv$ RHS

6 c LHS $\equiv \sec^2 A(\cot^2 A - \cos^2 A)$

$\equiv \frac{1}{\cos^2 A}\left(\frac{\cos^2 A}{\sin^2 A} - \cos^2 A\right)$

$\equiv \frac{1}{\sin^2 A} - 1 \equiv \text{cosec}^2 A - 1$

(use $1 + \cot^2\theta = \text{cosec}^2\theta$)

$\equiv 1 + \cot^2 A - 1$

$\equiv \cot^2 A \equiv$ RHS

d RHS $\equiv (\sec^2\theta - 1)(1 - \sin^2\theta)$

$\equiv \tan^2\theta \times \cos^2\theta$

(use $1 + \tan^2\theta \equiv \sec^2\theta$ and $\cos^2\theta + \sin^2\theta \equiv 1$)

$\equiv \frac{\sin^2\theta}{\cos^2\theta} \times \cos^2\theta \equiv \sin^2\theta$

$\equiv 1 - \cos^2\theta \equiv$ LHS

e LHS $\equiv \frac{1 - \tan^2 A}{1 + \tan^2 A} \equiv \frac{1 - \tan^2 A}{\sec^2 A}$

$\equiv \frac{1}{\sec^2 A}(1 - \tan^2 A)$

$\equiv \cos^2 A\left(1 - \frac{\sin^2 A}{\cos^2 A}\right)$

$\equiv \cos^2 A - \sin^2 A$

$\equiv (1 - \sin^2 A) - \sin^2 A$

$\equiv 1 - 2\sin^2 A \equiv$ RHS

f RHS $\equiv \sec^2\theta\,\text{cosec}^2\theta$

$\equiv \sec^2\theta(1 + \cot^2\theta)$

$\equiv \sec^2\theta + \frac{1}{\cos^2\theta} \times \frac{\cos^2\theta}{\sin^2\theta}$

$\equiv \sec^2\theta + \frac{1}{\sin^2\theta}$

$\equiv \sec^2\theta + \text{cosec}^2\theta \equiv$ LHS

Alternatively

LHS $\equiv \sec^2\theta + \text{cosec}^2\theta \equiv \frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta}$

$\equiv \frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta\sin^2\theta} \equiv \frac{1}{\cos^2\theta\sin^2\theta}$

$\equiv \sec^2\theta\,\text{cosec}^2\theta \equiv$ RHS

g LHS $\equiv \text{cosec}\,A\sec^2 A$

$\equiv \text{cosec}\,A(1 + \tan^2 A)$

$\equiv \text{cosec}\,A + \frac{1}{\sin A} \times \frac{\sin^2 A}{\cos^2 A}$

$\equiv \text{cosec}\,A + \frac{\sin A}{\cos^2 A}$

$\equiv \text{cosec}\,A + \frac{\sin A}{\cos A} \times \frac{1}{\cos A}$

$\equiv \text{cosec}\,A + \tan A\sec A \equiv$ RHS

h LHS $\equiv (\sec\theta - \sin\theta)(\sec\theta + \sin\theta)$

$\equiv \sec^2\theta - \sin^2\theta$

$\equiv (1 + \tan^2\theta) - (1 - \cos^2\theta)$

$\equiv 1 + \tan^2\theta - 1 + \cos^2\theta$

$\equiv \tan^2\theta + \cos^2\theta \equiv$ RHS

7 $3\tan^2\theta + 4\sec^2\theta = 5$

$\Rightarrow 3\tan^2\theta + 4(1 + \tan^2\theta) = 5$

$\Rightarrow 3\tan^2\theta + 4 + 4\tan^2\theta = 5$

$\Rightarrow 7\tan^2\theta = 1$

$\Rightarrow \tan^2\theta = \frac{1}{7}$

$\Rightarrow \cot^2\theta = 7$

$\Rightarrow \text{cosec}^2\theta - 1 = 7$

$\Rightarrow \text{cosec}^2\theta = 8$

$\Rightarrow \sin^2\theta = \frac{1}{8}$

As θ is obtuse (in the 2nd quadrant), $\sin\theta$ is positive.

So $\sin\theta = \sqrt{\frac{1}{8}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$

8 a $\sec^2\theta = 3\tan\theta \quad 0 \le \theta \le 360°$

$\Rightarrow 1 + \tan^2\theta = 3\tan\theta$

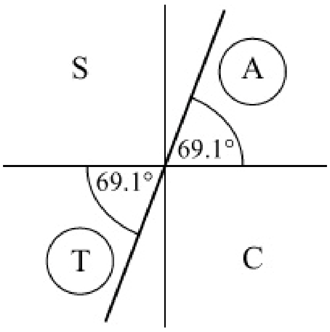
$\Rightarrow \tan^2\theta - 3\tan\theta + 1 = 0$

$\tan\theta = \dfrac{3 \pm \sqrt{5}}{2}$

(equation does not factorise)

For $\tan\theta = \dfrac{3+\sqrt{5}}{2}$,

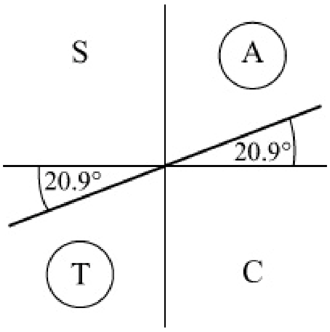
calculator value is 69.1° (3 s.f.)



Solutions are 69.1°, 249°

For $\tan\theta = \dfrac{3-\sqrt{5}}{2}$,

calculator value is 20.9° (3 s.f.)



Solutions are 20.9°, 201°

Set of solutions: 20.9°, 69.1°, 201°, 249° (3 s.f.)

b $\tan^2\theta - 2\sec\theta + 1 = 0 \quad -\pi \le \theta \le \pi$

$\Rightarrow (\sec^2\theta - 1) - 2\sec\theta + 1 = 0$

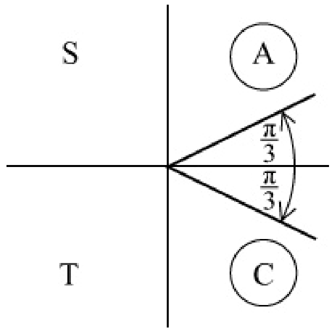
$\Rightarrow \sec^2\theta - 2\sec\theta = 0$

$\Rightarrow \sec\theta(\sec\theta - 2) = 0$

$\Rightarrow \sec\theta = 2 \quad$ (as $\sec\theta$ cannot be 0)

$\Rightarrow \cos\theta = \dfrac{1}{2}$

$\Rightarrow \theta = -\dfrac{\pi}{3}, \dfrac{\pi}{3}$



c $\operatorname{cosec}^2\theta + 1 = 3\cot\theta \quad -180° \le \theta \le 180°$

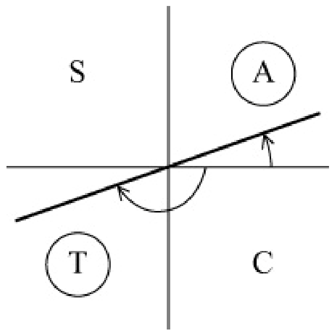
$\Rightarrow (1 + \cot^2\theta) + 1 = 3\cot\theta$

$\Rightarrow \cot^2\theta - 3\cot\theta + 2 = 0$

$\Rightarrow (\cot\theta - 1)(\cot\theta - 2) = 0$

$\Rightarrow \cot\theta = 1$ or $\cot\theta = 2$

$\Rightarrow \tan\theta = 1$ or $\tan\theta = \dfrac{1}{2}$



$\tan\theta = 1 \Rightarrow \theta = -135°, 45°$

$\tan\theta = \dfrac{1}{2} \Rightarrow \theta = -153°, 26.6°$ (3 s.f.)

8 d $\cot\theta = 1 - \operatorname{cosec}^2\theta \qquad 0 \le \theta \le 2\pi$

$\Rightarrow \cot\theta = 1 - (1 + \cot^2\theta)$

$\Rightarrow \cot\theta = -\cot^2\theta$

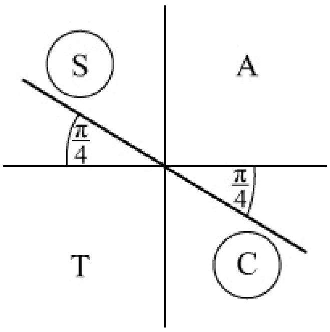
$\Rightarrow \cot^2\theta + \cot\theta = 0$

$\Rightarrow \cot\theta(\cot\theta + 1) = 0$

$\Rightarrow \cot\theta = 0$ or $\cot\theta = -1$

For $\cot\theta = 0$ refer to graph: $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

For $\cot\theta = -1$, $\tan\theta = -1$



So $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$

Set of solutions: $\frac{\pi}{2}, \frac{3\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}$

e $3\sec\frac{1}{2}\theta = 2\tan^2\frac{1}{2}\theta \qquad 0 \le \theta \le 360°$

$\Rightarrow 3\sec\frac{1}{2}\theta = 2\left(\sec^2\frac{1}{2}\theta - 1\right)$

(use $1 + \tan^2 A \equiv \sec^2 A$ with $A = \frac{1}{2}\theta$)

$\Rightarrow 2\sec^2\frac{1}{2}\theta - 3\sec\frac{1}{2}\theta - 2 = 0$

$\Rightarrow \left(2\sec\frac{1}{2}\theta + 1\right)\left(\sec\frac{1}{2}\theta - 2\right) = 0$

$\Rightarrow \sec\frac{1}{2}\theta = -\frac{1}{2}$ or $\sec\frac{1}{2}\theta = 2$

Only $\sec\frac{1}{2}\theta = 2$ applies as

$\sec A \le -1$ or $\sec A \ge 1$

$\Rightarrow \cos\frac{1}{2}\theta = \frac{1}{2}$

As $0 \le \theta \le 360°$ so $0 \le \frac{1}{2}\theta \le 180°$

Calculator value is 60°

This is the only value in the interval.

So $\frac{1}{2}\theta = 60°$

$\Rightarrow \theta = 120°$

f $(\sec\theta - \cos\theta)^2 = \tan\theta - \sin^2\theta \qquad 0 \le \theta \le \pi$

$\Rightarrow \sec^2\theta - 2\sec\theta\cos\theta + \cos^2\theta$

$= \tan\theta - \sin^2\theta$

$\Rightarrow \sec^2\theta - 2 + \cos^2\theta = \tan\theta - \sin^2\theta$

$\left(\sec\theta\cos\theta = \frac{1}{\cos\theta} \times \cos\theta = 1\right)$

$\Rightarrow (1 + \tan^2\theta) - 2 + (\cos^2\theta + \sin^2\theta)$

$= \tan\theta$

$\Rightarrow 1 + \tan^2\theta - 2 + 1 = \tan\theta$

$\Rightarrow \tan^2\theta - \tan\theta = 0$

$\Rightarrow \tan\theta(\tan\theta - 1) = 0$

$\Rightarrow \tan\theta = 0$ or $\tan\theta = 1$

$\tan\theta = 0 \Rightarrow \theta = 0, \pi$

$\tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$

Set of solutions: $0, \frac{\pi}{4}, \pi$

g $\tan^2 2\theta = \sec 2\theta - 1 \qquad 0 \le \theta \le 180°$

$\Rightarrow \sec^2 2\theta - 1 = \sec 2\theta - 1$

$\Rightarrow \sec^2 2\theta - \sec 2\theta = 0$

$\Rightarrow \sec 2\theta(\sec 2\theta - 1) = 0$

$\Rightarrow \sec 2\theta = 0$ (not possible)

or $\sec 2\theta = 1$

$\Rightarrow \cos 2\theta = 1 \qquad 0 \le 2\theta \le 360°$

Refer to graph of $y = \cos\theta$

$\Rightarrow 2\theta = 0°, 360°$

$\Rightarrow \theta = 0°, 180°$

8 h $\sec^2\theta-(1+\sqrt{3})\tan\theta+\sqrt{3}=1$

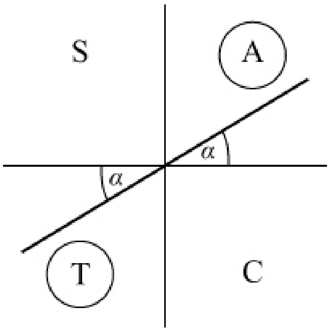
for $0\le\theta\le 2\pi$

$\Rightarrow (1+\tan^2\theta)-(1+\sqrt{3})\tan\theta+\sqrt{3}=1$

$\Rightarrow \tan^2\theta-(1+\sqrt{3})\tan\theta+\sqrt{3}=0$

$\Rightarrow (\tan\theta-\sqrt{3})(\tan\theta-1)=0$

$\Rightarrow \tan\theta=\sqrt{3}$ or $\tan\theta=1$



First answer for $\tan\theta=\sqrt{3}$ is $\frac{\pi}{3}$

Second solution is $\pi+\frac{\pi}{3}=\frac{4\pi}{3}$

First answer for $\tan\theta=1$ is $\frac{\pi}{4}$

Second solution is $\pi+\frac{\pi}{4}=\frac{5\pi}{4}$

Set of solutions: $\frac{\pi}{4}, \frac{\pi}{3}, \frac{5\pi}{4}, \frac{4\pi}{3}$

9 a $\tan^2 k=2\sec k$

$\Rightarrow (\sec^2 k-1)=2\sec k$

$\Rightarrow \sec^2 k-2\sec k-1=0$

$\Rightarrow \sec k=\frac{2\pm\sqrt{8}}{2}=\frac{2\pm 2\sqrt{2}}{2}=1\pm\sqrt{2}$

As $\sec k$ has no values between -1 and 1

$\sec k=1+\sqrt{2}$

b $\cos k=\frac{1}{1+\sqrt{2}}=\frac{\sqrt{2}-1}{\left(1+\sqrt{2}\right)\left(\sqrt{2}-1\right)}$

$=\frac{\sqrt{2}-1}{2-1}=\sqrt{2}-1$

c Solutions of $\tan^2 k=2\sec k,\ 0\le k\le 360°$

are solutions of $\cos k=\sqrt{2}-1$

Calculator solution is 65.5° (1 d.p.)

$\Rightarrow k=65.5°,\ 360°-65.5°$

$=65.5°,\ 294.5°$ (1 d.p.)

10 a As $a=4\sec x$

$\Rightarrow \sec x=\frac{a}{4}$

$\Rightarrow \cos x=\frac{4}{a}$

As $\cos x=b$

$\Rightarrow b=\frac{4}{a}$

b $c=\cot x$

$\Rightarrow c^2=\cot^2 x$

$\Rightarrow \frac{1}{c^2}=\tan^2 x$

$\Rightarrow \frac{1}{c^2}=\sec^2 x-1$

(use $1+\tan^2 x\equiv\sec^2 x$)

$\Rightarrow \frac{1}{c^2}=\frac{a^2}{16}-1 \quad \left(\sec x=\frac{a}{4}\right)$

$\Rightarrow 16=a^2c^2-16c^2$ (multiply by $16c^2$)

$\Rightarrow c^2(a^2-16)=16$

$\Rightarrow c^2=\frac{16}{a^2-16}$

11 a $x=\sec\theta+\tan\theta$

$\frac{1}{x}=\frac{1}{\sec\theta+\tan\theta}$

$=\frac{\sec\theta-\tan\theta}{(\sec\theta+\tan\theta)(\sec\theta-\tan\theta)}$

$=\frac{\sec\theta-\tan\theta}{\sec^2\theta-\tan^2\theta}$

$=\sec\theta-\tan\theta$

(as $1+\tan^2\theta\equiv\sec^2\theta$

$\Rightarrow \sec^2\theta-\tan^2\theta\equiv 1$)

11 b $x+\frac{1}{x}=\sec\theta+\tan\theta+\sec\theta-\tan\theta$

$=2\sec\theta$

$\Rightarrow \left(x+\frac{1}{x}\right)^2=4\sec^2\theta$

$\Rightarrow x^2+2x\times\frac{1}{x}+\frac{1}{x^2}=4\sec^2\theta$

$\Rightarrow x^2+\frac{1}{x^2}+2=4\sec^2\theta$

12 $2\sec^2\theta-\tan^2\theta=p$

$\Rightarrow 2(1+\tan^2\theta)-\tan^2\theta=p$

$\Rightarrow 2+2\tan^2\theta-\tan^2\theta=p$

$\Rightarrow \tan^2\theta=p-2$

$\Rightarrow \cot^2\theta=\frac{1}{p-2} \quad (p\neq 2)$

$\text{cosec}^2\theta=1+\cot^2\theta=1+\frac{1}{p-2}$

$=\frac{(p-2)+1}{p-2}=\frac{p-1}{p-2}, p\neq 2$