Exercise 3C

1 a $\frac{1}{\sin^3\theta}=\left(\frac{1}{\sin\theta}\right)^3=\operatorname{cosec}^3\theta$

b $\frac{4}{\tan^6\theta}=4\times\left(\frac{1}{\tan\theta}\right)^6=4\cot^6\theta$

c $\frac{1}{2\cos^2\theta}=\frac{1}{2}\times\left(\frac{1}{\cos\theta}\right)^2=\frac{1}{2}\sec^2\theta$

d $\frac{1-\sin^2\theta}{\sin^2\theta}=\frac{\cos^2\theta}{\sin^2\theta}$

(using $\sin^2\theta+\cos^2\theta\equiv1$)

So $\frac{1-\sin^2\theta}{\sin^2\theta}=\left(\frac{\cos\theta}{\sin\theta}\right)^2=\cot^2\theta$

e $\frac{\sec\theta}{\cos^4\theta}=\frac{1}{\cos\theta}\times\frac{1}{\cos^4\theta}=\frac{1}{\cos^5\theta}$

$=\left(\frac{1}{\cos\theta}\right)^5=\sec^5\theta$

f $\sqrt{\operatorname{cosec}^3\theta\cot\theta\sec\theta}$

$=\sqrt{\frac{1}{\sin^3\theta}\times\frac{\cos\theta}{\sin\theta}\times\frac{1}{\cos\theta}}=\sqrt{\frac{1}{\sin^4\theta}}$

$=\frac{1}{\sin^2\theta}=\left(\frac{1}{\sin\theta}\right)^2=\operatorname{cosec}^2\theta$

g $\frac{2}{\sqrt{\tan\theta}}=2\times\frac{1}{(\tan\theta)^{\frac{1}{2}}}=2\cot^{\frac{1}{2}}\theta$

h $\frac{\operatorname{cosec}^2\theta\tan^2\theta}{\cos\theta}=\frac{1}{\sin^2\theta}\times\frac{\sin^2\theta}{\cos^2\theta}\times\frac{1}{\cos\theta}$

$=\left(\frac{1}{\cos\theta}\right)^3=\sec^3\theta$

2 a $5\sin x=4\cos x$

$\Rightarrow\ 5=\frac{4\cos x}{\sin x}$ (divide by $\sin x$)

$\Rightarrow\ \frac{5}{4}=\cot x$ (divide by 4)

b $\tan x=-2$

$\Rightarrow\ \frac{1}{\tan x}=\frac{1}{-2}$

$\Rightarrow\ \cot x=-\frac{1}{2}$

c $3\frac{\sin x}{\cos x}=\frac{\cos x}{\sin x}$

$\Rightarrow\ 3\sin^2 x=\cos^2 x$

(multiply by $\sin x\cos x$)

$\Rightarrow\ 3=\frac{\cos^2 x}{\sin^2 x}$

(divide by $\sin^2 x$)

$\Rightarrow\ \left(\frac{\cos x}{\sin x}\right)^2=3$

$\Rightarrow\ \cot^2 x=3$

$\Rightarrow\ \cot x=\pm\sqrt{3}$

3 a $\sin\theta\cot\theta=\sin\theta\times\frac{\cos\theta}{\sin\theta}=\cos\theta$

b $\tan\theta\cot\theta=\tan\theta\times\frac{1}{\tan\theta}=1$

c $\tan2\theta\operatorname{cosec}2\theta=\frac{\sin2\theta}{\cos2\theta}\times\frac{1}{\sin2\theta}$

$=\frac{1}{\cos2\theta}=\sec2\theta$

d $\cos\theta\sin\theta(\cot\theta+\tan\theta)$

$=\cos\theta\sin\theta\left(\frac{\cos\theta}{\sin\theta}+\frac{\sin\theta}{\cos\theta}\right)$

$=\cos^2\theta+\sin^2\theta=1$

e $\sin^3 x\operatorname{cosec}x+\cos^3 x\sec x$

$=\sin^3 x\times\frac{1}{\sin x}+\cos^3 x\times\frac{1}{\cos x}$

$=\sin^2 x+\cos^2 x=1$

3 f $\sec A - \sec A\sin^2 A$

$= \sec A(1-\sin^2 A)$ (factorise)

$= \dfrac{1}{\cos A} \times \cos^2 A$

(using $\sin^2 A + \cos^2 A \equiv 1$)

$= \cos A$

g $\sec^2 x\cos^5 x + \cot x \operatorname{cosec} x \sin^4 x$

$= \dfrac{1}{\cos^2 x} \times \cos^5 x + \dfrac{\cos x}{\sin x} \times \dfrac{1}{\sin x} \times \sin^4 x$

$= \cos^3 x + \sin^2 x\cos x$

$= \cos x(\cos^2 x + \sin^2 x)$

$= \cos x$ (since $\cos^2 x + \sin^2 x \equiv 1$)

4 a $\text{LHS} \equiv \cos\theta + \sin\theta\tan\theta$

$\equiv \cos\theta + \sin\theta \dfrac{\sin\theta}{\cos\theta}$

$\equiv \dfrac{\cos^2\theta + \sin^2\theta}{\cos\theta}$

$\equiv \dfrac{1}{\cos\theta}$ (using $\sin^2\theta + \cos^2\theta \equiv 1$)

$\equiv \sec\theta \equiv \text{RHS}$

b $\text{LHS} \equiv \cot\theta + \tan\theta$

$\equiv \dfrac{\cos\theta}{\sin\theta} + \dfrac{\sin\theta}{\cos\theta}$

$\equiv \dfrac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta}$

$\equiv \dfrac{1}{\sin\theta\cos\theta}$

$\equiv \dfrac{1}{\sin\theta} \times \dfrac{1}{\cos\theta}$

$\equiv \operatorname{cosec}\theta\sec\theta \equiv \text{RHS}$

c $\text{LHS} \equiv \operatorname{cosec}\theta - \sin\theta$

$\equiv \dfrac{1}{\sin\theta} - \sin\theta$

$\equiv \dfrac{1-\sin^2\theta}{\sin\theta}$

$\equiv \dfrac{\cos^2\theta}{\sin\theta}$

$\equiv \cos\theta \times \dfrac{\cos\theta}{\sin\theta}$

$\equiv \cos\theta\cot\theta \equiv \text{RHS}$

d $\text{LHS} \equiv (1-\cos x)(1+\sec x)$

$\equiv 1 - \cos x + \sec x - \cos x\sec x$

(multiplying out)

$\equiv \sec x - \cos x$ (as $\cos x\sec x = 1$)

$\equiv \dfrac{1}{\cos x} - \cos x$

$\equiv \dfrac{1-\cos^2 x}{\cos x}$

$\equiv \dfrac{\sin^2 x}{\cos x}$

$\equiv \sin x \times \dfrac{\sin x}{\cos x}$

$\equiv \sin x\tan x \equiv \text{RHS}$

e $\text{LHS} \equiv \dfrac{\cos x}{1-\sin x} + \dfrac{1-\sin x}{\cos x}$

$\equiv \dfrac{\cos^2 x + (1-\sin x)^2}{(1-\sin x)\cos x}$

$\equiv \dfrac{\cos^2 x + (1-2\sin x + \sin^2 x)}{(1-\sin x)\cos x}$

$\equiv \dfrac{2-2\sin x}{(1-\sin x)\cos x}$

(using $\sin^2 x + \cos^2 x \equiv 1$)

$\equiv \dfrac{2(1-\sin x)}{(1-\sin x)\cos x}$

(factorising)

$\equiv \dfrac{2}{\cos x}$

$\equiv 2\sec x \equiv \text{RHS}$

f $\text{LHS} \equiv \dfrac{\cos\theta}{1+\cot\theta}$

$\equiv \dfrac{\cos\theta}{1+\frac{1}{\tan\theta}}$

$\equiv \dfrac{\cos\theta}{\frac{\tan\theta+1}{\tan\theta}}$

$\equiv \dfrac{\cos\theta\tan\theta}{1+\tan\theta}$

$\equiv \dfrac{\cos\theta \times \frac{\sin\theta}{\cos\theta}}{1+\tan\theta}$

$\equiv \dfrac{\sin\theta}{1+\tan\theta} \equiv \text{RHS}$

5 **a** $\sec\theta = \sqrt{2}$

$\Rightarrow \dfrac{1}{\cos\theta} = \sqrt{2}$

$\Rightarrow \cos\theta = \dfrac{1}{\sqrt{2}}$

Calculator value is $\theta = 45°$

$\cos\theta$ is positive

$\Rightarrow \theta$ is in 1st and 4th quadrants

Solutions are 45°, 315°

b $\operatorname{cosec}\theta = -3$

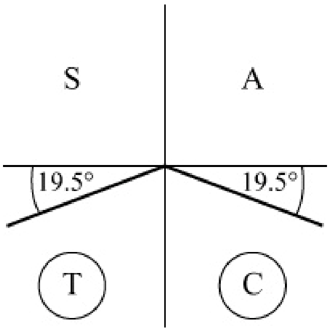
$\Rightarrow \dfrac{1}{\sin\theta} = -3$

$\Rightarrow \sin\theta = -\dfrac{1}{3}$

Calculator value is $\theta = -19.47°$ (2 d.p.)

$\sin\theta$ is negative

$\Rightarrow \theta$ is in 3rd and 4th quadrants



Solutions are 199°, 341° (3 s.f.)

c $5\cot\theta = -2$

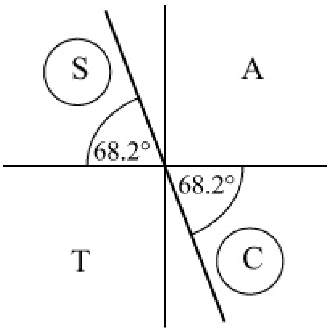
$\Rightarrow \cot\theta = -\dfrac{2}{5}$

$\Rightarrow \tan\theta = -\dfrac{5}{2}$

Calculator value is $\theta = -68.20°$ (2 d.p.)

$\tan\theta$ is negative

$\Rightarrow \theta$ is in 2nd and 4th quadrants



Solutions are 112°, 292° (3 s.f.)

d $\operatorname{cosec}\theta = 2$

$\Rightarrow \dfrac{1}{\sin\theta} = 2$

$\Rightarrow \sin\theta = \dfrac{1}{2}$

Calculator value is $\theta = 30°$

$\sin\theta$ is positive

$\Rightarrow \theta$ is in 1st and 2nd quadrants

Solutions are 30°, 150°

e $3\sec^2\theta = 4$

$\Rightarrow \sec^2\theta = \dfrac{4}{3}$

$\Rightarrow \cos^2\theta = \dfrac{3}{4}$

$\Rightarrow \cos\theta = \pm\dfrac{\sqrt{3}}{2}$

Calculator value for $\cos\theta = \dfrac{\sqrt{3}}{2}$ is $\theta = 30°$

As $\cos\theta$ is $\pm$, θ is in all four quadrants

Solutions are 30°, 150°, 210°, 330°

f $5\cos\theta = 3\cot\theta$

$\Rightarrow 5\cos\theta = 3\dfrac{\cos\theta}{\sin\theta}$

Do not cancel $\cos\theta$ on each side.

Multiply through by $\sin\theta$.

$\Rightarrow 5\cos\theta\sin\theta = 3\cos\theta$

$\Rightarrow 5\cos\theta\sin\theta - 3\cos\theta = 0$

$\Rightarrow \cos\theta(5\sin\theta - 3) = 0$ (factorise)

So $\cos\theta = 0$ or $\sin\theta = \dfrac{3}{5}$

When $\cos\theta = 0$, $\theta = 90°, 270°$

When $\sin\theta = \dfrac{3}{5}$, $\theta = 36.9°, 143°$ (3 s.f.)

Solutions are 36.9°, 90°, 143°, 270°

5 g $\cot^2\theta - 8\tan\theta = 0$

$\Rightarrow \dfrac{1}{\tan^2\theta} - 8\tan\theta = 0$

$\Rightarrow 1 - 8\tan^3\theta = 0$

$\Rightarrow 8\tan^3\theta = 1$

$\Rightarrow \tan^3\theta = \dfrac{1}{8}$

$\Rightarrow \tan\theta = \dfrac{1}{2}$

Calculator value is $\theta = 26.57°$ (2 d.p.)

$\tan\theta$ is positive

$\Rightarrow \theta$ is in 1st and 3rd quadrants

Solutions are $26.57°$ and $(180° + 26.57°)$

So solutions are $26.6°$, $207°$ (3 s.f.)

h $2\sin\theta = \operatorname{cosec}\theta$

$\Rightarrow 2\sin\theta = \dfrac{1}{\sin\theta}$

$\Rightarrow \sin^2\theta = \dfrac{1}{2}$

$\Rightarrow \sin\theta = \pm\dfrac{1}{\sqrt{2}}$

Calculator value for $\sin\theta = \dfrac{1}{\sqrt{2}}$ is $\theta = 45°$

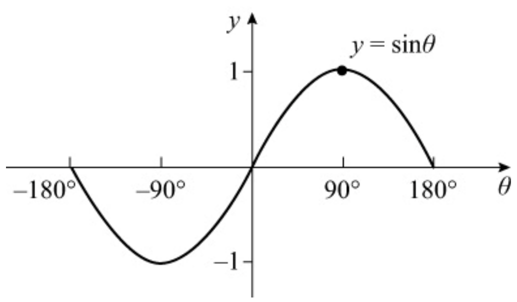
Solutions are in all four quadrants

Solutions are $45°$, $135°$, $225°$, $315°$

6 a $\operatorname{cosec}\theta = 1$

$\Rightarrow \sin\theta = 1$

$\Rightarrow \theta = 90°$



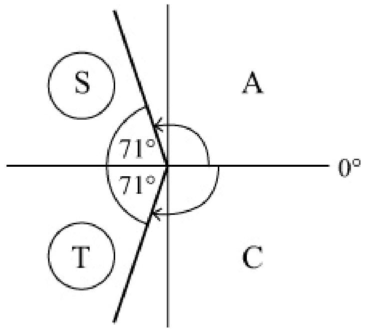
b $\sec\theta = -3$

$\Rightarrow \cos\theta = -\dfrac{1}{3}$

Calculator value is $\theta = 109°$ (3 s.f.)

$\cos\theta$ is negative

$\Rightarrow \theta$ is in 2nd and 3rd quadrants



Solutions are $109°$, $-109°$ (3 s.f.)

c $\cot\theta = 3.45$

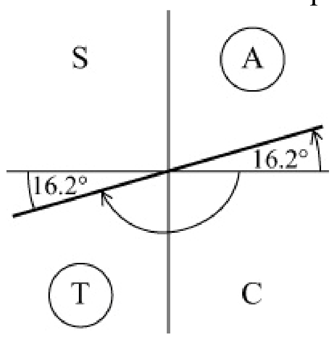
$\Rightarrow \dfrac{1}{\tan\theta} = 3.45$

$\Rightarrow \tan\theta = \dfrac{1}{3.45} = 0.2899$ (4 d.p.)

Calculator value is $\theta = 16.16°$ (2 d.p.)

$\tan\theta$ is positive

$\Rightarrow \theta$ is in 1st and 3rd quadrants



Solutions are $16.2°$ and $(-180° + 16.2°)$

So solutions are $16.2°$, $-164°$ (3 s.f.)

6 d $2\text{cosec}^2\theta - 3\text{cosec}\,\theta = 0$

$\Rightarrow \text{cosec}\,\theta(2\text{cosec}\,\theta - 3) = 0$ (factorise)

$\Rightarrow \text{cosec}\,\theta = 0$ or $\text{cosec}\,\theta = \frac{3}{2}$

$\Rightarrow \sin\theta = \frac{2}{3}$

$\text{cosec}\,\theta = 0$ has no solutions

Calculator value for $\sin\theta = \frac{2}{3}$ is $\theta = 41.8°$

θ is in 1st and 2nd quadrants

Solutions are 41.8°, (180 − 41.8)°

So solutions are 41.8°, 138° (3 s.f.)

e $\sec\theta = 2\cos\theta$

$\Rightarrow \frac{1}{\cos\theta} = 2\cos\theta$

$\Rightarrow \cos^2\theta = \frac{1}{2}$

$\Rightarrow \cos\theta = \pm\frac{1}{\sqrt{2}}$

Calculator value for $\cos\theta = \frac{1}{\sqrt{2}}$ is $\theta = 45°$

θ is in all quadrants, but remember that solutions required for $-180° \le \theta \le 180°$

Solutions are $\pm 45°$, $\pm 135°$

f $3\cot\theta = 2\sin\theta$

$\Rightarrow 3\frac{\cos\theta}{\sin\theta} = 2\sin\theta$

$\Rightarrow 3\cos\theta = 2\sin^2\theta$

$\Rightarrow 3\cos\theta = 2(1 - \cos^2\theta)$

(use $\sin^2\theta + \cos^2\theta \equiv 1$)

$\Rightarrow 2\cos^2\theta + 3\cos\theta - 2 = 0$

$\Rightarrow (2\cos\theta - 1)(\cos\theta + 2) = 0$

$\Rightarrow \cos\theta = \frac{1}{2}$ or $\cos\theta = -2$

As $\cos\theta = -2$ has no solutions, $\cos\theta = \frac{1}{2}$

Solutions are $\pm 60°$

g $\text{cosec}\,2\theta = 4$

$\Rightarrow \sin 2\theta = \frac{1}{4}$

Remember that solutions are required in the interval $-180° \le \theta \le 180°$

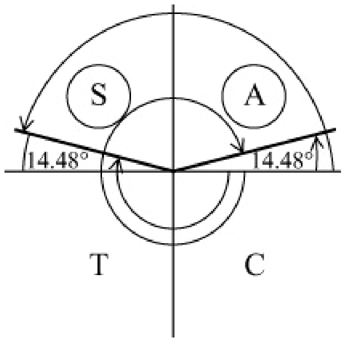
So $-360° \le 2\theta \le 360°$

Calculator value for $\sin 2\theta = \frac{1}{4}$ is

$2\theta = 14.48°$ (2 d.p.)

$\sin 2\theta$ is positive

$\Rightarrow 2\theta$ is in 1st and 2nd quadrants



$2\theta = -194.48°, -345.52°,$
$14.48°, 165.52°$

$\theta = -97.2°, -172.8°, 7.24°, 82.76°$
$= -173°, -97.2°, 7.24°, 82.8°$ (3 s.f.)

6 h $2\cot^2\theta - \cot\theta - 5 = 0$

As this quadratic in $\cot\theta$ does not factorise, use the quadratic formula

$\cot\theta = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

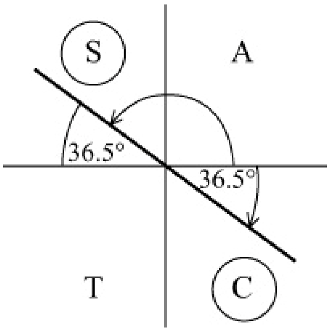
(You could change $\cot\theta$ to $\frac{1}{\tan\theta}$ and work with the quadratic $5\tan^2\theta + \tan\theta - 2 = 0$)

So $\cot\theta = \dfrac{1 \pm \sqrt{41}}{4}$

$= -1.3508,\ 1.8508$ (4 d.p.)

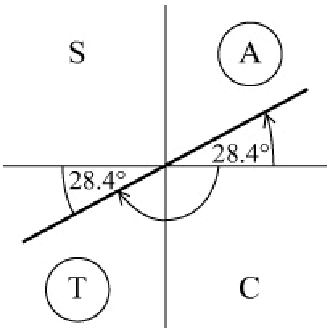
So $\tan\theta = -0.7403,\ 0.5403$ (4 d.p.)

The calculator value for $\tan\theta = -0.7403$ is $\theta = -36.51°$ (2 d.p.)



Solutions are $-36.5°,\ 143°$ (3 s.f.).

The calculator value for $\tan\theta = 0.5403$ is $\theta = 28.38°$ (2 d.p.)



Solutions are $28.4°,\ (-180 + 28.4)°$

Total set of solutions is

$-152°,\ -36.5°,\ 28.4°,\ 143°$ (3 s.f.)

7 a $\sec\theta = -1$

$\Rightarrow\ \cos\theta = -1$

$\Rightarrow\ \theta = \pi$

(refer to graph of $y = \cos\theta$)

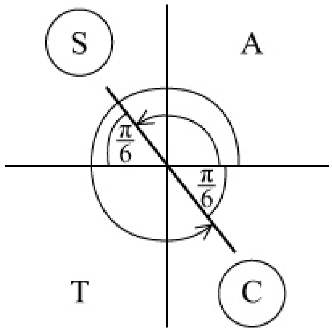
b $\cot\theta = -\sqrt{3}$

$\Rightarrow\ \tan\theta = -\dfrac{1}{\sqrt{3}}$

Calculator solution is $-\dfrac{\pi}{6}$

$\left(\text{you should know that } \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}\right)$

$-\dfrac{\pi}{6}$ is not in the interval



Solutions are $\pi - \dfrac{\pi}{6},\ 2\pi - \dfrac{\pi}{6} = \dfrac{5\pi}{6},\ \dfrac{11\pi}{6}$

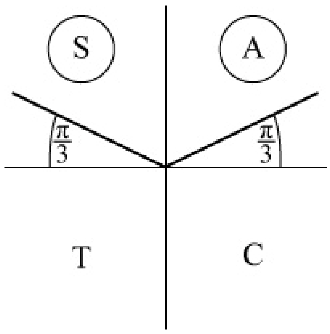
c $\operatorname{cosec}\dfrac{\theta}{2} = \dfrac{2\sqrt{3}}{3}$

$\Rightarrow\ \sin\dfrac{\theta}{2} = \dfrac{3}{2\sqrt{3}} = \dfrac{\sqrt{3}}{2}$

Remember that $0 \leq \theta \leq 2\pi$

so $0 \leq \dfrac{\theta}{2} \leq \pi$

First solution for $\sin\dfrac{\theta}{2} = \dfrac{\sqrt{3}}{2}$ is $\dfrac{\theta}{2} = \dfrac{\pi}{3}$



So $\dfrac{\theta}{2} = \dfrac{\pi}{3},\ \dfrac{2\pi}{3}$

$\Rightarrow\ \theta = \dfrac{2\pi}{3},\ \dfrac{4\pi}{3}$

7 d $\sec\theta = \sqrt{2}\tan\theta$

$\Rightarrow \dfrac{1}{\cos\theta} = \sqrt{2}\dfrac{\sin\theta}{\cos\theta}$

$\Rightarrow 1 = \sqrt{2}\sin\theta \quad (\cos\theta \neq 0)$

$\Rightarrow \sin\theta = \dfrac{1}{\sqrt{2}}$

Solutions are $\dfrac{\pi}{4}, \dfrac{3\pi}{4}$

8 a In the right-angled triangle ABD

$\dfrac{AB}{AD} = \cos\theta$

$\Rightarrow AD = \dfrac{6}{\cos\theta} = 6\sec\theta$

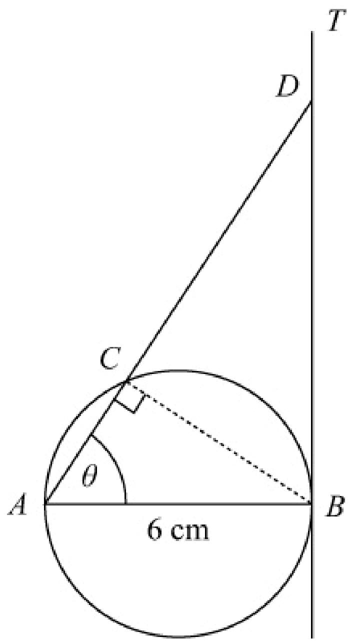
In the right-angled triangle ACB

$\dfrac{AC}{AB} = \cos\theta$

$\Rightarrow AC = 6\cos\theta$

$CD = AD - AC$

$= 6\sec\theta - 6\cos\theta = 6(\sec\theta - \cos\theta)$



b As $16 = 6\sec\theta - 6\cos\theta$

$\Rightarrow 8 = \dfrac{3}{\cos\theta} - 3\cos\theta$

$\Rightarrow 8\cos\theta = 3 - 3\cos^2\theta$

$\Rightarrow 3\cos^2\theta + 8\cos\theta - 3 = 0$

$\Rightarrow (3\cos\theta - 1)(\cos\theta + 3) = 0$

$\Rightarrow \cos\theta = \dfrac{1}{3}$ as $\cos\theta \neq -3$

From (a) $AC = 6\cos\theta = 6 \times \dfrac{1}{3} = 2$ cm

9 a $\dfrac{\operatorname{cosec} x - \cot x}{1 - \cos x} \equiv \dfrac{\dfrac{1}{\sin x} - \dfrac{\cos x}{\sin x}}{1 - \cos x}$

$\equiv \dfrac{1}{\sin x} \times \dfrac{1 - \cos x}{1 - \cos x}$

$\equiv \operatorname{cosec} x$

b By part a equation becomes

$\operatorname{cosec} x = 2$

$\Rightarrow \dfrac{1}{\sin x} = 2$

$\Rightarrow \sin x = \dfrac{1}{2}$

$\sin x$ is positive, so x is in

1st and 2nd quadrants

$x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

10 a $\dfrac{\sin x \tan x}{1 - \cos x} - 1 \equiv \dfrac{\sin^2 x}{\cos x(1 - \cos x)} - 1$

$\equiv \dfrac{\sin^2 x - \cos x + \cos^2 x}{\cos x(1 - \cos x)}$

$\equiv \dfrac{1 - \cos x}{\cos x(1 - \cos x)}$

$\equiv \dfrac{1}{\cos x}$

$\equiv \sec x$

b Need to solve $\sec x = -\dfrac{1}{2}$

$\Rightarrow \cos x = -2$

which has no solutions.

11 $\frac{1+\cot x}{1+\tan x}=5$

$\Rightarrow \frac{1+\frac{\cos x}{\sin x}}{1+\frac{\sin x}{\cos x}}=5$

$\Rightarrow \frac{\frac{\sin x+\cos x}{\sin x}}{\frac{\cos x+\sin x}{\cos x}}=5$

$\Rightarrow \frac{\sin x+\cos x}{\sin x}\times\frac{\cos x}{\cos x+\sin x}=5$

$\Rightarrow \frac{\cos x}{\sin x}=5$

$\Rightarrow \cot x=5$

$\Rightarrow \tan x=\frac{1}{5}$

Calculator solution is $11.3°$ (1 d.p.)

$\tan x$ is positive, so x is in

1st and 3rd quadrants

Solutions are $11.3°, 191.3°$ (1 d.p.)