

# 4753 (C3) Methods for Advanced Mathematics

## Section A

<p><b>1</b></p> $\int_0^{\pi/6} \sin 3x dx = \left[ -\frac{1}{3} \cos 3x \right]_0^{\pi/6}$ $= -\frac{1}{3} \cos \frac{\pi}{2} + \frac{1}{3} \cos 0$ $= \frac{1}{3}$	<p>B1</p> <p>M1</p> <p>A1cao [3]</p>	$\left[ -\frac{1}{3} \cos 3x \right] \text{ or } \left[ -\frac{1}{3} \cos u \right]$ <p>substituting correct limits in <math>\pm k \cos \dots</math></p> <p>0.33 or better.</p>
<p><b>2(i)</b></p> $100 = Ae^0 = A \Rightarrow A = 100$ $50 = 100 e^{-1500k}$ $\Rightarrow e^{-1500k} = 0.5$ $\Rightarrow -1500k = \ln 0.5$ $\Rightarrow k = -\ln 0.5 / 1500 = 4.62 \times 10^{-4}$	<p>M1A1</p> <p>M1</p> <p>M1</p> <p>A1 [5]</p>	<p><math>50 = A e^{-1500k}</math> ft their 'A' if used</p> <p>taking lns correctly</p> <p>0.00046 or better</p>
<p><b>(ii)</b></p> $1 = 100e^{-kt}$ $\Rightarrow -kt = \ln 0.01$ $\Rightarrow t = -\ln 0.01 / k$ $= 9966 \text{ years}$	<p>M1</p> <p>M1</p> <p>A1 [3]</p>	<p>ft their A and k</p> <p>taking lns correctly</p> <p>art 9970</p>
<p><b>3</b></p>	<p>M1</p> <p>B1</p> <p>A1 [3]</p>	<p>Can use degrees or radians</p> <p>reasonable shape (condone extra range)</p> <p>passes through <math>(-1, 2\pi)</math>, <math>(0, \pi)</math> and <math>(1, 0)</math></p> <p>good sketches – look for curve reasonably vertical at <math>(-1, 2\pi)</math> and <math>(1, 0)</math>, negative gradient at <math>(0, \pi)</math>. Domain and range must be clearly marked and correct.</p>
<p><b>4</b></p> $g(x) = 2 x-1 $ $\Rightarrow b = 2 0-1  = 2 \text{ or } (0, 2)$ $2 x-1  = 0$ $\Rightarrow x = 1, \text{ so } a = 1 \text{ or } (1, 0)$	<p>B1</p> <p>M1</p> <p>A1 [3]</p>	<p>Allow unsupported answers.</p> <p>www</p> <p><math> x  = 1</math> is A0</p> <p>www</p>

<p><b>5(i)</b> <math>e^{2y} = 1 + \sin x</math>  <math>\Rightarrow 2e^{2y} dy/dx = \cos x</math>  <math>\Rightarrow dy/dx = \frac{\cos x}{2e^{2y}}</math></p>	<p>M1 B1  A1 [3]</p>	<p>Their <math>2e^{2y} \times dy/dx</math>  <math>2e^{2y}</math>  o.e. cao</p>
<p><b>(ii)</b> <math>2y = \ln(1 + \sin x)</math>  <math>\Rightarrow y = \frac{1}{2} \ln(1 + \sin x)</math>  <math>\Rightarrow dy/dx = \frac{1}{2} \frac{\cos x}{1 + \sin x}</math>  <math>= \frac{\cos x}{2e^{2y}}</math> as before</p>	<p>B1 M1  B1 E1 [4]</p>	<p>chain rule (can be within 'correct' quotient rule with <math>dv/dx = 0</math>)  <math>1/u</math> or <math>1/(1 + \sin x)</math> soi  www</p>
<p><b>6</b></p> $f(x) = \frac{\frac{x+1}{x-1} + 1}{\frac{x+1}{x-1} - 1}$ $= \frac{x+1+x-1}{x+1-x+1}$ $= 2x/2 = x^*$ <p><math>f^{-1}(x) = f(x)</math></p> <p>Symmetrical about <math>y = x</math>.</p>	<p>M1  M1  E1  B1  B1 [5]</p>	<p>correct expression  without subsidiary denominators  e.g. <math>= \frac{x+1+x-1}{x-1} \times \frac{x-1}{x+1-x+1}</math>  stated, or shown by inverting</p>
<p><b>7(i)</b> (A) <math>(x-y)(x^2 + xy + y^2)</math>  <math>= x^3 + x^2y + xy^2 - yx^2 - xy^2 - y^3</math>  <math>= x^3 - y^3</math> *</p> <p>(B) <math>(x + \frac{1}{2}y)^2 + \frac{3}{4}y^2</math>  <math>= x^2 + xy + \frac{1}{4}y^2 + \frac{3}{4}y^2</math>  <math>= x^2 + xy + y^2</math></p>	<p>M1  E1  M1  E1 [4]</p>	<p>expanding - allow tabulation  www  <math>(x + \frac{1}{2}y)^2 = x^2 + \frac{1}{2}xy + \frac{1}{2}xy + \frac{1}{4}y^2</math> o.e.  cao www</p>
<p><b>(ii)</b> <math>x^3 - y^3 = (x-y)[(x + \frac{1}{2}y)^2 + \frac{3}{4}y^2]</math>  <math>(x + \frac{1}{2}y)^2 + \frac{3}{4}y^2 &gt; 0</math> [as squares <math>\geq 0</math>]</p> <p><math>\Rightarrow</math> if <math>x - y &gt; 0</math> then <math>x^3 - y^3 &gt; 0</math>  <math>\Rightarrow</math> if <math>x &gt; y</math> then <math>x^3 &gt; y^3</math> *</p>	<p>M1  M1  E1 [3]</p>	<p>substituting results of (i)</p>

<p><b>8(i)</b> A: <math>1 + \ln x = 0</math>  <math>\Rightarrow \ln x = -1</math> so A is <math>(e^{-1}, 0)</math>  <math>\Rightarrow x = e^{-1}</math>            B: <math>x = 0, y = e^{0-1} = e^{-1}</math> so B is <math>(0, e^{-1})</math></p> <p>C: <math>f(1) = e^{1-1} = e^0 = 1</math>  <math>g(1) = 1 + \ln 1 = 1</math></p>	<p>M1  A1 B1  E1 E1 [5]</p>	<p>SC1 if obtained using symmetry            condone use of symmetry            Penalise <math>A = e^{-1}, B = e^{-1}</math>, or co-ords wrong way round, but condone labelling errors.</p>
<p><b>(ii)</b> Either by inversion:            e.g. <math>y = e^{x-1} \quad x \leftrightarrow y</math>  <math>x = e^{y-1}</math>  <math>\Rightarrow \ln x = y - 1</math>  <math>\Rightarrow 1 + \ln x = y</math></p>	<p>M1 E1</p>	<p>taking lns or exps</p>
<p>or by composing            e.g. <math>f(g(x)) = f(1 + \ln x)</math>  <math>= e^{1 + \ln x - 1}</math>  <math>= e^{\ln x} = x</math></p>	<p>M1  E1 [2]</p>	<p><math>e^{1 + \ln x - 1}</math> or <math>1 + \ln(e^{x-1})</math></p>
<p><b>(iii)</b> <math>\int_0^1 e^{x-1} dx = [e^{x-1}]_0^1</math>  <math>= e^0 - e^{-1}</math>  <math>= 1 - e^{-1}</math></p>	<p>M1  M1 A1cao [3]</p>	<p><math>[e^{x-1}]</math> o.e. or <math>u = x - 1 \Rightarrow [e^u]</math>            substituting correct limits for <math>x</math> or <math>u</math>            o.e. not <math>e^0</math>, must be exact.</p>
<p><b>(iv)</b> <math>\int \ln x dx = \int \ln x \frac{d}{dx}(x) dx</math>  <math>= x \ln x - \int x \cdot \frac{1}{x} dx</math>  <math>= x \ln x - x + c</math>  <math>\Rightarrow \int_{e^{-1}}^1 g(x) dx = \int_{e^{-1}}^1 (1 + \ln x) dx</math>  <math>= [x + x \ln x - x]_{e^{-1}}^1</math>  <math>= [x \ln x]_{e^{-1}}^1</math>  <math>= 1 \ln 1 - e^{-1} \ln(e^{-1})</math>  <math>= e^{-1} *</math></p>	<p>M1  A1  A1cao  B1ft  DM1 E1 [6]</p>	<p>parts: <math>u = \ln x, du/dx = 1/x, v = x, dv/dx = 1</math>             condone no 'c'             fit their '<math>x \ln x - x</math>' (provided 'algebraic')             substituting limits dep B1            www</p>
<p><b>(v)</b> Area = <math>\int_0^1 f(x) dx - \int_{e^{-1}}^1 g(x) dx</math>  <math>= (1 - e^{-1}) - e^{-1}</math>  <math>= 1 - 2/e</math></p>	<p>M1  A1cao</p>	<p>Must have correct limits             0.264 or better.</p>
<p>or            Area OCB = area under curve - triangle  <math>= 1 - e^{-1} - \frac{1}{2} \times 1 \times 1</math>  <math>= \frac{1}{2} - e^{-1}</math></p> <p>or            Area OAC = triangle - area under curve  <math>= \frac{1}{2} \times 1 \times 1 - e^{-1}</math>  <math>= \frac{1}{2} - e^{-1}</math>            Total area = <math>2(\frac{1}{2} - e^{-1}) = 1 - 2/e</math></p>	<p>M1  A1cao [2]</p>	<p>OCA or OCB = <math>\frac{1}{2} - e^{-1}</math>             0.264 or better</p>

<b>9(i)</b> $a = 1/3$	B1 [1]	or 0.33 or better
<b>(ii)</b> $\frac{dy}{dx} = \frac{(3x-1)2x - x^2 \cdot 3}{(3x-1)^2}$ $= \frac{6x^2 - 2x - 3x^2}{(3x-1)^2}$ $= \frac{3x^2 - 2x}{(3x-1)^2}$ $= \frac{x(3x-2)}{(3x-1)^2}$ *	M1 A1    E1 [3]	quotient rule     www – must show both steps; penalise missing brackets.
<b>(iii)</b> $dy/dx = 0$ when $x(3x-2) = 0$ $\Rightarrow x = 0$ or $x = 2/3$ , so at P, $x = 2/3$ when $x = \frac{2}{3}$ , $y = \frac{(2/3)^2}{3 \times (2/3) - 1} = \frac{4}{9}$  when $x = 0.6$ , $dy/dx = -0.1875$ when $x = 0.8$ , $dy/dx = 0.1633$ Gradient increasing $\Rightarrow$ minimum	M1 A1 M1 A1cao  B1 B1 E1 [7]	if denom = 0 also then M0 o.e e.g. 0.6, but must be exact o.e e.g. 0.4, but must be exact -3/16, or -0.19 or better 8/49 or 0.16 or better o.e. e.g. 'from negative to positive'. Allow ft on their gradients, provided -ve and +ve respectively. Accept table with indications of signs of gradient.
<b>(iv)</b> $\int \frac{x^2}{3x-1} dx$ $u = 3x-1 \Rightarrow du = 3dx$ $\frac{(u+1)^2}{9} \frac{1}{3} du$ $= \frac{1}{27} \int \frac{(u+1)^2}{u} du = \frac{1}{27} \int \frac{u^2 + 2u + 1}{u} du$ $= \frac{1}{27} \int (u + 2 + \frac{1}{u}) du$ *  Area = $\int_{2/3}^1 \frac{x^2}{3x-1} dx$ When $x = 2/3$ , $u = 1$ , when $x = 1$ , $u = 2$ $= \frac{1}{27} \int_1^2 (u + 2 + 1/u) du$ $= \frac{1}{27} \left[ \frac{1}{2} u^2 + 2u + \ln u \right]_1^2$ $= \frac{1}{27} [(2+4+\ln 2) - (\frac{1}{2} + 2 + \ln 1)]$ $= \frac{1}{27} (3\frac{1}{2} + \ln 2) [= \frac{7+2\ln 2}{54}]$	B1  M1 M1 E1   B1 M1 A1cao [7]	$\frac{(u+1)^2}{9} \frac{1}{u}$ o.e. $\times 1/3 (du)$ expanding Condone missing $du$ 's   $\left[ \frac{1}{2} u^2 + 2u + \ln u \right]$ substituting correct limits, dep integration o.e., but must evaluate $\ln 1 = 0$ and collect terms.