C3 June 2018 (MA)

Q1a) $y = 2x(3x-1)^5$ **PRODUCT RULE:**

$$\frac{dy}{dx} = 2(3x-1)^5 + 2x(5)(3x-1)^4(3)$$

$$= 2(3x-1)^5 + 30x(3x-1)^4$$

$$= 2(3x-1)^4\left[3x-1+\frac{30x}{2}\right]$$

$$= 2(3x-1)^4\left[3x+15x-1\right]$$

$$= 2(3x-1)^4(18x-1) \quad //$$

b) $\underline{\frac{dy}{dx} \leq 0}$: $2(3x-1)^4(18x-1) \leq 0$

$(3x-1)^4$ will always be ≥ 0.

so $\frac{dy}{dx} = 0$ when $x = \frac{1}{3}$, $x = \frac{1}{18}$.

and $\frac{dy}{dx} < 0$ when $18x - 1 < 0$

$$x < \frac{1}{18} \quad //$$

So range of values: $\boxed{x \leq \frac{1}{18}, \quad x = \frac{1}{3}}$

Q2a) $f(x) = \dfrac{6(2x-5)}{(2x+5)(2x-5)} + \dfrac{2(2x+5)}{(2x+5)(2x-5)} + \dfrac{60}{(2x+5)(2x-5)}$

$= \dfrac{12x - 30 + 4x + 10 + 60}{(2x+5)(2x-5)}$

$= \dfrac{16x + 40}{(2x+5)(2x-5)} = \dfrac{8\cancel{(2x+5)}}{\cancel{(2x+5)}(2x-5)}$

$= \boxed{\dfrac{8}{2x-5}}$

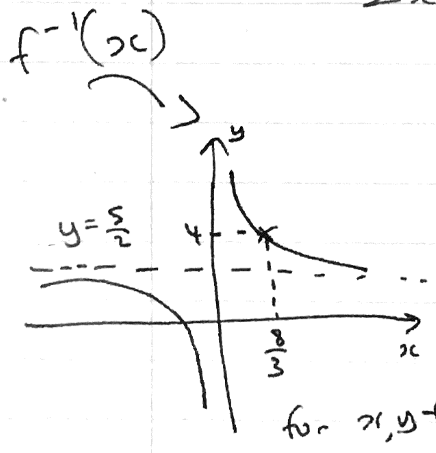
b) $y = \dfrac{8}{2x-5}$

$x \leftrightarrow y$; $x = \dfrac{8}{2y-5}$

now make y the subject again,

$2xy - 5x = 8$

$y = \dfrac{8+5x}{2x} = \boxed{\dfrac{4}{x} + \dfrac{5}{2}}$



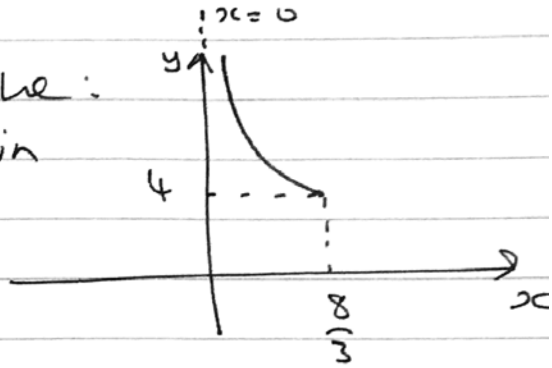
so domain : $\boxed{0 < x < \dfrac{8}{3}}$

for $x, y \in \mathbb{R}$.

remember, domain of $f(x)$ was $x > 4$
so range of $f^{-1}(x)$ is $y > 4$

so $f^{-1}(x)$ will look like:
hence giving us the domain

$0 < x < \frac{8}{3}$.



($f^{-1}(x)$ isn't actually defined at $x = \frac{8}{3}$)

Q3a) $\underline{t=0, V=17500}$: $17500 = 16000e^{0} + A$

$\therefore A = \boxed{1500}$

b) $\underline{t=2, V=13500}$: $13500 = 16000e^{-2k} + 1500$

$12000 = 16000e^{-2k}$

$\frac{12}{16} = \frac{3}{4} = e^{-2k}$ //

$\therefore -2k = \ln\frac{3}{4}$

$\therefore k = -\frac{1}{2}\ln\frac{3}{4} = \boxed{\ln\left(\frac{2}{\sqrt{3}}\right)}$

c) $6000 = 16000e^{-t\ln\left(\frac{2}{\sqrt{3}}\right)} + 1500$

$4500 = 16000e^{-t\ln\left(\frac{2}{\sqrt{3}}\right)}$

$\frac{9}{32} = e^{-t\ln\left(\frac{2}{\sqrt{3}}\right)} = \left(e^{\ln\left(\frac{2}{\sqrt{3}}\right)}\right)^{-t} = \left(\frac{2}{\sqrt{3}}\right)^{-t}$ //

$\therefore \left(\frac{2}{\sqrt{3}}\right)^{-t} = \frac{9}{32}$

$\ln\left(\frac{9}{32}\right) = \ln\left(\left(\frac{2}{\sqrt{3}}\right)^{-t}\right)$

$\ln\left(\frac{9}{32}\right) = -t\ln\left(\frac{2}{\sqrt{3}}\right)$

so $t = \dfrac{-\ln\left(\frac{9}{32}\right)}{\ln\left(\frac{2}{\sqrt{3}}\right)} = \boxed{8.82}$ years

Q4a) $y = e^{-2x} + x^2 - 3$

$\frac{dy}{dx} = -2e^{-2x} + 2x$

$\underline{x=0}$: $\frac{dy}{dx} = -2$ // so at normal $m = \frac{1}{2}$ //

$\left(-2 \times \frac{1}{2} = -1\right)$

$\Rightarrow y = e^0 + 0 - 3 = \underline{\underline{-2}}$ at A.

$\therefore \quad y - -2 = \frac{1}{2}(x - 0)$

$$\boxed{y = \frac{1}{2}x - 2}$$

b) $y = e^{-2x} + x^2 - 3 = \frac{1}{2}x - 2$

$$x^2 = \frac{1}{2}x + 1 - e^{-2x}$$

$$\therefore x = \sqrt{1 + \frac{1}{2}x - e^{-2x}}$$

c) $x_1 = 1$

$x_2 = \sqrt{1 + \frac{1}{2} - e^{-2}} = \boxed{1.168}$

similarly, $x_3 = \boxed{1.220}$

Q5a) the minimum point is at $(y = 3)$
$f(x)$ touches the y-axis at $(13 = y)$

So if $\boxed{k = 3}$ and $\boxed{k > 13}$ then $f(x) = k$ will only have one root as $f(x)$ will only meet $y = k$ once.

Intersecting at the minimum point does NOT imply two roots ... its only one point!

b) $2|5-x| + 3 = \frac{1}{2}x + 10$

one root is given by : $2(5-x) + 3 = \frac{1}{2}x + 10$

$\cancel{10} - 2x + 3 = \frac{1}{2}x + \cancel{10}$

$\frac{5}{2}x = 3$

$\boxed{x = \frac{6}{5}}$

another solution is given by : $2(x-5) + 3 = \frac{1}{2}x + 10$

$2x - 10 + 3 = \frac{1}{2}x + 10$

$\frac{3}{2}x = 17$

$\therefore \boxed{x = \frac{34}{3}}$

c) minimum of f(x) is (5, 3)

4f(x-1) means this point is translated one unit in the +ve x-direction and its y-coordinate multiplied by 4.

hence (6, 12) will be the new point. after the transformation (ie the image).

$p = 6, \ q = 12$

$-90 < x < 90$

Q6i)

$$\frac{\tan 2x + \tan 32}{1 - \tan 2x \tan 32} = 5$$

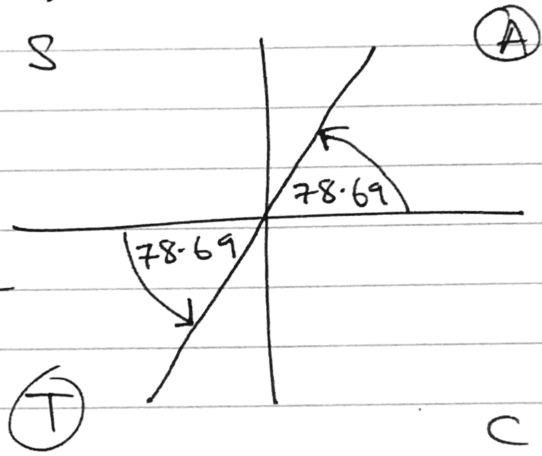
LHS $= \tan(2x + 32)$

$\therefore \tan(2x+32) = 5$

$2x + 32 = \tan^{-1}(5) = 78.69°$

NEW RANGE :

$-148 < 2x+32 < 212$



$\therefore 2x + 32 = -101.31°, 78.69°$

$2x = -133.31°, 46.69°$

$x = -66.65°, 23.35°$

(use exact value of $\tan^{-1}(5)$ to achieve correct answer to 2 d.p) iia) $\tan(3\theta - 45) = \dfrac{\tan 3\theta - 1}{1 + \tan 3\theta}$

$$\text{LHS} = \frac{\tan 3\theta - \tan 45}{1 + \tan 45 \tan 3\theta} = \frac{\tan 3\theta - 1}{1 + \tan 3\theta} \quad //$$

$\tan 45 = 1$.

b) $(1 + \tan 3\theta)\tan(\theta + 28) = \tan 3\theta - 1$

$$\tan(\theta + 28^\circ) = \frac{\tan 3\theta - 1}{1 + \tan 3\theta}$$

$$\tan(\theta + 28) = \tan(3\theta - 45)$$

so $\theta + 28 = 3\theta - 45$

$2\theta = 73 \quad \therefore \boxed{\theta = 36.5^\circ} //$

Remember, tan graphs are cyclical (every 180°).

So to find a second solution,

$$\theta + 28 + 180 = 3\theta - 45$$

$2\theta = 253$

$\therefore \boxed{\theta = 126.5^\circ}$

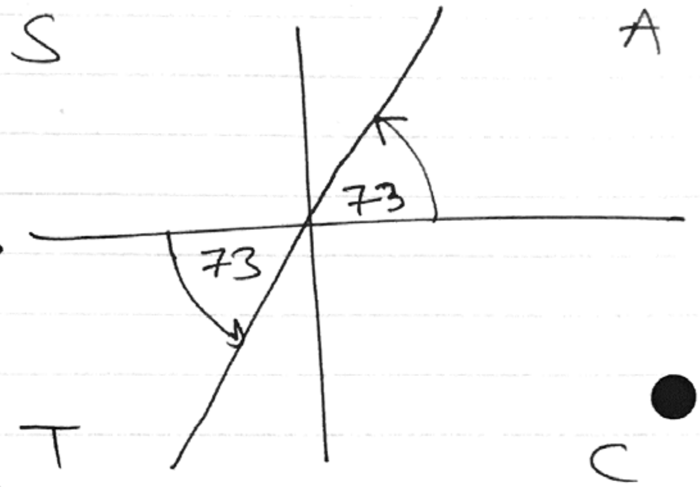
an alternative method to obtain the second solution via CAST can be seen below:

(from above) $2\theta = 73^\circ$

$\therefore 2\theta = 73, 180+73$

$2\theta = 73, 253$

$\therefore \boxed{\theta = 36.5^\circ, 126.5^\circ}$



(Q7a) $y = \dfrac{\ln(x^2+1)}{x^2+1}$ BY QUOTIENT RULE

$\dfrac{dy}{dx} = \dfrac{vu' - uv'}{v^2}$

$u = \ln(x^2+1)$

$u' = \dfrac{2x}{x^2+1}$

$v = x^2+1$

$v' = 2x$

$= \dfrac{2x - 2x\ln(x^2+1)}{(x^2+1)^2}$

$$\boxed{= \frac{2x(1-\ln(x^2+1))}{(x^2+1)^2}}$$

b) $\dfrac{2x(1-\ln(x^2+1))}{(x^2+1)^2} = \dfrac{dy}{dx} = 0$ at stationary points.

$\therefore \; 2x(1-\ln(x^2+1)) = 0.$

$x = 0 \longrightarrow y = \dfrac{\ln(1)}{0+1} = 0$

so one stationary point is $\boxed{(0, 0)}$.

$1 - \ln(x^2+1) = 0$

$\ln(x^2+1) = 1$

$e^1 = x^2+1$

$e - 1 = x^2$

$\therefore \; x = \pm\sqrt{e-1}$

at $x = \sqrt{e-1}$, $y = \dfrac{\ln(e-1+1)}{e-1+1} = \dfrac{1}{e}$

at $x = -\sqrt{e-1}$, $y = \dfrac{\ln(e)}{e} = \dfrac{1}{e}$

So the other stationary points are:

$\boxed{\left(\sqrt{e-1}, \dfrac{1}{e}\right)}$ and $\boxed{\left(-\sqrt{e-1}, \dfrac{1}{e}\right)}$

Q8a) $\sec\theta = \frac{1}{\cos\theta} = (\cos\theta)^{-1}$

$\frac{d}{d\theta}(\sec\theta) = \frac{d}{d\theta}((\cos\theta)^{-1})$

$= -(\cos\theta)^{-2} \times -\sin\theta$

$= \frac{\sin\theta}{\cos^2\theta} = \frac{\sin\theta}{\cos\theta} \times \frac{1}{\cos\theta}$

$= \tan\theta \times \sec\theta$

$= \sec\theta\tan\theta$

b) $x = e^{\sec y}$

$\frac{dx}{dy} = \sec y \tan y \, e^{\sec y}$

$\therefore \frac{dy}{dx} = \frac{1}{e^{\sec y}\sec y\tan y} = \frac{1}{x\sec y\tan y}.$

$x = e^{\sec y}$

$\therefore \ln x = \sec y$

$1 + \tan^2 y = \sec^2 y$

$\therefore 1 + \tan^2 y = (\ln x)^2$

$\therefore \tan^2 y = (\ln x)^2 - 1$

$\Rightarrow \tan y = \sqrt{(\ln x)^2 - 1}$

$\therefore \theta = \frac{\pi}{2} + 1.107 = \boxed{2.68}$

ci) $N(\theta) = \dfrac{30}{5 + 2(5\sin^2(2\theta - 1.107^c))}$

$N(\theta)$ is max when denominator is minimum.

$\sin^2(2\theta - 1.107^c) \geq 0$ for all $\theta \in \mathbb{R}$.

so the lowest value $\sin^2(2\theta - 1.107^c)$ can take is 0. So this is when $N(\theta)$ is max.

$\therefore [N(\theta)]_{max} = \dfrac{30}{5 + 2(5)(0)} = \dfrac{30}{5} = \boxed{6}$

ii) remember $(\sin^2(2\theta - 1.107^c) = 0)$ at max.

$\therefore \sin(2\theta - 1.107) = 0$

$2\theta - 1.107 = \sin^{-1}(0) = 0, \pi, 2\pi, 3\pi.$

range: $\boxed{-1.107 < 2\theta - 1.107 \leq 11.459}$

so $2\theta - 1.107^c = 3\pi$ (4π is not in this range)

(3π is the biggest solution in the boxed interval)

$2\theta = 3\pi + 1.107$

$\therefore \theta = \dfrac{3\pi + 1.107}{2} = \boxed{5.27}$