

## C3 June 2016 Model Answers

by Kprime2 TSR

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blank1. The functions  $f$  and  $g$  are defined by

$$f: x \rightarrow 7x - 1, \quad x \in \mathbb{R}$$

$$g: x \rightarrow \frac{4}{x-2}, \quad x \neq 2, x \in \mathbb{R}$$

(a) Solve the equation  $fg(x) = x$ 

(4)

(b) Hence, or otherwise, find the largest value of  $a$  such that  $g(a) = f^{-1}(a)$ 

(1)

$$\begin{aligned} 1(a). \quad fg(x) &= 7 \left( \frac{4}{x-2} \right) - 1 \\ &= \frac{28}{x-2} - 1 \end{aligned}$$

$$fg(x) = x \Rightarrow \frac{28}{x-2} - 1 = x$$

$$\textcircled{x(x-2)} \Rightarrow 28 - x + 2 = x(x-2)$$

$$\therefore x^2 - x - 30 = 0$$

$$(x-6)(x+5) = 0$$

$$\Rightarrow \underline{\underline{x=6}} \quad \underline{\underline{x=-5}}$$

$$\left. \begin{aligned} (b) \quad fg(x) &= a \\ \Rightarrow f^{-1}[fg(x)] &= f^{-1}(a) \\ \therefore g(x) &= f^{-1}(a) \end{aligned} \right\} \underline{\underline{x=6}}$$



2.

$$y = \frac{4x}{x^2 + 5}$$

(a) Find  $\frac{dy}{dx}$ , writing your answer as a single fraction in its simplest form. (4)

(b) Hence find the set of values of  $x$  for which  $\frac{dy}{dx} < 0$  (3)

$$2(a) \quad y = \frac{4x}{x^2 + 5}$$

$$\frac{dy}{dx} = \frac{4(x^2 + 5) - 4x(2x)}{(x^2 + 5)^2}$$

$$= \frac{4x^2 + 20 - 8x^2}{(x^2 + 5)^2}$$

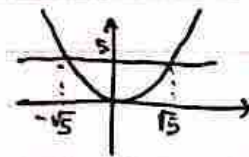
$$\therefore \frac{dy}{dx} = \frac{20 - 4x^2}{(x^2 + 5)^2}$$

$$(b) \quad 20 - 4x^2 < 0$$

$$\Rightarrow x^2 > 5$$

$$\underline{\underline{x > \sqrt{5}}}$$

$$\underline{\underline{x < -\sqrt{5}}}$$



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3. (a) Express  $2 \cos \theta - \sin \theta$  in the form  $R \cos(\theta + \alpha)$ , where  $R$  and  $\alpha$  are constants,  $R > 0$  and  $0 < \alpha < 90^\circ$ . Give the exact value of  $R$  and give the value of  $\alpha$  to 2 decimal places.

(3)

- (b) Hence solve, for  $0 \leq \theta < 360^\circ$ ,

$$\frac{2}{2 \cos \theta - \sin \theta - 1} = 15$$

Give your answers to one decimal place.

(5)

- (c) Use your solutions to parts (a) and (b) to deduce the smallest positive value of  $\theta$  for which

$$\frac{2}{2 \cos \theta + \sin \theta - 1} = 15$$

Give your answer to one decimal place.

(2)

$$\begin{aligned} 3(a) \quad R \cos(\theta + \alpha) &= R \cos \theta \cos \alpha - R \sin \theta \sin \alpha \\ &\equiv 2 \cos \theta - \sin \theta \end{aligned}$$

$$\Rightarrow R \cos \alpha = 2 \quad \& \quad R \sin \alpha = 1$$

$$R^2 = 2^2 + 1^2 = 5 \quad \Rightarrow \quad R = \sqrt{5}$$

$$\tan \alpha = \frac{1}{2} \quad \Rightarrow \quad \alpha = \arctan\left(\frac{1}{2}\right) = \underline{26.57^\circ} \text{ (2dp)}$$

$$\therefore \underline{2 \cos \theta - \sin \theta = \sqrt{5} \cos(\theta + 26.57^\circ)}$$



Question 3 continued

$$(b) \frac{2}{2\cos\theta - \sin\theta - 1} = 15$$

$$\therefore 2 = 30\cos\theta - 15\sin\theta - 15$$

$$17 = 15(2\cos\theta - \sin\theta)$$

$$\therefore \sqrt{5} \cos(\theta + 26.57^\circ) = \frac{17}{15}$$

$$0 \leq \theta < 360^\circ \quad \therefore \cos(\theta + 26.57^\circ) = \frac{17}{15\sqrt{5}}$$

$$26.57 \leq (\theta + 26.57^\circ) < 386.57^\circ$$

$\theta + 26.565... = \arccos\left(\frac{17}{15\sqrt{5}}\right) = 59.54...$

~~$\theta + 26.565... = 180 - 59.54... = 120.45...$~~

$$\theta + 26.565... = 360 - 59.54... = 300.453...$$

$$\therefore \underline{\underline{\theta = 33.0^\circ}}$$

$$\theta = \underline{\underline{273.9^\circ}}$$



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Question 3 continued

$$(c) 2 \cos \theta \pm \sin \theta \equiv R \cos(\theta \mp \alpha)$$

(c)

$$\therefore 2 \cos \theta + \sin \theta = \sqrt{5} \cos(\theta - 26.57)$$

(d) from part (b):

$$\cos(\theta - 26.57) = \frac{17}{15\sqrt{5}}$$

$$\begin{aligned} \therefore \theta - 26.565\dots &= \arccos\left(\frac{17}{15\sqrt{5}}\right) \\ &= 59.546\dots \end{aligned}$$

$$\therefore \theta = 59.546\dots + 26.565\dots$$

$$\Rightarrow \theta = \underline{\underline{86.1^\circ}} \text{ (1 d.p.)}$$

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4.

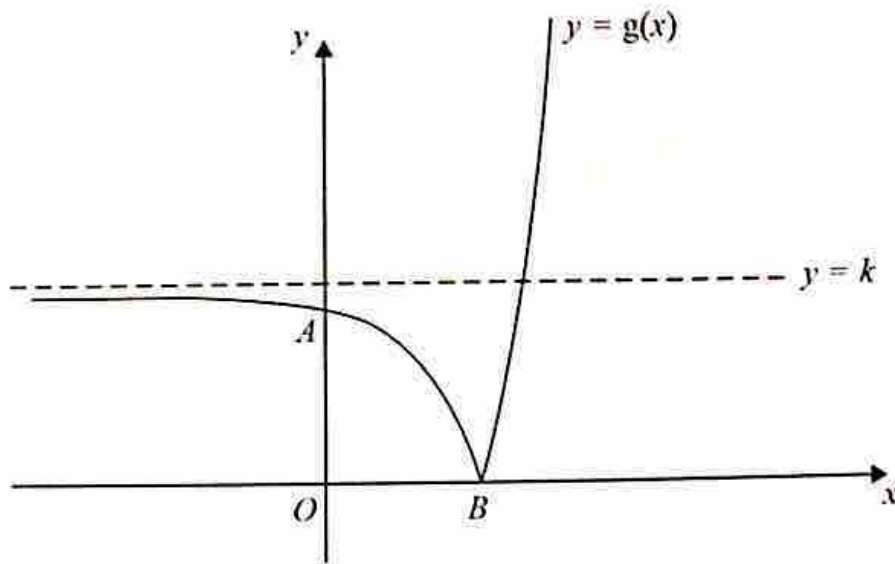


Figure 1

Figure 1 shows a sketch of part of the curve with equation  $y = g(x)$ , where

$$g(x) = |4e^{2x} - 25|, \quad x \in \mathbb{R}$$

The curve cuts the  $y$ -axis at the point  $A$  and meets the  $x$ -axis at the point  $B$ . The curve has an asymptote  $y = k$ , where  $k$  is a constant, as shown in Figure 1

(a) Find, giving each answer in its simplest form,

- (i) the  $y$  coordinate of the point  $A$ ,
- (ii) the exact  $x$  coordinate of the point  $B$ ,
- (iii) the value of the constant  $k$ .

(5)

The equation  $g(x) = 2x + 43$  has a positive root at  $x = \alpha$

(b) Show that  $\alpha$  is a solution of  $x = \frac{1}{2} \ln \left( \frac{1}{2}x + 17 \right)$

(2)

The iteration formula

$$x_{n+1} = \frac{1}{2} \ln \left( \frac{1}{2}x_n + 17 \right)$$

can be used to find an approximation for  $\alpha$

(c) Taking  $x_0 = 1.4$  find the values of  $x_1$  and  $x_2$   
Give each answer to 4 decimal places.

(2)

(d) By choosing a suitable interval, show that  $\alpha = 1.437$  to 3 decimal places.

(2)



Question 4 continued

$$4(a)(i) \quad g(0) = -21$$

$$\therefore \underline{y_A = 21}$$

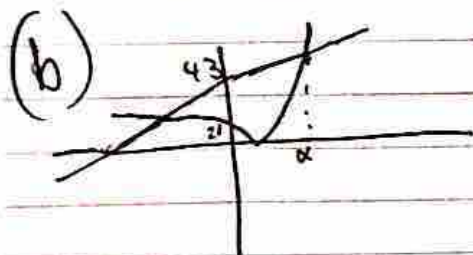
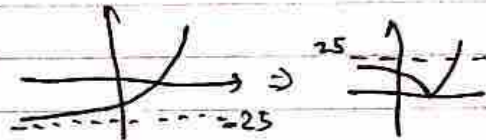
$$(ii) \quad g(x) = 0 \Rightarrow 4e^{2x} - 25 = 0$$

$$e^{2x} = \frac{25}{4}$$

$$e^x = \frac{5}{2}$$

$$\underline{x = \ln\left(\frac{5}{2}\right)}$$

$$(iii) \quad \underline{k = 25}$$



$$\Rightarrow 4e^{2x} - 25 = 2x + 43$$

$$\therefore 4e^{2x} = 2x + 68$$

$$\therefore e^{2x} = \frac{2x + 68}{4} = \frac{1}{2}x + 17$$

$$\therefore e^x = \sqrt{\frac{1}{2}x + 17}$$

$$\Rightarrow x = \ln\left(\sqrt{\frac{1}{2}x + 17}\right) = \ln\left[\left(\frac{1}{2}x + 17\right)^{\frac{1}{2}}\right]$$

$$\therefore \underline{x = \frac{1}{2} \ln\left(\frac{1}{2}x + 17\right)} \quad \text{as required.}$$



Question 4 continued

$$(c) \quad x_1 = \underline{\underline{1.4368}} \quad (4dp)$$

$$x_2 = \underline{\underline{1.4373}} \quad (4dp)$$

(d) We are solving  $4e^{2x} - 23 = 2x + 43$

$$\therefore 4e^{2x} - 2x - 68 = 0$$

$$\text{Let } f(x) = 4e^{2x} - 2x - 68$$

$$\text{Solve } f(x) = 0$$

$$f(1.4365) = -0.11296\dots$$

$$f(1.4375) = 0.026696\dots$$

There is a sign change in the interval  $[1.4365, 1.4375]$

$$\Rightarrow x \in [1.4365, 1.4375]$$

$$\therefore x \approx \underline{\underline{1.437}} \quad (3dp)$$

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5. (i) Find, using calculus, the  $x$  coordinate of the turning point of the curve with equation

$$y = e^{3x} \cos 4x, \quad \frac{\pi}{4} \leq x < \frac{\pi}{2}$$

Give your answer to 4 decimal places.

(5)

- (ii) Given  $x = \sin^2 2y$ ,  $0 < y < \frac{\pi}{4}$ , find  $\frac{dy}{dx}$  as a function of  $y$ .

Write your answer in the form

$$\frac{dy}{dx} = p \operatorname{cosec}(qy), \quad 0 < y < \frac{\pi}{4}$$

where  $p$  and  $q$  are constants to be determined.

(5)

$$5(i) \quad y = e^{3x} \cos 4x \quad \begin{array}{l} u = e^{3x} \\ v = \cos 4x \end{array}$$

$$\therefore \frac{dy}{dx} = (\cos 4x)(3e^{3x}) + e^{3x}(-4 \sin 4x)$$

$$= 3e^{3x} \cos 4x - 4e^{3x} \sin 4x$$

$$\frac{dy}{dx} = 0 \Rightarrow e^{3x}(3 \cos 4x - 4 \sin 4x) = 0$$

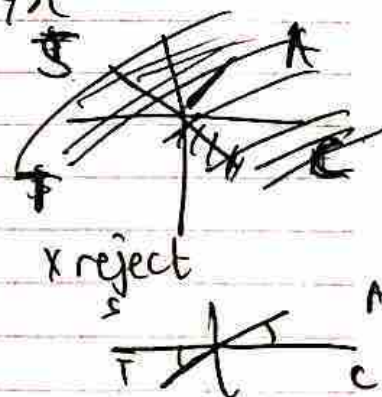
$$e^{3x} \neq 0$$

$$\therefore 3 \cos 4x - 4 \sin 4x = 0$$

$$\therefore 4 \sin 4x = 3 \cos 4x$$

$$\Rightarrow \tan 4x = \frac{3}{4}$$

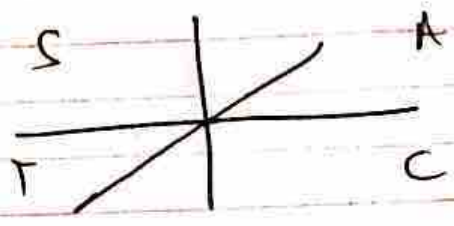
$$\pi \leq 4x < 2\pi \quad | \quad 4x = \arctan\left(\frac{3}{4}\right) = 0.64 \quad \times \text{reject}$$



Question 5 continued

~~4x = 2\pi + 0.6435 = 6.9266 \dots~~

4x = \pi + \arctan(\frac{3}{4})



\Rightarrow x = \frac{\pi}{4} + \frac{1}{4} \arctan(\frac{3}{4})

\therefore x = 0.9463 (4de)

(ii) x = \sin^2 2y

\frac{dx}{dy} = 2 \sin 2y \times 2 \cos 2y = 4 \sin 2y \cos 2y

\therefore \frac{dy}{dx} = \frac{1}{4 \sin 2y \cos 2y}

\therefore \frac{dy}{dx} = \frac{1}{2(2 \sin 2y \cos 2y)}

\therefore \frac{dy}{dx} = \frac{1}{2 \sin 4y} = \frac{1}{2} \operatorname{cosec} 4y

p = \frac{1}{2}      q = 4

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6.  $f(x) = \frac{x^4 + x^3 - 3x^2 + 7x - 6}{x^2 + x - 6}, \quad x > 2, x \in \mathbb{R}$

(a) Given that

$$\text{LHS } \frac{x^4 + x^3 - 3x^2 + 7x - 6}{x^2 + x - 6} \equiv x^2 + A + \frac{B}{x-2} \quad \text{RHS}$$

find the values of the constants  $A$  and  $B$ .

(4)

(b) Hence or otherwise, using calculus, find an equation of the normal to the curve with equation  $y = f(x)$  at the point where  $x = 3$

(5)

(a)

$$x > 2 :$$

$$x = 3 \Rightarrow \text{LHS} = 16$$

$$\text{RHS} = 9 + A + B$$

$$\therefore 9 + A + B = 16 \Rightarrow A + B = 7 \quad (1)$$

$$x = 4 \Rightarrow$$

$$\text{LHS} = 21$$

$$\text{RHS} = 16 + A + \frac{B}{2}$$

$$\therefore 16 + A + \frac{B}{2} = 21$$

$$\Rightarrow 2A + B = 10 \quad (2)$$

Solve simultaneously:  $A + A + B = 10$

$$\therefore A + 7 = 10$$

$$\Rightarrow A = 3$$

$$B = 4$$

$$\underline{\underline{A = 3}}$$

$$\underline{\underline{B = 4}}$$



Question 6 continued

$$(b) f(x) = x^2 + 3 + \frac{4}{x-2}$$

$$f(3) = 16$$

$$f'(x) = 2x - 4(x-2)^{-2}$$

$$f'(3) = 2$$

$$\therefore \text{gradient of normal} = -\frac{1}{2}$$

$$y - y_1 = m(x - x_1)$$

$$\therefore y - 16 = -\frac{1}{2}(x - 3)$$

$$32 - 2y = x - 3$$

$$y = \frac{35 - x}{2}$$



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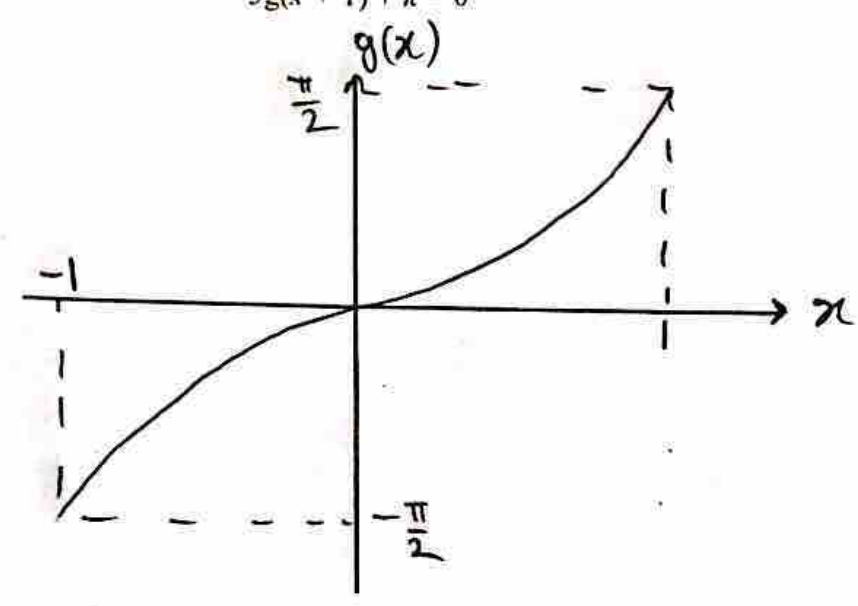


7. (a) For  $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ , sketch the graph of  $y = g(x)$  where

$$g(x) = \arcsin x \quad -1 \leq x \leq 1 \tag{2}$$

(b) Find the exact value of  $x$  for which

$$3g(x+1) + \pi = 0 \tag{3}$$



$$(b) \quad 3g(x+1) = -\pi$$

$$g(x+1) = -\frac{\pi}{3}$$

$$g^{-1}(g(x+1)) = g^{-1}\left(-\frac{\pi}{3}\right)$$

$$\therefore x+1 = g^{-1}\left(-\frac{\pi}{3}\right)$$

$$\therefore x+1 = \sin\left(-\frac{\pi}{3}\right)$$

$$x = \sin\left(-\frac{\pi}{3}\right) - 1$$

$$\Rightarrow x = \frac{-2 - \sqrt{3}}{2}$$

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8. (a) Prove that

$$2 \cot 2x + \tan x \equiv \cot x \quad x \neq \frac{n\pi}{2}, n \in \mathbb{Z}$$

(4)

(b) Hence, or otherwise, solve, for  $-\pi \leq x < \pi$ ,

$$6 \cot 2x + 3 \tan x = \operatorname{cosec}^2 x - 2$$

Give your answers to 3 decimal places.

(Solutions based entirely on graphical or numerical methods are not acceptable.)

(6)

8(a)

$$\text{LHS} = 2 \cot 2x + \tan x = \frac{2 \cos 2x}{\sin 2x} + \frac{\sin x}{\cos x}$$

$$= \frac{2}{\tan 2x} + \tan x$$

$$= \frac{2}{\frac{2 \tan x}{1 - \tan^2 x}} + \tan x$$

$$= \frac{1 - \tan^2 x}{\tan x} + \tan x$$

$$= \frac{1 - \tan^2 x}{\tan x} + \frac{\tan^2 x}{\tan x}$$

$$= \frac{1 - \tan^2 x + \tan^2 x}{\tan x} = \frac{1}{\tan x} = \cot x = \text{RHS}$$

as required.



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Question 8 continued

$$(b) 6 \cot 2x + 3 \tan x = \operatorname{cosec}^2 x - 2$$

$$\therefore 3(2 \cot 2x + \tan x) = \operatorname{cosec}^2 x - 2$$

$$\therefore 3 \cot x = \operatorname{cosec}^2 x - 2$$

$$s^2 + c^2 = 1$$

$$\frac{s^2}{c^2} =$$

$$1 + \cot^2 = \operatorname{cosec}^2$$

$$3 \cot x = \cot^2 x - 1$$

$$\therefore \cot^2 x - 3 \cot x - 1 = 0$$

$$\cot x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

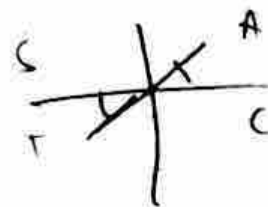
$$\therefore \cot x = \frac{3 \pm \sqrt{13}}{2}$$

$$\therefore \tan x = \frac{2}{3 \pm \sqrt{13}}$$

$$\tan x = \frac{2}{3 + \sqrt{13}}$$

$$x = \arctan\left(\frac{2}{3 + \sqrt{13}}\right) = 0.294$$

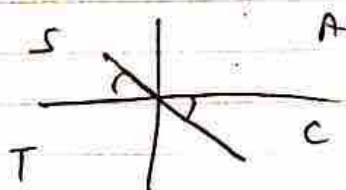
$$x = -\pi + 0.294001\dots = -2.848$$



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Question 8 continued

$$\tan x = \frac{2}{3 - \sqrt{13}}$$



$$x = \arctan\left(\frac{2}{3 - \sqrt{13}}\right) = -1.277$$

$$x = \pi - 1.277 \dots = 1.865$$

$$\therefore x = \underline{\underline{-2.848^{\circ}}}$$

$$x = \underline{\underline{0.294^{\circ}}}$$

$$x = \underline{\underline{-1.277^{\circ}}}$$

$$x = \underline{\underline{1.865^{\circ}}}$$

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9. The amount of an antibiotic in the bloodstream, from a given dose, is modelled by the formula

$$x = De^{-0.2t}$$

where  $x$  is the amount of the antibiotic in the bloodstream in milligrams,  $D$  is the dose given in milligrams and  $t$  is the time in hours after the antibiotic has been given.

A first dose of 15 mg of the antibiotic is given.

- (a) Use the model to find the amount of the antibiotic in the bloodstream 4 hours after the dose is given. Give your answer in mg to 3 decimal places. (2)

A second dose of 15 mg is given 5 hours after the first dose has been given. Using the same model for the second dose,

- (b) show that the **total** amount of the antibiotic in the bloodstream 2 hours after the second dose is given is 13.754 mg to 3 decimal places. (2)

No more doses of the antibiotic are given. At time  $T$  hours after the second dose is given, the total amount of the antibiotic in the bloodstream is 7.5 mg.

- (c) Show that  $T = a \ln\left(b + \frac{b}{c}\right)$ , where  $a$  and  $b$  are integers to be determined. (4)

$$9(a) \quad D=15 \quad t=4$$

$$x = 15e^{-0.2 \times 4} = 6.740 \text{ (3dp)}$$

(b) 7 hours since first dose:

$$x = 15e^{-0.2 \times 7} = 3.698 \dots$$

2 hours since second dose:

$$x = 15e^{-0.2 \times 2} = 10.054 \dots$$

$$\therefore \sum x = 15e^{-0.2 \times 7} + 15e^{-0.2 \times 2}$$

$$= 13.7537 \dots = 13.754 \text{ (3dp)}$$

as required.



Question 9 continued

5+T hours since first dose

T hours since second dose

$$\therefore \Sigma x = 15e^{-0.2(5+T)} + 15e^{-0.2T} = 7.5$$

$$\times e^{0.2T}$$

$$\Rightarrow 15e^{-0.2T-1-0.2T} + 15 = 7.5e^{0.2T}$$

$$\therefore 15e^{-1} + 15 = 7.5e^{0.2T}$$

$$\frac{\circ}{\circ} 7.5 \Rightarrow \frac{2}{e} + 2 = e^{0.2T}$$

$$\ln(e^{0.2T}) = \ln\left(\frac{2}{e} + 2\right)$$

$$\therefore 0.2T = \ln\left(\frac{2}{e} + 2\right)$$

$$\Rightarrow T = 5 \ln\left(2 + \frac{2}{e}\right)$$

$$\underline{\underline{a=5}} \quad \underline{\underline{b=2}}$$