C3 June 13 (original ull)

1.
$$g(x) = \frac{6x+12}{x^2+3x+2} - 2, \quad x \geqslant 0$$

(a) Show that $g(x) = \dfrac{4-2x}{x+1}$, $x \geqslant 0$

(3)

(b)

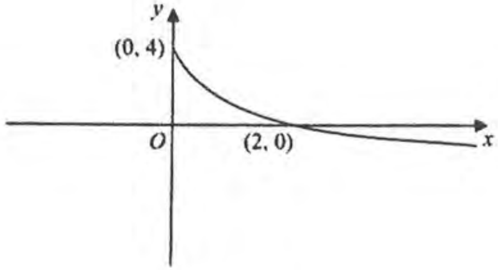

Figure 1

Figure 1 shows a sketch of the curve with equation $y = g(x)$, $x \geqslant 0$

The curve meets the y-axis at $(0, 4)$ and crosses the x-axis at $(2, 0)$.

On separate diagrams sketch the graph with equation

(i) $y = 2g(2x)$,

(ii) $y = g^{-1}(x)$.

Show on each sketch the coordinates of each point at which the graph meets or crosses the axes.

(5)

$$g(x) = \frac{6\cancel{(x+2)}}{(x+1)\cancel{(x+2)}} - \frac{2(x+1)\cancel{(x+2)}}{(x+1)\cancel{(x+2)}}$$

$$= \frac{6-2x-2}{x+1} = \frac{4-2x}{x+1} \quad \#$$

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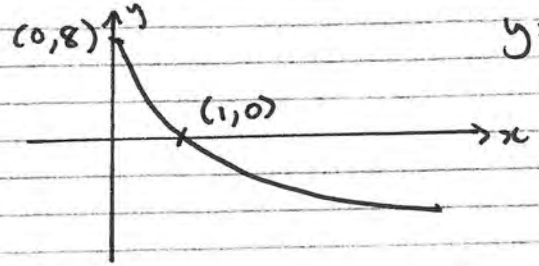
Question 1 continued

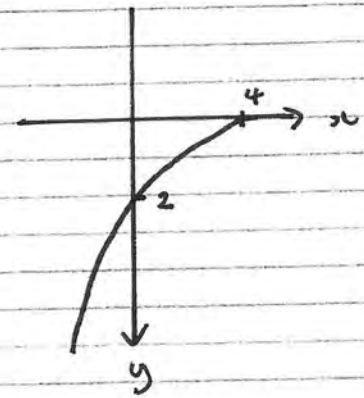
b) $\uparrow 2g(2x) \downarrow$ $\rightarrow 2 \leftarrow$



$y = 2g(2x)$

ii) $g^{-1}(x)$

↷90°



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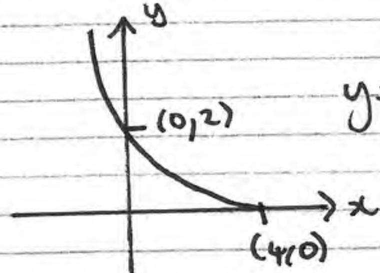

$y = g^{-1}(x)$

2. Given that $\tan 40° = p$, find in terms of p

(a) $\cot 40°$ **(1)**

(b) $\sec 40°$ **(2)**

(c) $\tan 85°$ **(2)**

a) $\cot 40 = \frac{1}{\tan 40} = \frac{1}{p}$

b) $\sec^2 40 = 1 + \tan^2 40$

$\therefore \sec 40 = \sqrt{1+p^2}$

c) $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$\tan(45+40) = \frac{\tan 45 + \tan 40}{1 - \tan 45 \tan 40}$

$\therefore \tan 85 = \frac{1+p}{1-p}$

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3.

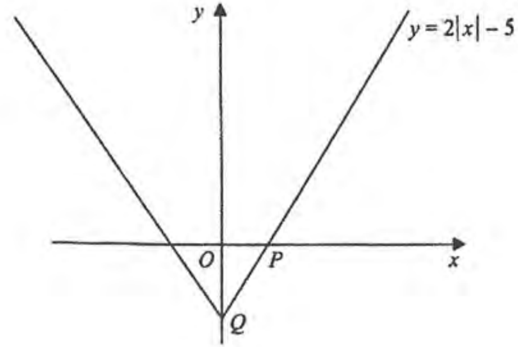

Figure 2

Figure 2 shows a sketch of the graph with equation $y = 2|x| - 5$.

The graph intersects the positive x-axis at the point P and the negative y-axis at the point Q.

(a) State the coordinates of P and the coordinates of Q. **(2)**

(b) Solve the equation

$$2|x| - 5 = 3 - x$$

(3)

a) $P(2.5, 0)$ $Q(0, -5)$

b) $2|x| = 8 - x \Rightarrow |x| = 4 - \frac{1}{2}x$

$\therefore x = 4 - \frac{1}{2}x$ $\qquad x = \frac{1}{2}x - 4$

$\frac{3}{2}x = 4$ $\qquad \frac{1}{2}x = -4$

$x = \frac{8}{3}$ $\qquad x = -8$

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4. (a) On the same diagram, sketch and clearly label the graphs with equations

$$y = e^x \quad \text{and} \quad y = 10 - x$$

Show on your sketch the coordinates of each point at which the graphs cut the axes. **(3)**

(b) Explain why the equation $e^x - 10 + x = 0$ has only one solution. **(1)**

(c) Show that the solution of the equation

$$e^x - 10 + x = 0$$

lies between $x = 2$ and $x = 3$ **(2)**

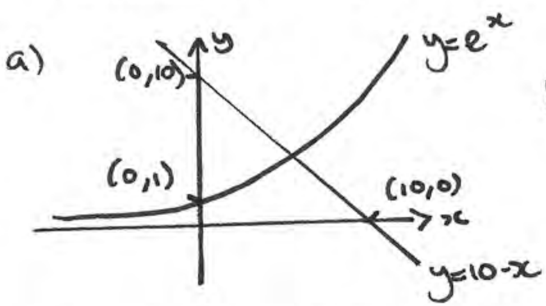
(d) Use the iterative formula

$$x_{n+1} = \ln(10 - x_n), \quad x_1 = 2$$

to calculate the values of x_2, x_3 and x_4.

Give your answers to 4 decimal places. **(3)**

a)



b) One point of intersection.

c) $f(2) = -0.6 < 0$ ∴ by sign change rule
$f(3) = 13 > 0$ Solution lies between 2 and 3.

d) $x_1 = 2$
$x_2 = 2.0794$
$x_3 = 2.0695$
$x_4 = 2.0707$

5. (i) (a) Show that $\dfrac{d}{dx}\left(x^{\frac{1}{2}} \ln x\right) = \dfrac{\ln x}{2\sqrt{x}} + \dfrac{1}{\sqrt{x}}$ **(3)**

The curve with equation $y = x^{\frac{1}{2}} \ln x$, $x > 0$ has one turning point at the point P.

(b) Find the exact coordinates of P. Give your answer in its simplest form. **(4)**

(ii) A curve C has equation $y = \dfrac{x-k}{x+k}$, where k is a positive constant.

Find $\dfrac{dy}{dx}$, and show that C has no turning points. **(4)**

a) $u = x^{\frac{1}{2}}$ $\quad v = \ln x$
$u' = \frac{1}{2}x^{-\frac{1}{2}}$ $\quad v' = \frac{1}{x}$

$\therefore \dfrac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} \ln x + x^{\frac{1}{2}} \dfrac{1}{x}$

$= \dfrac{\ln x}{2\sqrt{x}} + \dfrac{1}{\sqrt{x}}$ #

b) TP when $\dfrac{dy}{dx} = 0$ $\therefore \dfrac{\ln x}{2\sqrt{x}} = -\dfrac{1}{\cancel{\sqrt{x}}}$

$\Rightarrow \ln x = -2 \Rightarrow x = e^{-2}$

$y = (e^{-2})^{\frac{1}{2}} \ln e^{-2} = e^{-1} \times -2 = -2e^{-1}$ $\quad (e^{-2}, -2e^{-1})$

(ii) $u = x - k$ $\quad v = x + k$ $\Rightarrow \dfrac{dy}{dx} = \dfrac{(x+k)-(x-k)}{(x+k)^2}$
$u' = 1$ $\quad v' = 1$

$\therefore \dfrac{dy}{dx} = \dfrac{2k}{(x+k)^2}$ $\quad \dfrac{dy}{dx} = 0$ if $k = 0$ but k is a positive constant $\therefore$ no TP!

6.

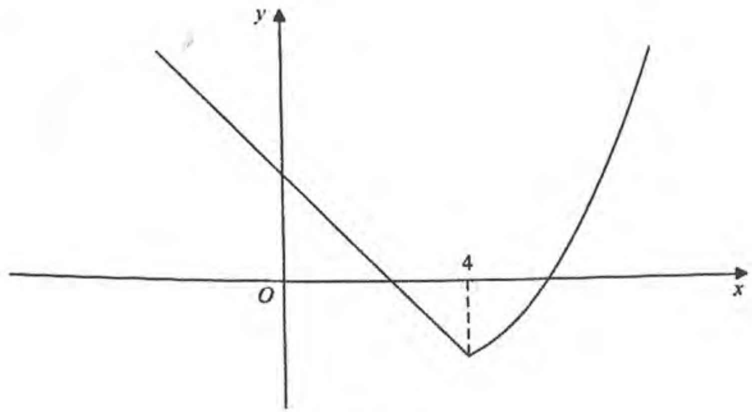


Figure 3

Figure 3 shows a sketch of the graph of $y = f(x)$ where

$$f(x) = \begin{cases} 5 - 2x, & x \leqslant 4 \\ e^{2x-8} - 4, & x > 4 \end{cases}$$

(a) State the range of $f(x)$. **(1)**

(b) Determine the exact value of $ff(0)$. **(2)**

(c) Solve $f(x) = 21$

Give each answer as an exact answer. **(5)**

(d) Explain why the function f does not have an inverse. **(1)**

a) $x = 4 \quad f(4) = -3 \quad \therefore$ range $y = f(x) \geqslant -3$.

b) $f(0) = 5 \quad \therefore ff(0) = f(5) = e^2 - 4$.

c) $5 - 2x = 21 \quad \therefore 2x = -16 \quad \therefore x = -8$

$e^{2x-8} - 4 = 21 \Rightarrow e^{2x-8} = 25 \Rightarrow 2x - 8 = \ln 25$

$\therefore x = 4 + \frac{1}{2}\ln 25$

d) f is not a one to one function.

7. (a) Prove that

$$\frac{\cos x}{1 - \sin x} + \frac{1 - \sin x}{\cos x} = 2\sec x, \qquad x \neq (2n+1)\frac{\pi}{2}, \quad n \in \mathbb{Z}$$

(4)

(b) Hence find, for $0 < x < \frac{\pi}{4}$, the exact solution of

$$\frac{\cos x}{1 - \sin x} + \frac{1 - \sin x}{\cos x} = 8\sin x$$

(4)

$\dfrac{\cos x(\cos x)}{(1-\sin x)(\cos x)} + \dfrac{(1-\sin x)(1-\sin x)}{(1-\sin x)(\cos x)}$

$= \dfrac{\cos^2 x + 1 - 2\sin x + \sin^2 x}{(1-\sin x)\cos x}$

$\sin^2 + \cos^2 = 1 \Rightarrow \dfrac{2 - 2\sin x}{(1-\sin x)(\cos x)} = \dfrac{2\cancel{(1-\sin x)}}{\cancel{(1-\sin x)}\cos x}$

$= \dfrac{2}{\cos x} = 2\sec x$ #

b) $2\sec x = 8\sin x \Rightarrow \dfrac{2}{\cos x} = 8\sin x$

$\Rightarrow 2 = 8\sin x\cos x \Rightarrow 2 = 4(2\sin x\cos x)$

$= \sin 2x = \frac{1}{2} \Rightarrow 2x = \sin^{-1}(\frac{1}{2})$

$\therefore 2x = \frac{\pi}{6}, \frac{5\pi}{6} \quad \therefore x = \frac{\pi}{12}, \cancel{\frac{5\pi}{12}}$

8\. (a) Express $9\cos\theta - 2\sin\theta$ in the form $R\cos(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

Give the exact value of R and give the value of α to 4 decimal places. **(3)**

(b) (i) State the maximum value of $9\cos\theta - 2\sin\theta$

(ii) Find the value of θ, for $0 < \theta < 2\pi$, at which this maximum occurs. **(3)**

Ruth models the height H above the ground of a passenger on a Ferris wheel by the equation

$$H = 10 - 9\cos\left(\frac{\pi t}{5}\right) + 2\sin\left(\frac{\pi t}{5}\right)$$

where H is measured in metres and t is the time in minutes after the wheel starts turning.

(c) Calculate the maximum value of H predicted by this model, and the value of t, when this maximum first occurs. Give your answers to 2 decimal places. **(4)**

(d) Determine the time for the Ferris wheel to complete two revolutions. **(2)**

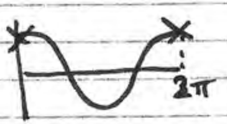
a) $R\cos(\theta+\alpha) = R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

$9\cos\theta \quad - \quad 2\sin\theta$

$\frac{R\sin\alpha = 2}{R\cos\alpha = 9}$ $\therefore \tan\alpha = \frac{2}{9}$ $\alpha = 0.2187$

$R^2 = 2^2 + 9^2 \therefore R = \sqrt{85}$

b) $\sqrt{85}\cos(\theta + 0.2187) \therefore \max = \sqrt{85}$.



max at $0, 2\pi$

$\therefore \theta = 2\pi - 0.2187$

$\theta = 6.065$

Question 8 continued

c) $\max H = 10 - (-\sqrt{85}) = \underline{19.22}$

this occurs when $\sqrt{85}\cos(\theta + 0.2187)$ is at its min value $-\sqrt{85}$. this is when $\theta + 0.2187 = \pi$

$\therefore \theta = \underline{2.92}$

$2.92 = \frac{\pi t}{5} \quad \therefore t = \underline{4.65}$

d) two revolutions $= 4\pi$

$\frac{\cancel{\pi} t}{5} = 4\cancel{\pi} \quad \underline{t = 20}$

9.

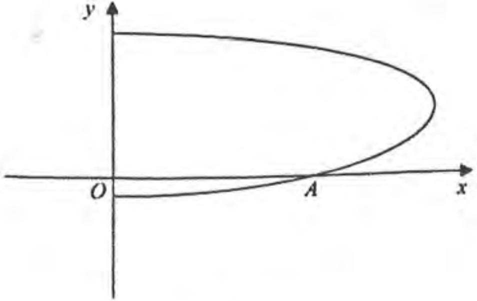


Figure 4

Figure 4 shows a sketch of the curve with equation $x = (9 + 16y - 2y^2)^{\frac{1}{2}}$.

The curve crosses the x-axis at the point A.

(a) State the coordinates of A. **(1)**

(b) Find an expression for $\frac{dx}{dy}$, in terms of y. **(3)**

(c) Find an equation of the tangent to the curve at A. **(4)**

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Question 9 continued

a) $A(3,0)$

b) $\frac{dx}{dy} = \frac{1}{2}(9+16y-2y^2)^{-\frac{1}{2}}(16-4y)$

$\therefore \frac{dx}{dy} = \frac{8-2y}{\sqrt{9+16y-2y^2}}$

c) $\frac{dy}{dx} = \frac{\sqrt{9+16y-2y^2}}{8-2y}$ $\therefore M_t\Big|_{y=0} = \frac{\sqrt{9}}{8} = \frac{3}{8}$

$(3,0)$ $y - 0 = \frac{3}{8}(x-3)$ $\therefore 8y = 3x - 9$