

C3 JUNE 11

1. Differentiate with respect to x

(a) $\ln(x^2 + 3x + 5)$

(2)

(b) $\frac{\cos x}{x^2}$

(3)

a)
$$\frac{2x+3}{x^2+3x+5}$$

b) $u = \cos x \quad v = x^2$
 $u' = -\sin x \quad v' = 2x$

$$\frac{vu' - uv'}{v^2}$$

$$\frac{dy}{dx} = \frac{x^2(-\sin x) - (\cos x)2x}{x^4}$$

$$= \frac{-x^2 \sin x - 2x \cos x}{x^4} \quad (\text{not required})$$

$$= \frac{-x \sin x - 2 \cos x}{x^3} \quad (\text{not required})$$

2.

$$f(x) = 2 \sin(x^2) + x - 2, \quad 0 \leq x < 2\pi$$

(a) Show that $f(x) = 0$ has a root α between $x = 0.75$ and $x = 0.85$

(2)

The equation $f(x) = 0$ can be written as $x = [\arcsin(1 - 0.5x)]^{\frac{1}{2}}$.

(b) Use the iterative formula

$$x_{n+1} = [\arcsin(1 - 0.5x_n)]^{\frac{1}{2}}, \quad x_0 = 0.8$$

to find the values of x_1 , x_2 and x_3 , giving your answers to 5 decimal places.

(3)

(c) Show that $\alpha = 0.80157$ is correct to 5 decimal places.

(3)

$$\begin{aligned} \text{a) } f(0.75) &= -0.18339 \dots < 0 && \text{Change of} \\ f(0.85) &= 0.1725 \dots > 0 && \text{Sign} \\ &&& \Rightarrow 0.75 < \alpha < 0.85 \end{aligned}$$

$$\begin{aligned} \text{b) } x_0 &= 0.8 \\ x_1 &= 0.80219 \\ x_2 &= 0.80133 \\ x_3 &= 0.80167 \end{aligned}$$

$$\begin{aligned} \text{c) } f(0.801575) &= 8 \times 10^{-6} > 0 \\ f(0.801565) &= -2.7 \times 10^{-5} < 0 \end{aligned}$$

change of sign $\Rightarrow \alpha \in (0.801565, 0.801575)$

$$\therefore \alpha = \underline{0.80157} \text{ (5dp)}$$

3.

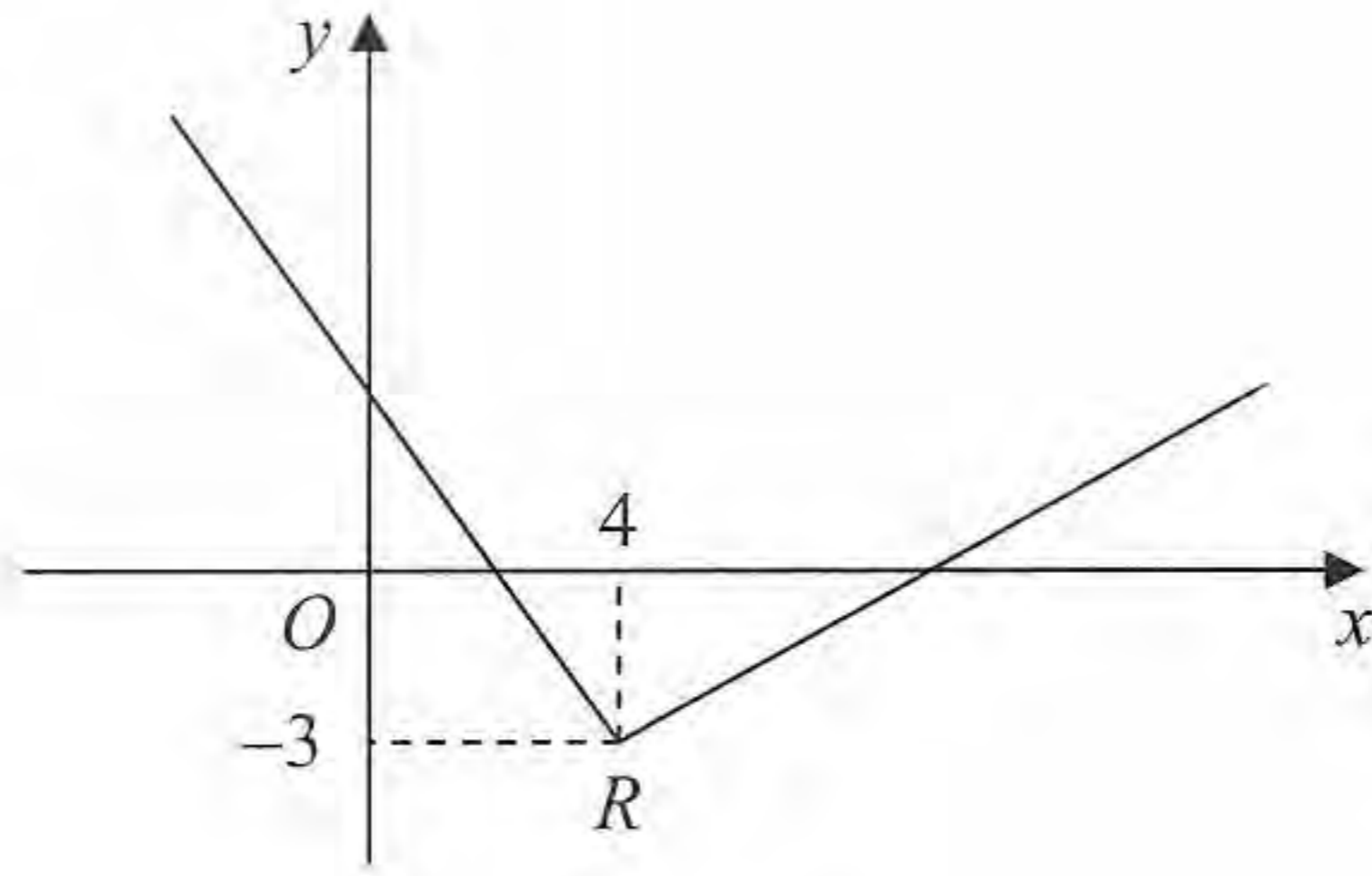


Figure 1

Figure 1 shows part of the graph of $y = f(x)$, $x \in \mathbb{R}$.

The graph consists of two line segments that meet at the point $R(4, -3)$, as shown in Figure 1.

Sketch, on separate diagrams, the graphs of

(a) $y = 2f(x+4)$, (3)

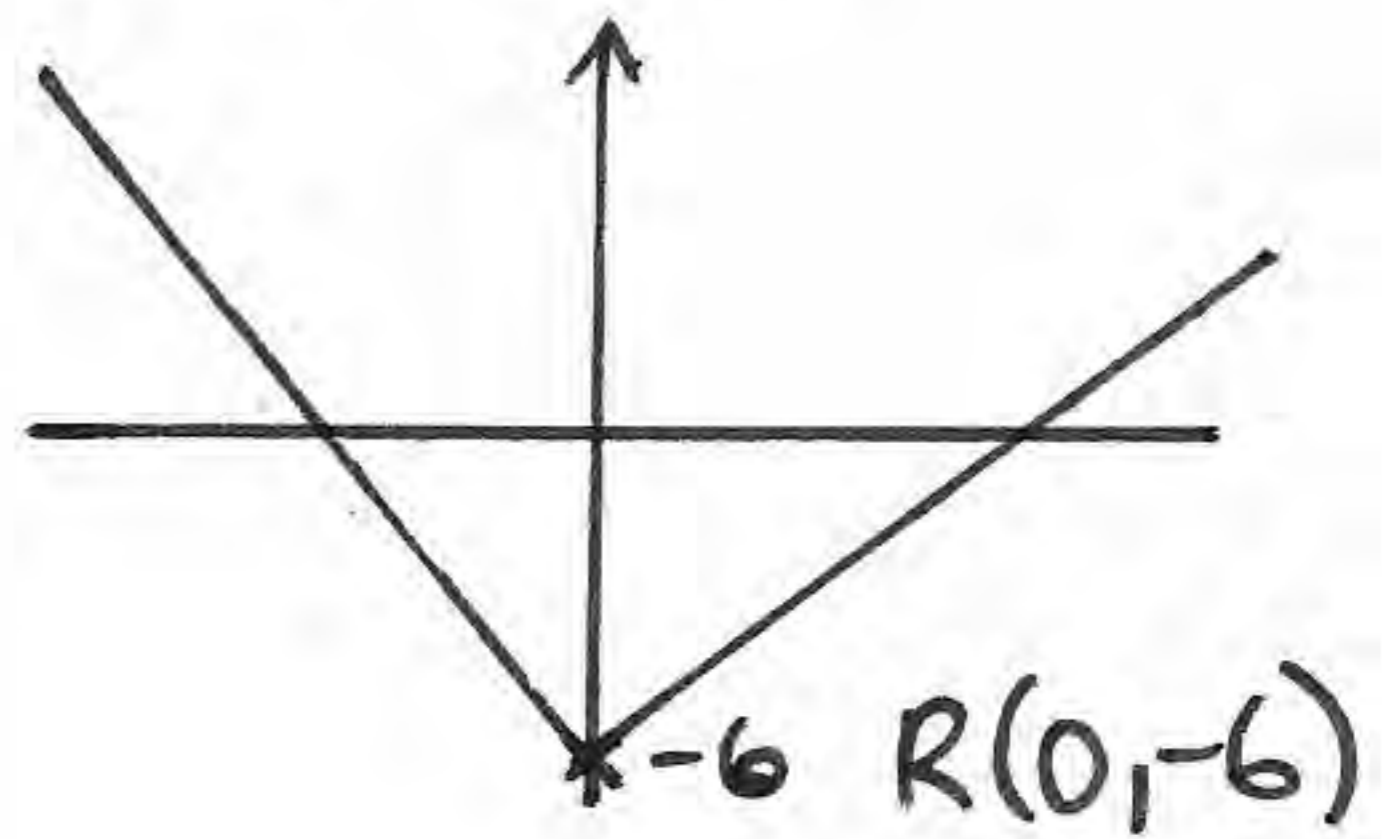
(b) $y = |f(-x)|$. (3)

On each diagram, show the coordinates of the point corresponding to R .

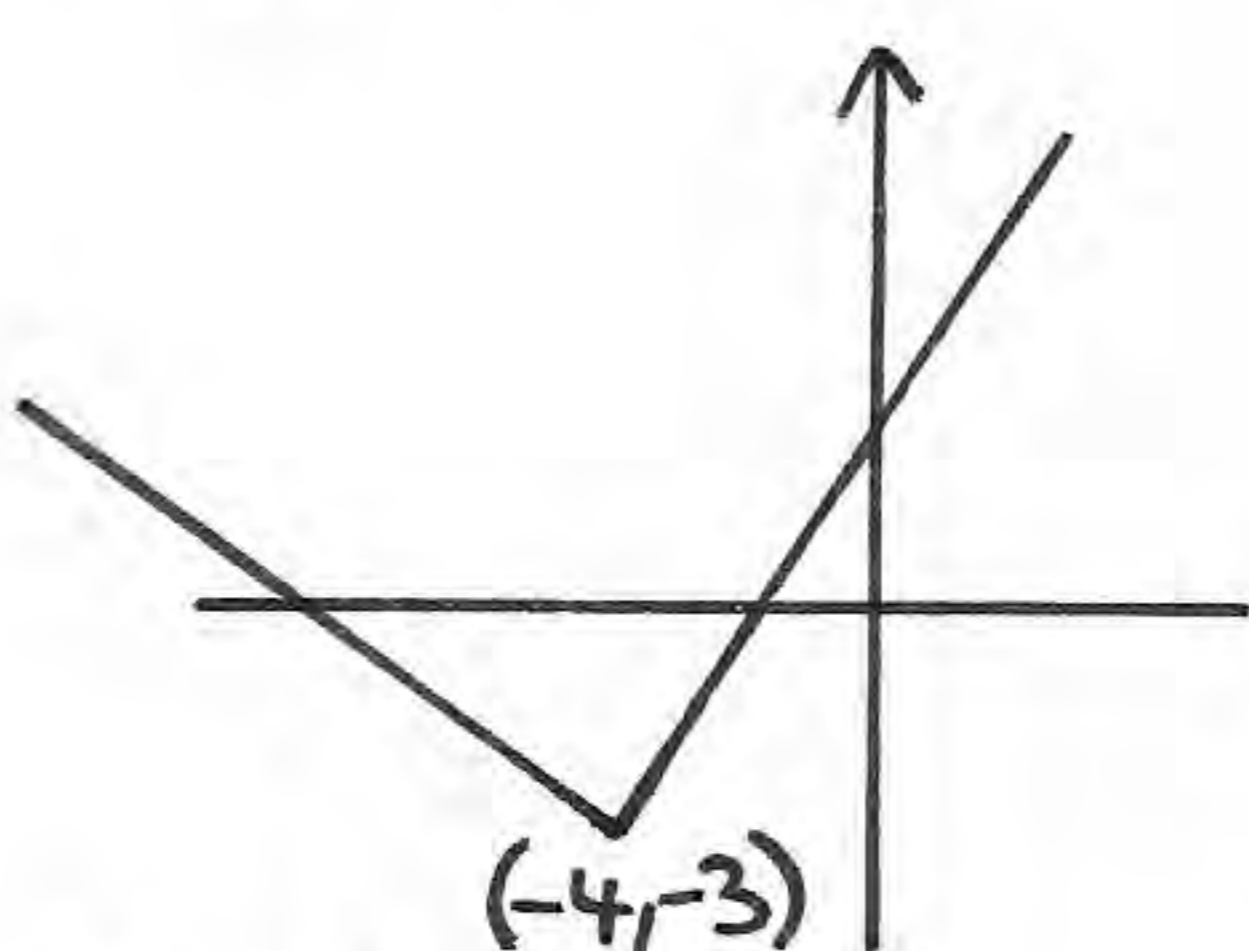
a) $y = 2f(x+4)$

\uparrow
 \downarrow

$\leftarrow 4$

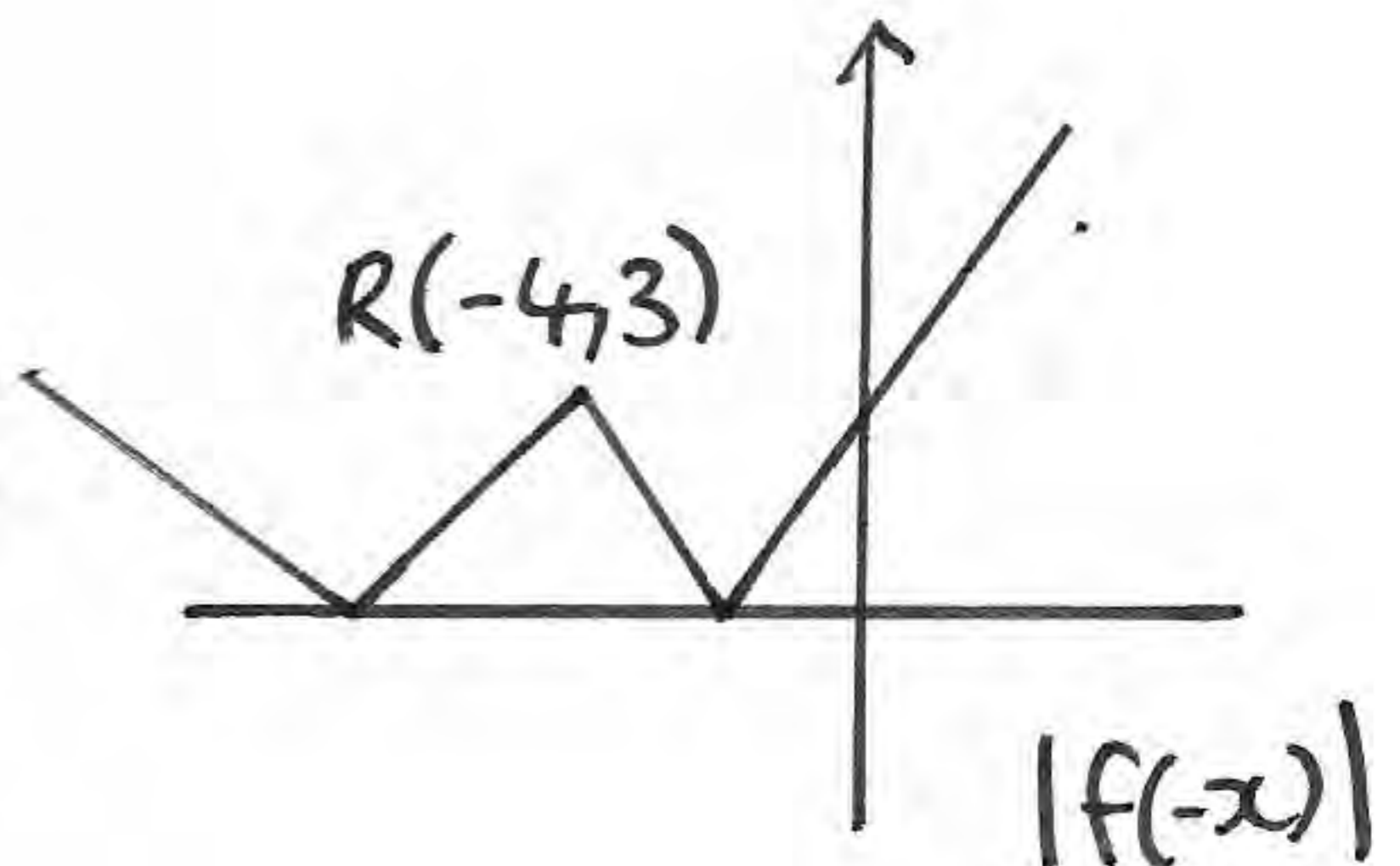


b) $y = |f(-x)|$



$f(-x)$ \leftarrow flip \rightarrow

\Rightarrow



4. The function f is defined by

$$f: x \mapsto 4 - \ln(x+2), \quad x \in \mathbb{R}, x \geq -1$$

(a) Find $f^{-1}(x)$.

(3)

(b) Find the domain of f^{-1} .

(1)

The function g is defined by

$$g: x \mapsto e^{x^2} - 2, \quad x \in \mathbb{R}$$

(c) Find $fg(x)$, giving your answer in its simplest form.

(3)

(d) Find the range of fg .

(1)

$$a) y = 4 - \ln(x+2) \Rightarrow x = 4 - \ln(y+2)$$

$$\Rightarrow \ln(y+2) = 4 - x \Rightarrow y+2 = e^{4-x} \Rightarrow y = -2 + e^{4-x} = f^{-1}(x)$$

$$b) \begin{aligned} f(-1) &= 4 - \ln(1) = 4 \\ f(0) &= 4 - \ln 2 < 4 \end{aligned} \Rightarrow \text{range } y \leq 4 \text{ for } f(x)$$
$$\therefore \text{domain } x \leq 4 \text{ for } f^{-1}(x)$$

$$c) fg(x) = f(e^{x^2} - 2) = 4 - \ln(e^{x^2} - 2 + 2)$$
$$= 4 - \ln(e^{x^2}) = 4 - x^2$$

$$d) \begin{array}{ccccccc} x & \longrightarrow & g & \longrightarrow & f & \longrightarrow & fg(x) \\ x \in \mathbb{R} & & \geq -1 & & & & \Rightarrow \text{range } y \leq 4 \end{array}$$

5. The mass, m grams, of a leaf t days after it has been picked from a tree is given by

$$m = p e^{-kt}$$

where k and p are positive constants.

When the leaf is picked from the tree, its mass is 7.5 grams and 4 days later its mass is 2.5 grams.

(a) Write down the value of p .

(1)

(b) Show that $k = \frac{1}{4} \ln 3$.

(4)

(c) Find the value of t when $\frac{dm}{dt} = -0.6 \ln 3$.

(6)

$$a) \quad t=0 \quad m=7.5 \quad \Rightarrow \quad 7.5 = p$$

$$b) \quad t=4 \quad m=2.5 \quad \Rightarrow \quad 2.5 = 7.5 e^{-4k}$$

$$\Rightarrow \frac{1}{3} = e^{-4k} \quad \Rightarrow \quad \ln\left(\frac{1}{3}\right) = -4k \quad \Rightarrow \quad -\ln 3 = -4k$$

$$\Rightarrow k = \frac{1}{4} \ln 3 \quad \#$$

$$c) \quad m = 7.5 e^{(-\frac{1}{4} \ln 3)t}$$

$$\frac{dm}{dt} = 7.5 \left(-\frac{1}{4} \ln 3\right) e^{(-\frac{1}{4} \ln 3)t} = -0.6 \ln 3$$

$$\Rightarrow \cancel{\frac{15}{8} \ln 3} e^{(-\frac{1}{4} \ln 3)t} = \cancel{-0.6 \ln 3}$$

$$\Rightarrow e^{(-\frac{1}{4} \ln 3)t} = \frac{8}{25} \quad \Rightarrow \quad \left(-\frac{1}{4} \ln 3\right)t = \ln\left(\frac{8}{25}\right)$$

$$\Rightarrow t = \frac{-4 \ln\left(\frac{8}{25}\right)}{\ln 3} \quad \Rightarrow \quad t = \underline{4.15} \text{ (3sf)}$$

6. (a) Prove that

$$\frac{1}{\sin 2\theta} - \frac{\cos 2\theta}{\sin 2\theta} = \tan \theta, \quad \theta \neq 90n^\circ, \quad n \in \mathbb{Z} \quad (4)$$

(b) Hence, or otherwise,

(i) show that $\tan 15^\circ = 2 - \sqrt{3}$, (3)

(ii) solve, for $0 < x < 360^\circ$,

$$\operatorname{cosec} 4x - \cot 4x = 1 \quad (5)$$

$$\begin{aligned} \text{a) } \frac{1 - \cos 2\theta}{\sin 2\theta} &= \frac{1 - (1 - 2\sin^2 \theta)}{2 \sin \theta \cos \theta} = \frac{2 \sin^2 \theta}{2 \sin \theta \cos \theta} \\ &= \frac{\sin \theta}{\cos \theta} = \tan \theta \quad \# \end{aligned}$$

$$\text{b) } \tan 15 = \frac{1}{\sin 30} - \frac{\cos 30}{\sin 30} = \frac{1}{\frac{1}{2}} - \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \frac{2 - \sqrt{3}}{1} \quad \#$$

$$\text{c) } \operatorname{cosec} 4x - \cot 4x = 1$$

$$\Rightarrow \frac{1}{\sin 4x} - \frac{\cos 4x}{\sin 4x} = 1 \Rightarrow \underline{\theta = 2x} \Rightarrow \tan 2x = 1$$

$$\Rightarrow 2x = \tan^{-1}(1) = 45, 225, 405, 585, \dots$$

$$x = \underline{22.5}, \underline{112.5}, \underline{202.5}, \underline{292.5}^\circ$$

7.

$$f(x) = \frac{4x-5}{(2x+1)(x-3)} - \frac{2x}{x^2-9}, \quad x \neq \pm 3, x \neq -\frac{1}{2}$$

(a) Show that

$$f(x) = \frac{5}{(2x+1)(x+3)}$$

(5)

The curve C has equation $y=f(x)$. The point $P\left(-1, -\frac{5}{2}\right)$ lies on C .

(b) Find an equation of the normal to C at P .

(8)

$$\begin{aligned} \text{a) } f(x) &= \frac{4x-5}{(2x+1)(x-3)} - \frac{2x}{(x+3)(x-3)} \\ &= \frac{(4x-5)(x+3) - 2x(2x+1)}{(2x+1)(x-3)(x+3)} \end{aligned}$$

$$= \frac{4x^2 + 7x - 15 - 4x^2 - 2x}{(2x+1)(x-3)(x+3)} = \frac{5x - 15}{(2x+1)(x-3)(x+3)}$$

$$= \frac{5(x-3)}{(2x+1)(x-3)(x+3)} = \frac{5}{(2x+1)(x+3)} \quad \#$$

$$\text{b) } f(x) = 5(2x^2 + 7x + 3)^{-1}$$

$$f'(x) = -5(2x^2 + 7x + 3)^{-2} \times (4x + 7)$$

$$f'(x) = \frac{-5(4x+7)}{(2x^2+7x+3)^2} \quad x=-1 \quad m_t = \frac{-15}{4}$$

$$m_n = \frac{4}{15}$$

$$\therefore y + \frac{5}{2} = \frac{4}{15}(x+1)$$

8. (a) Express $2\cos 3x - 3\sin 3x$ in the form $R\cos(3x + \alpha)$, where R and α are constants, $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. Give your answers to 3 significant figures. (4)

$$f(x) = e^{2x} \cos 3x$$

- (b) Show that $f'(x)$ can be written in the form

$$f'(x) = R e^{2x} \cos(3x + \alpha)$$

where R and α are the constants found in part (a). (5)

- (c) Hence, or otherwise, find the smallest positive value of x for which the curve with equation $y = f(x)$ has a turning point. (3)

$$R \cos(3x + \alpha) = R \cos 3x \cos \alpha - R \sin 3x \sin \alpha$$

$$2 \cos 3x \quad - 3 \sin 3x$$

$$\frac{R \sin \alpha}{R \cos \alpha} = \frac{3}{2} \Rightarrow \alpha = \tan^{-1}\left(\frac{3}{2}\right) = 0.983 \text{ (3sf)}$$

$$R^2 = 3^2 + 2^2 = 13 \Rightarrow R = 3.61$$

$$\sqrt{13} \cos(3x + 0.983)$$

b) $u = e^{2x} \quad v = \cos 3x \quad vu' + uv'$

$$u' = 2e^{2x} \quad v' = -3\sin 3x$$

$$f'(x) = 2e^{2x} \cos 3x - 3e^{2x} \sin 3x$$

$$= e^{2x} (2\cos 3x - 3\sin 3x)$$

$$= e^{2x} (\sqrt{13} \cos(3x + 0.983))$$

c) $f(x)$ has TP when $f'(x) = 0$

$$e^{2x} \sqrt{13} (\cos(3x + 0.983)) = 0$$

$$\Rightarrow \cos(3x + 0.983\dots) = 0$$

$$\Rightarrow 3x + 0.983\dots = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \dots$$

$$3x = \frac{\pi}{2} - 0.983\dots$$

$$x = \frac{\frac{\pi}{2} - 0.983\dots}{3} = \underline{0.196} \text{ (3sf)}$$