Core 3 - June 05 Solutions

1 a) $\sin^2\theta + \cos^2\theta = 1 \qquad \div \cos^2\theta$

$\tan^2\theta + 1 = \sec^2\theta$ #

b) $2\tan^2\theta + \sec\theta = 1$

$2(\sec^2\theta - 1) + \sec\theta = 1$

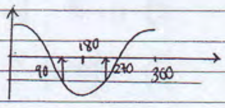
$2\sec^2\theta + \sec\theta - 3 = 0$

$(2\sec\theta + 3)(\sec\theta - 1) = 0$

$\sec\theta = \frac{-3}{2} \qquad \sec\theta = 1$

$\cos\theta = -\frac{2}{3} \qquad \cos\theta = 1$

$\theta = 131.8^\circ \qquad \theta = 0^\circ$



$\theta = 0^\circ, 131.8^\circ, 228.2^\circ, 360^\circ$

2a) I) $\frac{d}{dx}(3\sin^2 x + \sec 2x) = \frac{d}{dx}\left(3\sin^2 x + (\cos 2x)^{-1}\right)$

$= 6\sin x\cos x - (\cos 2x)^{-2} \cdot -2\sin 2x$

$= 6\sin x\cos x + \frac{2\sin 2x}{\cos^2 2x}$

II) $\frac{d}{dx}\left((x+\ln(2x))^3\right) = 3(x+\ln(2x))^2 \cdot \left(1+\frac{2}{2x}\right)$

$= 3(x+\ln(2x))^2 \cdot \left(1+\frac{1}{x}\right)$

b) $y = \frac{5x^2-10x+9}{(x-1)^2} \; \begin{matrix} u \\ v \end{matrix} \qquad u = 5x^2-10x+9 \quad u' = 10x-10$

$v = (x-1)^2 \quad v' = 2(x-1)$

$\frac{dy}{dx} = \frac{(x-1)^2(10x-10) - (5x^2-10x+9)(2(x-1))}{(x-1)^4}$

$= \frac{(x-1)(10x-10) - 2(5x^2-10x+9)}{(x-1)^3}$

$= \frac{10x^2-10x-10x+10-10x^2+20x-18}{(x-1)^3}$

$= \frac{-8}{(x-1)^3}$ #

3 a) $\frac{5x+1}{x^2+x-2} - \frac{3}{x+2}$

$= \frac{5x+1}{(x+2)(x-1)} - \frac{3}{x+2}$

$= \frac{5x+1-3(x-1)}{(x+2)(x-1)}$

$= \frac{2x+4}{(x+2)(x-1)} = \frac{2}{x-1}$ #

b) $y = \frac{2}{x-1}$

$x - 1 = \frac{2}{y}$

$x = \frac{2}{y} + 1 \Rightarrow f^{-1}(x) = \frac{2}{x} + 1$

c) $fg(x) = \frac{2}{x^2+4}$

$fg(x) = \frac{1}{4} \qquad \frac{2}{x^2+4} = \frac{1}{4}$

$x^2 + 4 = 8$

$x^2 = 4 \Rightarrow x = \pm 2$ ($x = 2$ as $x > 1$)

4. a) $f(x) = 3e^x - \frac{1}{2}\ln x - 2$

$f'(x) = 3e^x - \frac{1}{2x}$

b) turning point $f'(x) = 0$

$3e^\alpha - \frac{1}{2\alpha} = 0$

$\frac{1}{\alpha} = 6e^\alpha \Rightarrow \alpha = \frac{1}{6}e^{-\alpha}$ #

c) $x_1 = \frac{1}{6}e^{-1} = 0.0613$

$x_2 = \frac{1}{6}e^{-0.061313} = 0.1568$

$x_3 = \frac{1}{6}e^{-0.156754} = 0.1425$

$x_4 = \frac{1}{6}e^{-0.142485} = 0.1445$

d) $f'(0.14435) = 0.00206$

$f'(0.14425) = -0.00069$

change of sign indicates root $\therefore \alpha = 0.1443$ (4dp) #

5. a) $\cos 2A = \cos^2 A - \sin^2 A$

$= 1 - \sin^2 A - \sin^2 A = 1 - 2\sin^2 A$ #

b) $2\sin 2\theta - 3\cos 2\theta - 3\sin\theta + 3$

$= 2 \cdot 2\sin\theta\cos\theta - 3(\cos^2\theta - \sin^2\theta) - 3\sin\theta + 3$

$= 4\sin\theta\cos\theta - 3\cos^2\theta + 3\sin^2\theta - 3\sin\theta + 3$

$= 4\sin\theta\cos\theta - (3 - 3\sin^2\theta) + 3\sin^2\theta - 3\sin\theta + 3$

$= 4\sin\theta\cos\theta - 3 + 3\sin^2\theta + 3\sin^2\theta - 3\sin\theta + 3$

$= 4\sin\theta\cos\theta + 6\sin^2\theta - 3\sin\theta$

PMT

(5)

$$= \sin\theta\,(4\cos\theta + 6\sin\theta - 3)$$

c) $4\cos\theta + 6\sin\theta = R\sin(\theta+\alpha)$

$$= R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$$

$R\cos\alpha = 6 \quad R\sin\alpha = 4$

$$\tan\alpha = \frac{4}{6} = \frac{2}{3} \Rightarrow \alpha = 0.588^c$$

$$R^2 = 6^2 + 4^2 = 36 + 16$$

$$R = \sqrt{52} = 2\sqrt{13}$$

$$\therefore 4\cos\theta + 6\sin\theta = 2\sqrt{13}\sin(\theta + 0.588)$$

d) $2\sin 2\theta = 3(\cos 2\theta + \sin\theta - 1)$

$$2\sin 2\theta - 3\cos 2\theta - 3\sin\theta + 3 = 0$$

$$\sin\theta(4\cos\theta + 6\sin\theta - 3) = 0 \quad \text{(part b))}$$

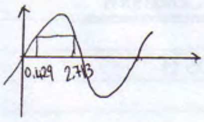
$$\sin\theta\,(2\sqrt{13}\sin(\theta + 0.588) - 3) = 0$$

$\therefore \sin\theta = 0 \qquad 2\sqrt{13}\sin(\theta + 0.588) - 3 = 0$

$\theta = 0 \qquad \sin(\theta + 0.588) = \frac{3}{2\sqrt{13}}$

$$\theta + 0.588 = 0.429^c \qquad 0.588^c \leq \theta + 0.588 < \pi + 0.588$$

(6)

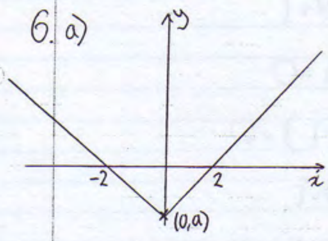


$$\therefore \theta + 0.588 = 2.713^c$$

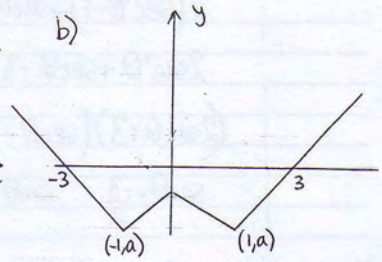
$$\theta = 2.125^c$$

$$\therefore \theta = 0^c, 2.125^c$$

6. a)

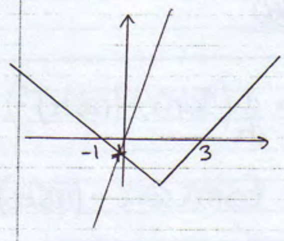


b)



c) $a = -2$, $b = -1$

d) $|x-1| - 2 = 5x$



one point of intersection

$$-(x-1) - 2 = 5x$$

$$-x + 1 - 2 = 5x$$

$$-1 = 6x$$

$$x = -\tfrac{1}{6}$$

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7. a) when $t = 0$ $p = 300$ $\quad 300 = \frac{2800ae^0}{1 + ae^0}$

$$300 = \frac{2800a}{1+a}$$

$$300 + 300a = 2800a$$

$$300 = 2500a$$

$$a = \frac{300}{2500} = 0.12 \quad \#$$

b) when $p = 1850$ $\quad 1850 = \frac{2800 \times 0.12e^{0.2t}}{1 + 0.12e^{0.2t}}$

$$1850 = \frac{336e^{0.2t}}{1 + 0.12e^{0.2t}}$$

$$1850 + 222e^{0.2t} = 336e^{0.2t}$$

$$1850 = 114e^{0.2t}$$

$$0.2t = \ln\left(\frac{1850}{114}\right)$$

$$t = \frac{1}{0.2}\ln\left(\frac{1850}{114}\right) = 13.9$$

so 14 years

c) $P = \frac{2800 \times 0.12e^{0.2t}}{1 + 0.12e^{0.2t}} = \frac{336e^{0.2t}}{1 + 0.12e^{0.2t}} \div \frac{e^{0.2t}}{e^{0.2t}}$

$$= \frac{336}{e^{-0.2t} + 0.12} \quad \#$$

d) as $t \to \infty$ $e^{-0.2t} \to 0$ so $p \to \frac{336}{0.12} = 2800$.

so max value of $p = 2800$ #