C3 JAN 14 (R)

1.

$$f(x) = \sec x + 3x - 2, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

(a) Show that there is a root of $f(x) = 0$ in the interval $[0.2, 0.4]$ **(2)**

(b) Show that the equation $f(x) = 0$ can be written in the form

$$x = \frac{2}{3} - \frac{1}{3\cos x}$$

(1)

The solution of $f(x) = 0$ is α, where $\alpha = 0.3$ to 1 decimal place.

(c) Starting with $x_0 = 0.3$, use the iterative formula

$$x_{n+1} = \frac{2}{3} - \frac{1}{3\cos x_n}$$

to calculate the values of x_1, x_2 and x_3, giving your answers to 4 decimal places. **(3)**

(d) State the value of α correct to 3 decimal places. **(1)**

a) $f(0.2) = -0.38 < 0$ ∴ by sign change
$f(0.4) = 0.29 > 0$ rule $\alpha \in [0.2, 0.4]$

b) $\frac{1}{\cos x} + 3x - 2 = 0 \Rightarrow 3x = 2 - \frac{1}{\cos x}$

$\div 3 \Rightarrow x = \frac{2}{3} - \frac{1}{3\cos x}$ #

c) $x_0 = 0.3$
$x_1 = 0.3177$
$x_2 = 0.3158$
$x_3 = 0.3160$

d) $\alpha = 0.316$ (3dp)

2.

$$f(x) = \frac{15}{3x+4} - \frac{2x}{x-1} + \frac{14}{(3x+4)(x-1)}, \quad x > 1$$

(a) Express $f(x)$ as a single fraction in its simplest form. **(4)**

(b) Hence, or otherwise, find $f'(x)$, giving your answer as a single fraction in its simplest form. **(3)**

a) $\frac{15(x-1) - 2x(3x+4) + 14}{(3x+4)(x-1)}$

$= \frac{15x - 15 - 6x^2 - 8x + 14}{(3x+4)(x-1)}$

$= \frac{-6x^2 + 7x - 1}{(3x+4)(x-1)}$

$= -\frac{(6x^2 - 7x + 1)}{(3x+4)(x-1)}$

$= -\frac{(6x-1)\cancel{(x-1)}}{(3x+4)\cancel{(x-1)}} = \frac{1-6x}{4+3x}$

b) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$ $\quad u = 1-6x \quad v = 4+3x$
$u' = -6 \quad v' = 3$

$f'(x) = \frac{-6(4+3x) - 3(1-6x)}{(4+3x)^2} = \frac{-24 - \cancel{18x} - 3 + \cancel{18x}}{(4+3x)^2}$

$= \frac{-27}{(4+3x)^2}$

3. (a) By writing $\operatorname{cosec} x$ as $\frac{1}{\sin x}$, show that

$$\frac{d(\operatorname{cosec} x)}{dx} = -\operatorname{cosec} x \cot x$$

(3)

Given that $y = e^{3x} \operatorname{cosec} 2x$, $0 < x < \frac{\pi}{2}$,

(b) find an expression for $\frac{dy}{dx}$.

(3)

The curve with equation $y = e^{3x} \operatorname{cosec} 2x$, $0 < x < \frac{\pi}{2}$, has a single turning point.

(c) Show that the x-coordinate of this turning point is at $x = \frac{1}{2}\arctan k$ where the value of the constant k should be found.

(2)

a) $\frac{d}{dx} \operatorname{cosec} x = \frac{d}{dx}(\sin x)^{-1}$

$= -(\sin x)^{-2} \times \cos x$

$= -\frac{1}{\sin x} \times \frac{\cos x}{\sin x} = -\operatorname{cosec} x \cot x$

b) $y = e^{3x} \operatorname{cosec} 2x$

$u = e^{3x}$ $\quad v = \operatorname{cosec} 2x$

$u' = 3e^{3x}$ $\quad v' = -2\operatorname{cosec} 2x \cot 2x$

$y' = vu' + uv'$

$y' = 3e^{3x} \operatorname{cosec} 2x - 2e^{3x} \operatorname{cosec} 2x \cot 2x$

$= e^{3x} \operatorname{cosec} 2x (3 - 2\cot 2x)$

c) TP when $y' = 0$

$e^{3x} \operatorname{cosec} 2x = 0$ $\quad$ $3 - 2\cot 2x = 0$

No solutions $\quad$ $2\cot 2x = 3$

$\cot 2x = \frac{3}{2}$

$\Rightarrow \tan 2x = \frac{2}{3}$

$\Rightarrow 2x = \arctan\left(\frac{2}{3}\right)$

$\therefore x = \frac{1}{2}\arctan\left(\frac{2}{3}\right)$

4. A pot of coffee is delivered to a meeting room at 11am. At a time t minutes after 11am the temperature, θ °C, of the coffee in the pot is given by the equation

$$\theta = A + 60e^{-kt}$$

where A and k are positive constants.

Given also that the temperature of the coffee at 11am is 85 °C and that 15 minutes later it is 58 °C,

(a) find the value of A.

(1)

(b) Show that $k = \frac{1}{15}\ln\left(\frac{20}{11}\right)$

(3)

(c) Find, to the nearest minute, the time at which the temperature of the coffee reaches 50 °C.

(4)

$\theta = A + 60e^{-kt}$ $\quad t = 0, \theta = 85$ $\quad \therefore A = 25$

b) $t = 15$, $\theta = 58$ $\quad 58 = 25 + 60e^{-15k}$

$\Rightarrow 33 = 60e^{-15k}$ $\quad \Rightarrow e^{-15k} = \frac{11}{20}$

$\Rightarrow -15k = \ln\left(\frac{11}{20}\right)$ $\quad \Rightarrow -15k = \ln\left(\frac{20}{11}\right)^{-1}$

$\Rightarrow -15k = -\ln\left(\frac{20}{11}\right)$ $\quad \therefore k = \frac{1}{15}\ln\left(\frac{20}{11}\right)$

c) $50 = 25 + 60e^{-\frac{1}{15}\ln\left(\frac{20}{11}\right)t}$

$\Rightarrow e^{-\frac{1}{15}\ln\left(\frac{20}{11}\right)t} = \frac{5}{12}$ $\quad \Rightarrow -\frac{1}{15}\ln\left(\frac{20}{11}\right)t = \ln\left(\frac{5}{12}\right)$

$\Rightarrow \frac{1}{15}\ln\left(\frac{20}{11}\right)t = \ln\left(\frac{12}{5}\right)$ $\quad \therefore t = \frac{15\ln\left(\frac{12}{5}\right)}{\ln\left(\frac{20}{11}\right)}$

$\therefore t = 21.96 \simeq 22\text{min}$

5.

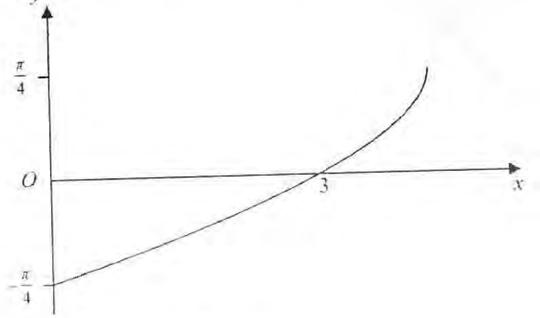


Figure 1

The curve shown in Figure 1 has equation

$$x = 3\sin y + 3\cos y, \qquad -\frac{\pi}{4} < y < \frac{\pi}{4}$$

(a) Express the equation of the curve in the form

$x = R\sin(y + \alpha)$, where R and α are constants, $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

(3)

(b) Find the coordinates of the point on the curve where the value of $\frac{dy}{dx}$ is $\frac{1}{2}$.

Give your answers to 3 decimal places.

(6)

$R\sin(y+\alpha) = R\sin y\cos\alpha + R\cos y\sin\alpha$

$x = 3\sin y + 3\cos y$

$\frac{R\sin\alpha}{R\cos\alpha} = \frac{3}{3}$ $\quad \tan\alpha = 1 \quad \therefore \alpha = \frac{\pi}{4}$

$R = \sqrt{18} = 3\sqrt{2}$

$\therefore x = 3\sqrt{2}\sin\left(y + \frac{\pi}{4}\right)$

$\frac{dx}{dy} = 3\sqrt{2}\cos\left(y + \frac{\pi}{4}\right) = 2 \quad \left(\frac{dy}{dx} = \frac{1}{2}\right)$

$\therefore \cos\left(y + \frac{\pi}{4}\right) = \frac{2}{3\sqrt{2}} \quad y + \frac{\pi}{4} = 1.0799 \therefore y = 0.295$

$x = 3\sqrt{2}\left(\sin\left(y + \frac{\pi}{4}\right)\right) \therefore x = 3.742$

$(3.742, 0.295)$

6. Given that a and b are constants and that $0 < a < b$,

(a) on separate diagrams, sketch the graph with equation

(i) $y = |2x + a|$,

(ii) $y = |2x + a| - b$.

Show on each sketch the coordinates of each point at which the graph crosses or meets the axes.

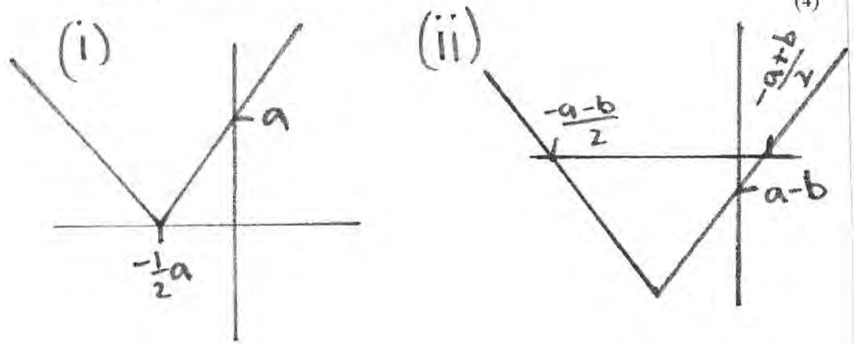
(6)

(b) Solve, for x, the equation

$$|2x + a| - b = \frac{1}{3}x$$

giving any answers in terms of a and b.

(4)



b) $|2x + a| = \frac{1}{3}x + b$

$2x + a = \frac{1}{3}x + b \quad \vdots \quad -2x - a = \frac{1}{3}x + b$

$\frac{5}{3}x = b - a \quad \vdots \quad -a - b = \frac{7}{3}x$

$x = \frac{3b - 3a}{5} \quad \vdots \quad x = \frac{-3a - 3b}{7}$

7. (i) (a) Prove that

$$\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$$

(You may use the double angle formulae and the identity $\cos(A+B) \equiv \cos A \cos B - \sin A \sin B$) **(4)**

(b) Hence solve the equation

$$2\cos 3\theta + \cos 2\theta + 1 = 0$$

giving answers in the interval $0 \leqslant \theta \leqslant \pi$.
Solutions based entirely on graphical or numerical methods are not acceptable. **(6)**

(ii) Given that $\theta = \arcsin x$ and that $0 < \theta < \frac{\pi}{2}$, show that

$$\cot\theta = \frac{\sqrt{(1-x^2)}}{x}, \quad 0 < x < 1$$ **(3)**

a) $\cos 3\theta = \cos(2\theta+\theta) = \cos 2\theta \cos\theta - \sin 2\theta \sin\theta$

$= (2\cos^2\theta - 1)\cos\theta - 2\sin^2\theta\cos\theta$

$= 2\cos^3\theta - \cos\theta - 2\cos\theta(1-\cos^2\theta)$

$= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta$

$= 4\cos^3\theta - 3\cos\theta$ #

b) $8\cos^3\theta - 6\cos\theta + \cos 2\theta + 1 = 0$

$8\cos^3\theta - 6\cos\theta + 2\cos^2\theta \cancel{-1} \cancel{+1} = 0$

$= 2\cos\theta(4\cos^2\theta + \cos\theta - 3) = 0$

$= 2\cos\theta(4\cos\theta - 3)(\cos\theta + 1) = 0$

$\therefore \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$

$\therefore \cos\theta = \frac{3}{4} \Rightarrow \theta = 0.723^c$

$\therefore \cos\theta = -1 \Rightarrow \theta = \pi$

ii) $\theta = \arcsin x \Rightarrow \sin\theta = x$

$\sin^2\theta = x^2 \Rightarrow 1 - \sin^2\theta = 1 - x^2 \Rightarrow \cos^2\theta = 1 - x^2$

$\Rightarrow \cos\theta = \sqrt{1-x^2}$

$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\sqrt{1-x^2}}{x}$ #

8. The function f is defined by

$$f : x \to 3 - 2e^{-x}, \quad x \in \mathbb{R}$$

(a) Find the inverse function, $f^{-1}(x)$ and give its domain. **(5)**

(b) Solve the equation $f^{-1}(x) = \ln x$. **(4)**

The equation $f(t) = ke^t$, where k is a positive constant, has exactly one real solution.

(c) Find the value of k. **(4)**

a) $y = 3 - 2e^{-x} \Rightarrow x = 3 - 2e^{-y}$

$\Rightarrow 2e^{-y} = 3 - x \Rightarrow e^{-y} = \frac{3-x}{2}$

$\Rightarrow -y = \ln\left|\frac{3-x}{2}\right| \Rightarrow y = \ln\left|\frac{2}{3-x}\right|$ $\underline{x<3}$

b) $\ln\left|\frac{2}{3-x}\right| = \ln x \Rightarrow \frac{2}{3-x} = x \Rightarrow 2 = 3x - x^2$

$x^2 - 3x + 2 = 0 \quad (x-2)(x-1) = 0$

$\Rightarrow \underline{x=2}, \underline{x=1}$

c) $f(t) = 3 - 2e^{-t} = ke^t$

$3 - \frac{2}{e^t} = ke^t \quad (\times e^t) \quad 3e^t - 2 = ke^{2t}$

$ke^{2t} - 3e^t + 2 = 0$

One solution $\Rightarrow b^2 - 4ac = 0 \quad 9 - 8k = 0 \quad k = \frac{9}{8}$