

C3 JAN 2010

PMT

1. Express

$$\frac{x+1}{3x^2-3} - \frac{1}{3x+1}$$

as a single fraction in its simplest form.

(4)

$$\frac{x+1}{3(x^2-1)} - \frac{1}{3x+1} = \frac{1(x+1)}{3(x+1)(x-1)} - \frac{1}{3x+1}$$

$$= \frac{1}{3(x-1)} - \frac{1}{3x+1} = \frac{1}{3(x-1)} \times \frac{3x+1}{3x+1} - \frac{1}{3x+1} \times \frac{3(x-1)}{3(x-1)}$$

$$= \frac{3x+1-3x+3}{3(x-1)(3x+1)} = \frac{4}{3(x-1)(3x+1)}$$

2.

$$f(x) = x^3 + 2x^2 - 3x - 11$$

(a) Show that $f(x) = 0$ can be rearranged as

$$x = \sqrt{\left(\frac{3x+11}{x+2}\right)}, \quad x \neq -2.$$

(2)

The equation $f(x) = 0$ has one positive root α .

The iterative formula $x_{n+1} = \sqrt{\left(\frac{3x_n+11}{x_n+2}\right)}$ is used to find an approximation to α .

(b) Taking $x_1 = 0$, find, to 3 decimal places, the values of x_2 , x_3 and x_4 .

(3)

(c) Show that $\alpha = 2.057$ correct to 3 decimal places.

(3)

$$(a) \quad x^3 + 2x^2 = 3x + 11$$

$$x^2(x+2) = 3x+11 \quad x^2 = \frac{3x+11}{x+2}$$

$$x = \sqrt{\frac{3x+11}{x+2}}$$

$$(b) \quad x_1 = 0$$

$$x_2 = 2.345$$

$$x_3 = 2.037$$

$$x_4 = 2.059$$

$$(c) \quad f(2.0575) = 0.00414$$

$$f(2.0565) = -0.0138$$

by change of sign law $\alpha = 2.057$ (3dp)

3. (a) Express $5 \cos x - 3 \sin x$ in the form $R \cos(x + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$.

(4)

(b) Hence, or otherwise, solve the equation

$$5 \cos x - 3 \sin x = 4$$

for $0 \leq x < 2\pi$, giving your answers to 2 decimal places.

(5)

$$(a) \begin{aligned} R \sin \alpha &= 3 \\ R \cos \alpha &= 5 \end{aligned} \Rightarrow \tan \alpha = \frac{3}{5} \Rightarrow \alpha = 0.5404$$

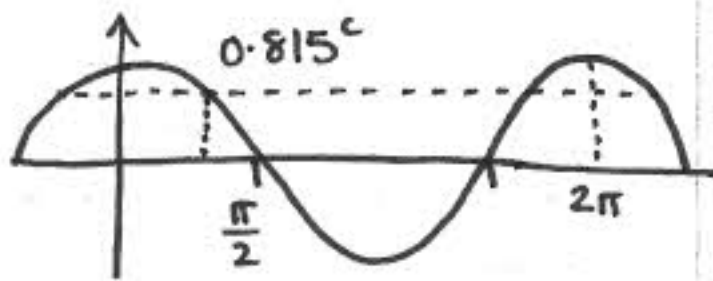
$$R = \sqrt{3^2 + 5^2} = \sqrt{34} \Rightarrow \sqrt{34} \cos(x + 0.5404)$$

$$(b) \sqrt{34} \cos(x + 0.5404) = 4$$

$$x + 0.5404 = \cos^{-1}\left(\frac{4}{\sqrt{34}}\right)$$

$$\begin{aligned} x + 0.5405 &= 0.815^\circ \\ &= 2\pi - 0.815 \end{aligned}$$

$$x = \underline{0.27^\circ}, \underline{4.93^\circ}$$



4. (i) Given that $y = \frac{\ln(x^2+1)}{x}$, find $\frac{dy}{dx}$.

(4)

(ii) Given that $x = \tan y$, show that $\frac{dy}{dx} = \frac{1}{1+x^2}$.

(5)

(i) $u = \ln(x^2+1)$ $v = x$

$u' = \frac{2x}{x^2+1}$ $v' = 1$

$$\frac{d}{dx}(uv) = \frac{\frac{2x^2}{x^2+1} - \ln(x^2+1)}{x^2} = \frac{2}{x^2+1} - \frac{\ln(x^2+1)}{x^2}$$

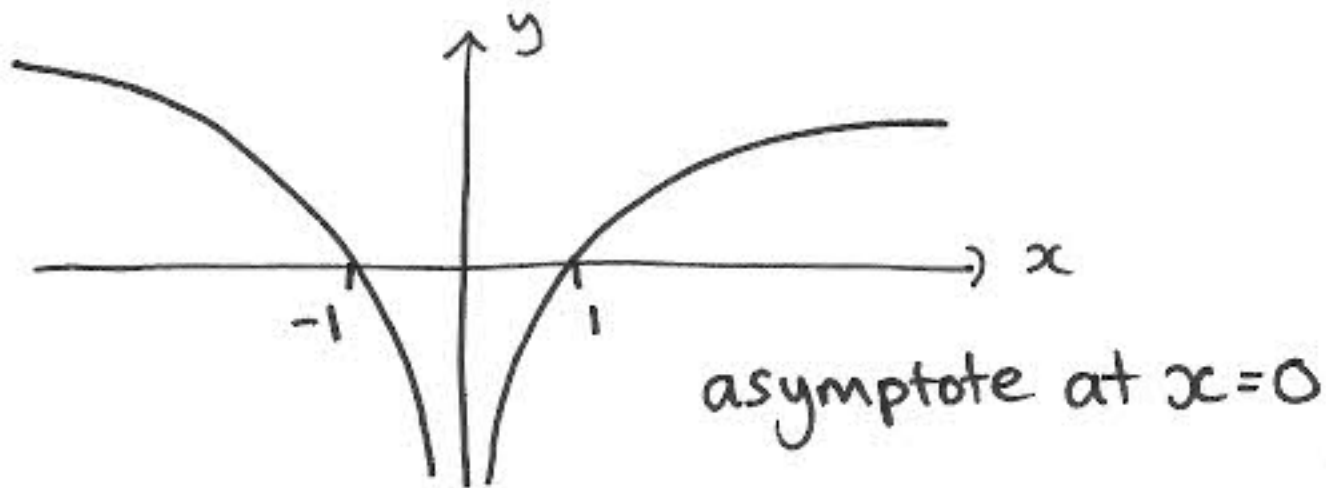
(ii) $x = \tan y$

$$\frac{dx}{dy} = \sec^2 y \Rightarrow \frac{dx}{dy} = 1 + \tan^2 y \Rightarrow \frac{dx}{dy} = 1 + x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+x^2}$$

5. Sketch the graph of $y = \ln|x|$, stating the coordinates of any points of intersection with the axes.

(3)



6.

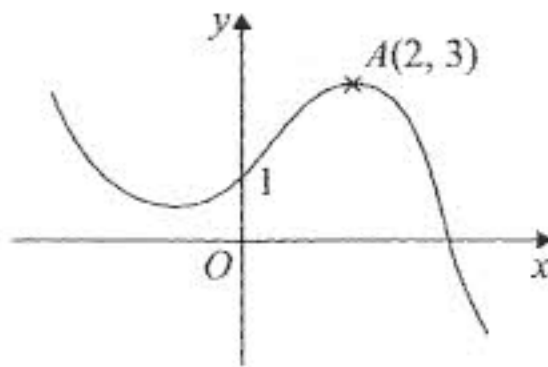


Figure 1

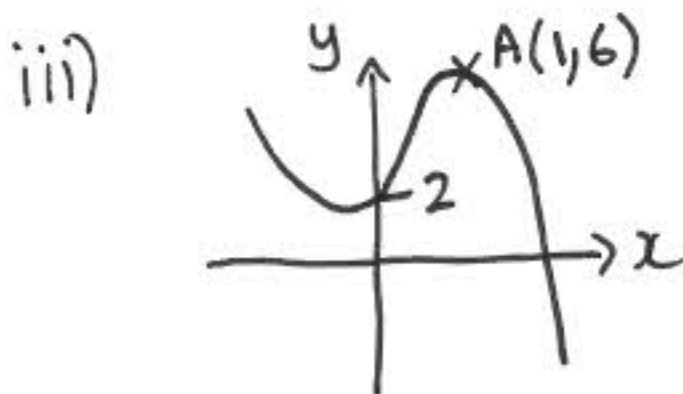
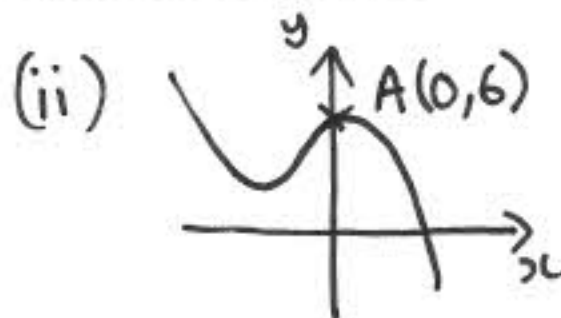
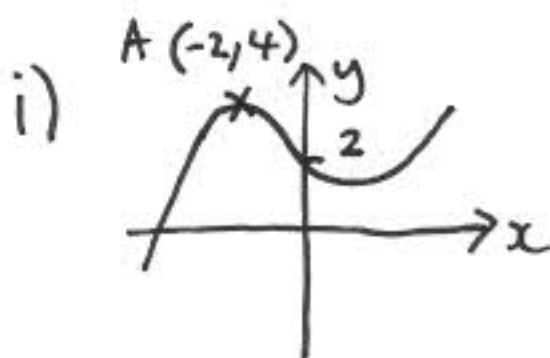
Figure 1 shows a sketch of the graph of $y = f(x)$.

The graph intersects the y -axis at the point $(0, 1)$ and the point $A(2, 3)$ is the maximum turning point.

Sketch, on separate axes, the graphs of

- (i) $y = f(-x) + 1$, \leftarrow flip $\uparrow +1$
(ii) $y = f(x + 2) + 3$, $2 \leftarrow$ $\uparrow +3$
(iii) $y = 2f(2x)$. $\times 2 \updownarrow \rightarrow \div 2 \leftarrow$

On each sketch, show the coordinates of the point at which your graph intersects the y -axis and the coordinates of the point to which A is transformed.



(9)

7. (a) By writing $\sec x$ as $\frac{1}{\cos x}$, show that $\frac{d(\sec x)}{dx} = \sec x \tan x$.

(3)

Given that $y = e^{2x} \sec 3x$,

(b) find $\frac{dy}{dx}$.

(4)

The curve with equation $y = e^{2x} \sec 3x$, $-\frac{\pi}{6} < x < \frac{\pi}{6}$, has a minimum turning point at (a, b) .

(c) Find the values of the constants a and b , giving your answers to 3 significant figures.

(4)

$$(a) \frac{d}{dx} \sec x = \frac{d}{dx} \left(\frac{1}{\cos x} \right) \quad u=1 \quad v=\cos x \\ u'=0 \quad v'=-\sin x$$

$$\frac{d}{dx}(uv) = \frac{0 + \sin x}{\cos^2 x} = \frac{1}{\cos x} \times \frac{\sin x}{\cos x} = \underline{\sec x \tan x}$$

$$(b) \frac{d}{dx} e^{2x} \sec 3x \quad u=e^{2x} \quad v=\sec 3x \\ u'=2e^{2x} \quad v'=3\sec 3x \tan 3x$$

$$\frac{d}{dx}(uv) = 2e^{2x} \sec 3x + 3e^{2x} \sec 3x \tan 3x \\ = e^{2x} \sec 3x (2 + 3 \tan 3x)$$

$$(c) \text{Minimum point} \Rightarrow \frac{dy}{dx} = 0 \quad e^{2x} \sec 3x \neq 0$$

$$\Rightarrow 2 + 3 \tan 3x = 0 \Rightarrow \tan 3x = -\frac{2}{3}$$

$$3x = \tan^{-1}\left(-\frac{2}{3}\right) \Rightarrow 3x = -0.588 \quad (+k\pi)$$

$$x = -0.196 \quad y = \frac{e^{2x}}{\cos 3x} = 0.812 \quad a = -0.196 \\ b = \underline{0.812}$$

8. Solve

$$\operatorname{cosec}^2 2x - \cot 2x = 1$$

for $0 \leq x \leq 180^\circ$.

(7)

$$(\cancel{1} + \cot^2 2x) - \cot 2x = \cancel{1}$$

$$\cot^2 2x - \cot 2x = 0$$

$$\cot 2x (\cot 2x - 1) = 0$$

$$\Rightarrow \frac{1}{\tan 2x} = 0 \Rightarrow \text{no solution}$$

$$\tan 2x \rightarrow \infty$$

$$2x = 90^\circ, 270^\circ$$

$$x = 45^\circ, 135^\circ$$

$$\Rightarrow \cot 2x - 1 = 0 \Rightarrow \cot 2x = 1$$

$$\Rightarrow \frac{1}{\tan 2x} = 1 \Rightarrow \tan 2x = 1$$

$$\Rightarrow 2x = \tan^{-1}(1) = 45^\circ$$

$$2x = 45^\circ, 225^\circ$$

$$x = \underline{22.5^\circ}, \underline{112.5^\circ}, \underline{45^\circ}, \underline{135^\circ}$$

9. (i) Find the exact solutions to the equations

(a) $\ln(3x - 7) = 5$

(3)

(b) $3^x e^{7x+2} = 15$

(5)

(ii) The functions f and g are defined by

$$f(x) = e^{2x} + 3, \quad x \in \mathbb{R}$$

$$g(x) = \ln(x - 1), \quad x \in \mathbb{R}, x > 1$$

(a) Find f^{-1} and state its domain.

(4)

(b) Find fg and state its range.

(3)

$$(a) 3x - 7 = e^5 \Rightarrow 3x = 7 + e^5 \Rightarrow x = \frac{7 + e^5}{3}$$

$$(b) \ln 3^x e^{7x+2} = \ln 15$$

$$\ln 3^x + \ln e^{7x+2} = x \ln 3 + 7x + 2 = \ln 15$$

$$x(7 + \ln 3) = \ln(15) - 2 \Rightarrow x = \frac{\ln(15) - 2}{7 + \ln 3}$$

$$(ii) y = e^{2x} + 3 \Rightarrow x = e^{2y} + 3 \Rightarrow e^{2y} = x - 3$$

$$\Rightarrow 2y = \ln(x - 3) \Rightarrow y = f^{-1}(x) = \frac{\ln(x - 3)}{2} \quad x > 3$$

$$(b) fg(x) = e^{2 \ln(x-1)} + 3 = (e^{\ln(x-1)})^2 + 3$$

$$fg(x) = (x-1)^2 + 3 \quad fg(x) > 3$$