C3 June 12

1. Express

$$\frac{2(3x+2)}{9x^2-4} - \frac{2}{3x+1}$$

as a single fraction in its simplest form.

(4)

$$\frac{2\cancel{(3x+2)}}{(3x-2)\cancel{(3x+2)}} - \frac{2}{3x+1}$$

$$= \frac{2(3x+1)-2(3x-2)}{(3x-2)(3x+1)} = \frac{6}{(3x-2)(3x+1)}$$

2. $$f(x) = x^3 + 3x^2 + 4x - 12$$

(a) Show that the equation $f(x) = 0$ can be written as

$$x = \sqrt{\left(\frac{4(3-x)}{(3+x)}\right)}, \qquad x \neq -3$$

(3)

The equation $x^3 + 3x^2 + 4x - 12 = 0$ has a single root which is between 1 and 2

(b) Use the iteration formula

$$x_{n+1} = \sqrt{\left(\frac{4(3-x_n)}{(3+x_n)}\right)}, \; n \geqslant 0$$

with $x_0 = 1$ to find, to 2 decimal places, the value of x_1, x_2 and x_3.

(3)

The root of $f(x) = 0$ is α.

(c) By choosing a suitable interval, prove that $\alpha = 1.272$ to 3 decimal places.

(3)

a) $x^3 + 3x^2 + 4x - 12 = 0$

$x^3 + 3x^2 = 12 - 4x$

$x^2(x+3) = 4(3-x)$

$x^2 = \dfrac{4(3-x)}{(x+3)} \Rightarrow x = \sqrt{\dfrac{4(3-x)}{3+x}}$ #

b) $x_0 = 1$; $x_1 = 1.41$; $x_2 = 1.20$; $x_3 = 1.31$

c) $f(1.2715) = -0.0082 < 0$

$f(1.2725) = 0.0083 > 0$

by sign change rule $\alpha \in (1.2715, 1.2725)$

$\therefore \alpha = 1.272$ (3dp)

3.

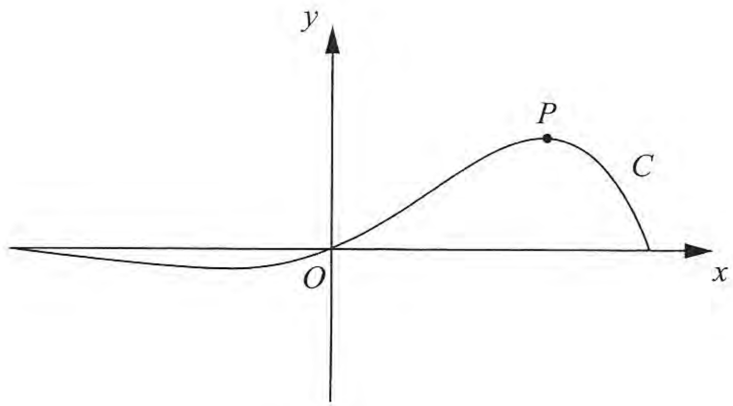


Figure 1

Figure 1 shows a sketch of the curve C which has equation

$$y = e^{x\sqrt{3}} \sin 3x, \quad -\frac{\pi}{3} \leqslant x \leqslant \frac{\pi}{3}$$

(a) Find the x coordinate of the turning point P on C, for which $x > 0$
Give your answer as a multiple of π.

(6)

(b) Find an equation of the normal to C at the point where $x = 0$

(3)

$\frac{d}{dx}(uv) = vu' + uv'$ $\quad u = e^{x\sqrt{3}}$ $\quad v = \sin 3x$

$u' = \sqrt{3}e^{x\sqrt{3}}$ $\quad v' = 3\cos 3x$

$\frac{dy}{dx} = \sqrt{3}e^{x\sqrt{3}}\sin 3x + 3e^{x\sqrt{3}}\cos 3x$

$= e^{x\sqrt{3}}(\sqrt{3}\sin 3x + 3\cos 3x)$

at TP $\frac{dy}{dx} = 0 \Rightarrow \sqrt{3}\sin 3x = -3\cos 3x$

$\therefore \tan 3x = -\frac{3}{\sqrt{3}}$

$\therefore 3x = -\frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{3}, \ldots \Rightarrow x = -\frac{\pi}{9}, \frac{2\pi}{9}, \frac{5\pi}{9}$

$\therefore x = \frac{2\pi}{9}$ $\quad y = e^{\frac{2\sqrt{3}\pi}{9}}\sin\left(3 \times \frac{2\pi}{9}\right) = \frac{\sqrt{3}}{2}e^{\frac{2\sqrt{3}\pi}{9}}$

(y not asked for!!)

b) $x = 0 \quad y = 0$

$\left.\frac{dy}{dx}\right|_{x=0} = e^0(0+3) = 3 \qquad m_t = 3 \Rightarrow m_n = -\frac{1}{3}$

$\therefore \; y - 0 = -\frac{1}{3}(x - 0) \Rightarrow y = -\frac{1}{3}x$

4.

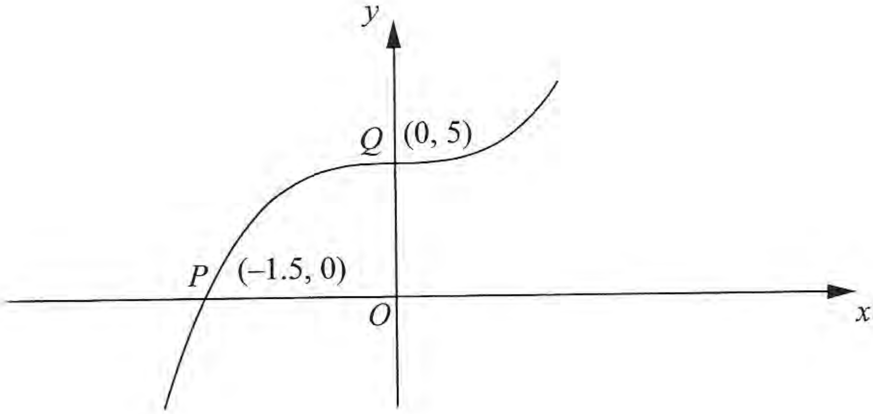


Figure 2

Figure 2 shows part of the curve with equation $y = \mathrm{f}(x)$
The curve passes through the points $P(-1.5,\ 0)$ and $Q(0,\ 5)$ as shown.

On separate diagrams, sketch the curve with equation

(a) $y = |\mathrm{f}(x)|$ **(2)**

(b) $y = \mathrm{f}(|x|)$ **(2)**

(c) $y = 2\mathrm{f}(3x)$ **(3)**

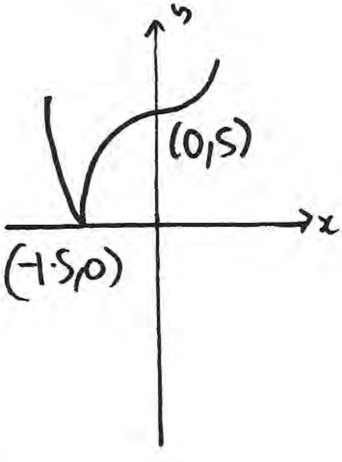
Indicate clearly on each sketch the coordinates of the points at which the curve crosses or meets the axes.

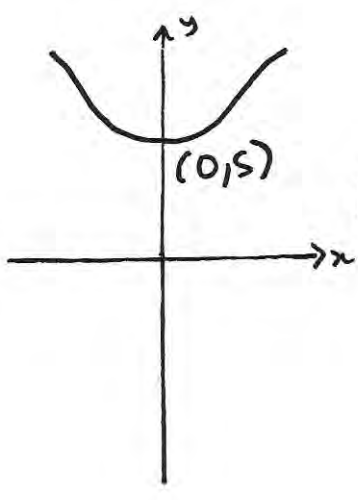
a) $|f(x)|$

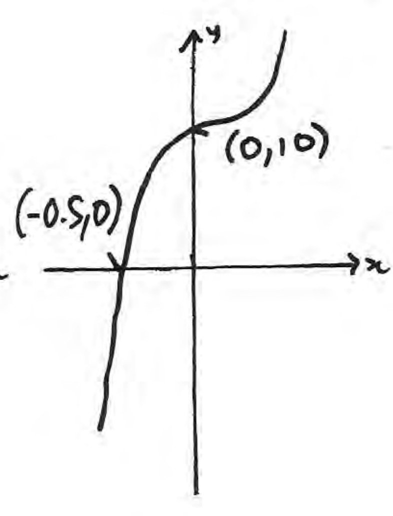
b) $f(|x|)$

c) $2f(3x)$

$\updownarrow 2 \quad \rightarrow 3 \leftarrow$







5. (a) Express $4\operatorname{cosec}^2 2\theta - \operatorname{cosec}^2\theta$ in terms of $\sin\theta$ and $\cos\theta$.

(2)

(b) Hence show that

$$4\operatorname{cosec}^2 2\theta - \operatorname{cosec}^2\theta = \sec^2\theta$$

(4)

(c) Hence or otherwise solve, for $0 < \theta < \pi$,

$$4\operatorname{cosec}^2 2\theta - \operatorname{cosec}^2\theta = 4$$

giving your answers in terms of π.

(3)

$$\frac{4}{\sin^2 2\theta} - \frac{1}{\sin^2\theta} = \frac{4}{(2\sin\theta\cos\theta)} - \frac{1}{\sin^2\theta}$$

$$= \frac{4}{4\sin^2\theta\cos^2\theta} - \frac{1}{\sin^2\theta} \Rightarrow \frac{1}{\sin^2\theta\cos^2\theta} - \frac{1}{\sin^2\theta}$$

b) $$\frac{1-\cos^2\theta}{\sin^2\theta\cos^2\theta} = \frac{\sin^2\theta}{\sin^2\theta\cos^2\theta} = \frac{1}{\cos^2\theta} = \sec^2\theta$$ #

c) $\sec^2\theta = 4 \Rightarrow \dfrac{1}{\cos^2\theta} = 4 \Rightarrow \cos^2\theta = \dfrac{1}{4}$

$\cos\theta = \frac{1}{2}$ $\qquad\qquad$ $\cos\theta = -\frac{1}{2}$

$\theta = \frac{\pi}{3}$ $\qquad\qquad$ $\theta = \frac{2\pi}{3}$

6. The functions f and g are defined by

$$f : x \mapsto e^x + 2, \qquad x \in \mathbb{R}$$

$$g : x \mapsto \ln x, \qquad x > 0$$

(a) State the range of f. **(1)**

(b) Find $fg(x)$, giving your answer in its simplest form. **(2)**

(c) Find the exact value of x for which $f(2x+3) = 6$ **(4)**

(d) Find f^{-1}, the inverse function of f, stating its domain. **(3)**

(e) On the same axes sketch the curves with equation $y = f(x)$ and $y = f^{-1}(x)$, giving the coordinates of all the points where the curves cross the axes. **(4)**

a) $y = e^x + 2$ range $y > 2$

b) $fg(x) = f(\ln x) = e^{\ln x} + 2 = x + 2$

c) $f(2x+3) = e^{2x+3} + 2 = 6$

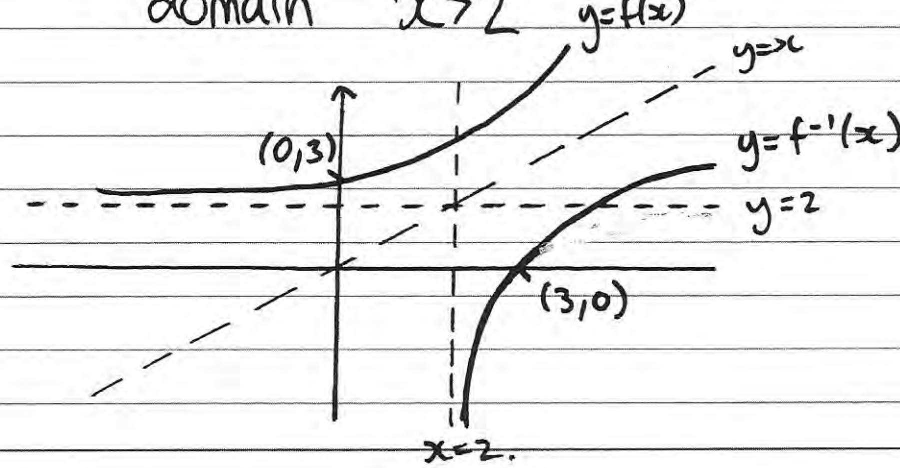
$e^{2x+3} = 4 \Rightarrow 2x+3 = \ln 4 \Rightarrow x = \dfrac{-3+\ln 4}{2}$

d) $y = e^x + 2 \Rightarrow x = e^y + 2$

$\Rightarrow e^y = x - 2 \Rightarrow y = \ln(x-2)$

domain $x > 2$

e)



7. (a) Differentiate with respect to x,

(i) $x^{\frac{1}{2}} \ln(3x)$

(ii) $\dfrac{1-10x}{(2x-1)^5}$, giving your answer in its simplest form.

(6)

(b) Given that $x = 3\tan 2y$ find $\dfrac{dy}{dx}$ in terms of x.

(5)

i) $u = x^{\frac{1}{2}}$ $\quad v = \ln 3x$ $\quad \dfrac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}\ln 3x + \dfrac{x^{\frac{1}{2}}}{x}$

$u' = \frac{1}{2}x^{-\frac{1}{2}}$ $\quad v' = \dfrac{3}{3x} = \dfrac{1}{x}$ $\quad = \dfrac{\ln 3x}{2x^{\frac{1}{2}}} + \dfrac{1}{\sqrt{x}}$

ii) $u = 1-10x$ $\quad v = (2x-1)^5$ $\quad \dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{vu' - uv'}{v^2}$

$u' = -10$ $\quad v' = 10(2x-1)^4$

$$\frac{dy}{dx} = \frac{-10(2x-1)^5 - 10(1-10x)(2x-1)^4}{(2x-1)^{10}}$$

$$= \frac{-10(2x-1) - 10(1-10x)}{(2x-1)^6} = \frac{80x}{(2x-1)^6}$$

iii) $x = 3\tan 2y$

$\dfrac{dx}{dy} = 6\sec^2 2y \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{6\sec^2 2y}$

$\therefore \dfrac{dy}{dx} = \frac{1}{6}\cos^2 2y$

$\tan 2y = \frac{x}{3} \Rightarrow 2y = \arctan\left(\frac{x}{3}\right) \Rightarrow y = \frac{1}{2}\arctan\left(\frac{x}{3}\right)$

$\therefore \dfrac{dy}{dx} = \frac{1}{6}\cos^2\left[\arctan\left(\frac{x}{3}\right)\right]$

iii) $x = 3\tan 2y$

$$\frac{dx}{dy} = 6\sec^2 2y \Rightarrow \frac{dy}{dx} = \frac{1}{6\sec^2 2y}$$

$$\frac{\sin^2}{\cos^2} + \frac{\cos^2}{\cos^2} = \frac{1}{\cos^2} \Rightarrow \tan^2 + 1 = \sec^2$$

$$\frac{x}{3} = \tan 2y \Rightarrow \tan^2 2y = \frac{x^2}{9} \Rightarrow \sec^2 2y = 1 + \frac{x^2}{9}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{6\left(1+\frac{x^2}{9}\right)} = \frac{9}{6(9+x^2)} = \frac{3}{2(9+x^2)}$$

8.

$$f(x) = 7\cos 2x - 24\sin 2x$$

Given that $f(x) = R\cos(2x + \alpha)$, where $R > 0$ and $0 < \alpha < 90^\circ$,

(a) find the value of R and the value of α.

(3)

(b) Hence solve the equation

$$7\cos 2x - 24\sin 2x = 12.5$$

for $0 \leqslant x < 180^\circ$, giving your answers to 1 decimal place.

(5)

(c) Express $14\cos^2 x - 48\sin x\cos x$ in the form $a\cos 2x + b\sin 2x + c$, where a, b, and c are constants to be found.

(2)

(d) Hence, using your answers to parts (a) and (c), deduce the maximum value of

$$14\cos^2 x - 48\sin x\cos x$$

(2)

$R\cos(2x+\alpha) = R\cos 2x\cos\alpha - R\sin 2x\sin\alpha$

$f(x) = 7\cos 2x - 24\sin 2x$

$\frac{R\sin\alpha = 24}{R\cos\alpha = 7} \Rightarrow \tan\alpha = \frac{24}{7}$ $\quad \alpha = 73.74$

$\alpha = \underline{73.7}$

$R = \sqrt{24^2 + 7^2}$ $\quad R = \underline{25}$

$f(x) = 25\cos(2x + 73.74)$

b) $25\cos(2x + 73.74) = 12.5$

$\Rightarrow \cos(2x + 73.74) = \frac{1}{2}$

$\Rightarrow 2x + 73.74 = 60^\circ, 300^\circ, 420^\circ$

$2x = -13.74, 226.26, 346.26$

$\therefore \quad x = \quad , 173.1 \; ; \; 113.1$

b) $14\cos^2 x - 7 - 48\sin x\cos x + 7$

$7(2\cos^2 x - 1) - 24(2\sin x\cos x) + 7$

$7\cos 2x - 24\sin 2x + 7$

d) $25\cos(2x + 73.74) + 7$

max value $= 25 + 7 = \underline{32}$