C3 June 2008
Model Solutions

① $y = 4e^{2x+1}$

a) If $y = 8$, $\quad 8 = 4e^{2x+1}$

$2 = e^{2x+1}$

$\ln 2 = 2x+1 \Rightarrow \underline{\frac{\ln 2 - 1}{2} = x}$

b) $m_{tangent} = \left.\frac{dy}{dx}\right|_{x = \frac{\ln 2 - 1}{2}}$

$\frac{dy}{dx} = 8e^{2x+1}$, when $x = \frac{\ln 2 -1}{2}$ $\; 2x+1 = \ln 2$

$\Rightarrow \frac{dy}{dx} = 8e^{\ln 2} = 16.$

Tangent has equation $\quad y - 8 = 16\left(x - \frac{\ln 2 - 1}{2}\right)$

$y = 16x - (8\ln 2 - 8) + 8$

$\underline{y = 16x + 16 - 8\ln 2}$

$a = 16, \quad b = 16 - 8\ln 2$

② a) $f(x) = 5\cos x + 12\sin x$

$f(x) = R\cos(x-\alpha)$

$= R\cos x\cos\alpha + R\sin x\sin\alpha$

$\left.\right\}$ $R\cos\alpha = 5$, $R\sin\alpha = 12$

$R^2 = 5^2 + 12^2 \Rightarrow \underline{R = 13}$

$\tan\alpha = \frac{12}{5} \Rightarrow \underline{\alpha = 1.176}$

$f(x) = 13\cos(x - 1.176)$

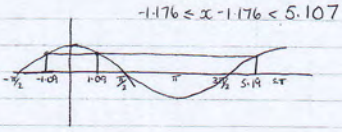
b) $5\cos x + 12\sin x = 6$ $\qquad 0 \leq x < 2\pi$

$13\cos(x - 1.176) = 6$ $\qquad -1.176 \leq x - 1.176 < 5.107$

$\cos(x-1.176) = \frac{6}{13}$

$x - 1.176 = \cos^{-1}\left(\frac{6}{13}\right) = 1.09,$

-1.09



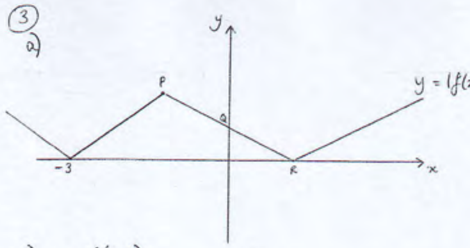
$\Rightarrow x = \underline{0.086}, \underline{2.266}$

c) i) Maximum value $= 13$

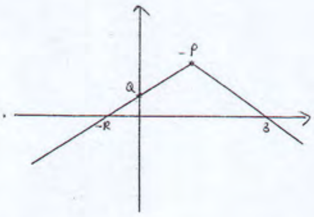
ii) Maximum of $\cos x$ is at $x = 0$ or $x = 2\pi$

$\therefore$ maximum of $\cos(x - 1.176)$ is at $\underline{x = 1.176}$

③ a)



b) $y = f(-x)$



c) $f(x) = 2 - |x+1|$

At R, $y = 0$: $\quad 2 - |x+1| = 0$

$|x+1| = 2$

$\Rightarrow x = 1$ $\qquad \underline{R = (1,0)}$

At Q, $x = 0$: $\quad 2 - |0+1| = y$

$y = 2 - 1 = 1$ $\qquad \underline{Q = (0,1)}$

$f(x)$ takes $|x+1|$, which has minimum value at $(-1,0)$ and reflects in x-axis before moving up 2, $\therefore$ P is the point $\underline{(-1,2)}$

d) $f(x) = \frac{1}{2}x$ Intersection is in 2 places:

Ⓐ $\quad 2 - (x+1) = \frac{1}{2}x$

$1 = \frac{3}{2}x \Rightarrow \underline{x = \frac{2}{3}}$

Ⓑ $\quad 2 - -(x+1) = \frac{1}{2}x$

$2 + (x+1) = \frac{1}{2}x$

$\frac{1}{2}x = -3$

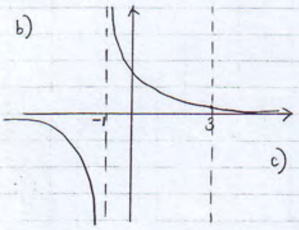
$\underline{x = -6}$

④ a) $f(x) = \frac{2(x-1)}{(x-3)(x+1)} - \frac{1}{(x-3)}$

$= \frac{2(x-1)}{(x-3)(x+1)} - \frac{(x+1)}{(x-3)(x+1)} = \frac{2(x-1)-(x+1)}{(x-3)(x+1)}$

$= \frac{\cancel{x-3}}{\cancel{(x-3)}(x+1)} = \underline{\frac{1}{x+1}}$ QED

b)



When $x = 3$, $y = \frac{1}{3+1} = \frac{1}{4}$.

$\therefore$ Range is $\underline{0 < f(x) < \frac{1}{4}}$

c) $y = \frac{1}{x+1} \qquad x = \frac{1}{y+1} \Rightarrow xy + x = 1$

$xy = 1 - x$

$f^{-1}(x) = \underline{y = \frac{1-x}{x}}$

Domain is $\underline{0 < x < \frac{1}{4}}$

d) $g(x) = 2x^2 - 3 \qquad fg(x) = \frac{1}{(2x^2-3)+1} = \frac{1}{2x^2-2}$

$fg(x) = \frac{1}{8}$

$\Rightarrow \frac{1}{2x^2-2} = \frac{1}{8} \Rightarrow 2x^2 - 2 = 8 \Rightarrow 2x^2 = 10$

$\underline{x = \pm\sqrt{5}}$

⑤ a) $\sin^2\theta + \cos^2\theta \equiv 1$

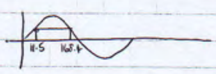
$(\div \sin^2\theta) \quad \frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} \equiv \frac{1}{\sin^2\theta} \Rightarrow \underline{1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta}$ QED

b) $2\cot^2\theta - 9\operatorname{cosec}\theta = 3$

$2(\operatorname{cosec}^2\theta - 1) - 9\operatorname{cosec}\theta = 3$

$2\operatorname{cosec}^2\theta - 9\operatorname{cosec}\theta - 5 = 0$

$(2\operatorname{cosec}\theta + 1)(\operatorname{cosec}\theta - 5) = 0$



$\Rightarrow \operatorname{cosec}\theta = -\frac{1}{2}$ or $\operatorname{cosec}\theta = 5$

$\sin\theta = -2$ ✗ $\qquad \sin\theta = \frac{1}{5} \Rightarrow \underline{\theta = 11.5°}, \underline{168.4°}$

⑥ a) i) $y = e^{3x}(\sin x + 2\cos x) \qquad u = e^{3x} \qquad v = \sin x + 2\cos x$

$y' = e^{3x}(\cos x - 2\sin x) + 3e^{3x}(\sin x + 2\cos x) \quad u' = 3e^{3x} \qquad v' = \cos x - 2\sin x$

$= e^{3x}(\cos x - 2\sin x + 3\sin x + 6\cos x)$

$= \underline{e^{3x}(7\cos x + \sin x)}$

⑥ a)

ii) $y = x^3\ln(5x+2) \qquad u = x^3 \quad v = \ln(5x+2)$

$u' = 3x^2 \quad v' = \frac{5}{5x+2}$

$y' = \frac{5x^3}{5x+2} + 3x^2(\ln(5x+2)) = \underline{x^2\left(\frac{5x}{5x+2} + 3\ln(5x+2)\right)}$

b) $y = \frac{3x^2+6x-7}{(x+1)^2} \qquad u = 3x^2+6x-7 \qquad v = (x+1)^2 \qquad v^2 = (x+$

$u' = 6x+6 = 6(x+1) \qquad v' = 2(x+1)$

$y' = \frac{6(x+1)^3 - 2(x+1)(3x^2+6x-7)}{(x+1)^4} = \frac{\cancel{(x+1)}[6(x+1)^2 - 2(3x^2+6x-7)]}{(x+1)^{\cancel{4}\,3}}$

$= \frac{6(x^2+2x+1) - 2(3x^2+6x-7)}{(x+1)^3}$

$= \underline{\frac{20}{(x+1)^3}}$ QED

c) $\frac{dy}{dx} = 20(x+1)^{-3}$

$\frac{d^2y}{dx^2} = -60(x+1)^{-4} = \frac{-60}{(x+1)^4}$

If $\frac{-60}{(x+1)^4} = -\frac{15}{4} \Rightarrow (x+1)^4 = 16$

$x + 1 = \pm 2$

$\underline{x = 1}$ or $\underline{x = -3}$

⑦ $f(x) = 3x^3 - 2x - 6$

a) $f(1.4) = -0.568 \; (<0)$

$f(1.45) = 0.245875 \; (>0)$ $\left.\right\}$ Change of sign $\Rightarrow$ root between 1.4 and 1.45.

b) $3x^3 - 2x - 6 = 0 \qquad 3x^3 = 2x + 6$

$x^3 = \frac{2x}{3} + 2$

$x^2 = \frac{2}{3} + \frac{2}{x} \Rightarrow x = \sqrt{\frac{2}{x} + \frac{2}{3}}$ QED.

c) $x_0 = 1.43$

$x_1 = 1.4371$

$x_2 = 1.4347$

$x_3 = 1.4355$

d) $1.4345 < 1.435 < 1.4355$

$f(1.4345) = -0.013 \; (<0)$

$f(1.4355) = 0.0032 \; (>0)$ Change of sign $\Rightarrow$ root in interval.