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Edexcel Modular Mathematics for AS and A-Level

Geometric sequences and series
Exercise A, Question 1

Question:

Which of the following are geometric sequences? For the ones that are, give the value of r in the sequence:

(a) $1, 2, 4, 8, 16, 32, \ldots$

(b) $2, 5, 8, 11, 14, \ldots$

(c) $40, 36, 32, 28, \ldots$

(d) $2, 6, 18, 54, 162, \ldots$

(e) $10, 5, 2.5, 1.25, \ldots$

(f) $5, -5, 5, -5, 5, \ldots$

(g) $3, 3, 3, 3, 3, 3, 3, \ldots$

(h) $4, -1, 0.25, -0.0625, \ldots$

Solution:

(a) $1 \xrightarrow{\times 2} 2 \xrightarrow{\times 2} 4 \xrightarrow{\times 2} 8 \xrightarrow{\times 2} 16 \xrightarrow{\times 2} 32$

Geometric $r = 2$

(b) $2 \xrightarrow{+3} 5 \xrightarrow{+3} 8 \xrightarrow{+3} 11 \xrightarrow{+3} 14$

Not geometric (this is an arithmetic sequence)

(c) $40 \xrightarrow{-4} 36 \xrightarrow{-4} 32 \xrightarrow{-4} 28$

Not geometric (arithmetic)

(d) $2 \xrightarrow{\times 3} 6 \xrightarrow{\times 3} 18 \xrightarrow{\times 3} 54$

Geometric $r = 3$

(e) $10 \xrightarrow{\times \frac{1}{2}} 5 \xrightarrow{\times \frac{1}{2}} 2.5 \xrightarrow{\times \frac{1}{2}} 1.25$

Geometric $r = \frac{1}{2}$

(f) $5 \xrightarrow{\times -1} -5 \xrightarrow{\times -1} 5 \xrightarrow{\times -1} -5$

Geometric $r = -1$

(g) $3 \xrightarrow{\times 1} 3 \xrightarrow{\times 1} 3 \xrightarrow{\times 1} 3 \xrightarrow{\times 1} 3$

Geometric $r = 1$

(h) $4 \xrightarrow{\times -\frac{1}{4}} -1 \xrightarrow{\times -\frac{1}{4}} 0.25 \xrightarrow{\times -\frac{1}{4}} -0.0625$

Geometric $r = -\frac{1}{4}$

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Geometric sequences and series
Exercise A, Question 2

Question:

Continue the following geometric sequences for three more terms:

(a) 5, 15, 45, …

(b) 4, − 8, 16, …

(c) 60, 30, 15, …

(d) 1, $\frac{1}{4}$, $\frac{1}{16}$, …

(e) 1, p, p^2, …

(f) x, $-2x^2$, $4x^3$, …

Solution:

(a) $5 \xrightarrow{\times 3} 15 \xrightarrow{\times 3} 45 \xrightarrow{\times 3} 135 \xrightarrow{\times 3} 405 \xrightarrow{\times 3} 1215$

(b) $4 \xrightarrow{\times -2} -8 \xrightarrow{\times -2} 16 \xrightarrow{\times -2} -32 \xrightarrow{\times -2} 64 \xrightarrow{\times -2} -128$

(c) $60 \xrightarrow{\times \frac{1}{2}} 30 \xrightarrow{\times \frac{1}{2}} 15 \xrightarrow{\times \frac{1}{2}} 7.5 \xrightarrow{\times \frac{1}{2}} 3.75 \xrightarrow{\times \frac{1}{2}} 1.875$

(d) $1 \xrightarrow{\times \frac{1}{4}} \frac{1}{4} \xrightarrow{\times \frac{1}{4}} \frac{1}{16} \xrightarrow{\times \frac{1}{4}} \frac{1}{64} \xrightarrow{\times \frac{1}{4}} \frac{1}{256} \xrightarrow{\times \frac{1}{4}} \frac{1}{1024}$

(e) $1 \xrightarrow{\times p} p \xrightarrow{\times p} p^2 \xrightarrow{\times p} p^3 \xrightarrow{\times p} p^4 \xrightarrow{\times p} p^5$

(f) $x \xrightarrow{\times -2x} -2x^2 \xrightarrow{\times -2x} 4x^3 \xrightarrow{\times -2x} -8x^4 \xrightarrow{\times -2x} 16x^5 \xrightarrow{\times -2x} -32x^6$

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Geometric sequences and series
Exercise A, Question 3

Question:

If 3, x and 9 are the first three terms of a geometric sequence. Find:

(a) The exact value of x.

(b) The exact value of the 4th term.

Solution:

(a) 3 x 9

Common ratio $= \frac{\text{term } 2}{\text{term } 1}$ or $\frac{\text{term } 3}{\text{term } 2} = \frac{x}{3}$ or $\frac{9}{x}$

Therefore,

$\frac{x}{3} = \frac{9}{x}$ (cross multiply)

$x^2 = 27$ (√)

$x = \sqrt{27}$

$x = \sqrt{9 \times 3}$

$x = 3\sqrt{3}$

(b) Term 4 = term 3 × r

Term 3 = 9 and $r = \frac{\text{term } 2}{\text{term } 1} = \frac{3\sqrt{3}}{3} = \sqrt{3}$

So term 4 = $9\sqrt{3}$

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Geometric sequences and series
Exercise B, Question 1

Question:

Find the sixth, tenth and *n*th terms of the following geometric sequences:

(a) 2, 6, 18, 54, …

(b) 100, 50, 25, 12.5, …

(c) 1, − 2, 4, − 8, …

(d) 1, 1.1, 1.21, 1.331, …

Solution:

(a) $2 \xrightarrow{\times 3} 6 \xrightarrow{\times 3} 18 \xrightarrow{\times 3} 54$

In this series $a = 2$ and $r = 3$
6th term $= ar^{6-1} = ar^5 = 2 \times 3^5 = 486$
10th term $= ar^{10-1} = ar^9 = 2 \times 3^9 = 39366$
nth term $= ar^{n-1} = 2 \times 3^{n-1}$

(b) $100 \xrightarrow{\times \frac{1}{2}} 50 \xrightarrow{\times \frac{1}{2}} 25 \xrightarrow{\times \frac{1}{2}} 12.5$

In this series $a = 100$, $r = \frac{1}{2}$

6th term $= ar^{6-1} = ar^5 = 100 \times \left(\frac{1}{2} \right)^5 = \frac{25}{8}$

10th term $= ar^{10-1} = ar^9 = 100 \times \left(\frac{1}{2} \right)^9 = \frac{25}{128}$

nth term $= ar^{n-1} = 100 \times \left(\frac{1}{2} \right)^{n-1} = \frac{4 \times 25}{2^{n-1}} = \frac{25}{2^{n-3}}$

(c) $1 \xrightarrow{\times -2} -2 \xrightarrow{\times -2} 4 \xrightarrow{\times -2} -8$

In this series $a = 1$ and $r = -2$
6th term $= ar^{6-1} = ar^5 = 1 \times (-2)^5 = -32$
10th term $= ar^{10-1} = ar^9 = 1 \times (-2)^9 = -512$
nth term $= ar^{n-1} = 1 \times (-2)^{n-1} = (-2)^{n-1}$

(d) $1 \xrightarrow{\times 1.1} 1.1 \xrightarrow{\times 1.1} 1.21 \xrightarrow{\times 1.1} 1.331$

In this series $a = 1$ and $r = 1.1$
6th term is $ar^{6-1} = ar^5 = 1 \times (1.1)^5 = 1.61051$
10th term is $ar^{10-1} = ar^9 = 1 \times (1.1)^9 = 2.35795$ (5 d.p.)
nth term is $ar^{n-1} = 1 \times (1.1)^{n-1} = (1.1)^{n-1}$

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Geometric sequences and series

Exercise B, Question 2

Question:

The nth term of a geometric sequence is $2 \times \left(5 \right)^{n}$. Find the first and 5th terms.

Solution:

nth term $= 2 \times \left(5 \right)^{n}$
1st term $\left(n = 1 \right) = 2 \times 5^{1} = 10$
5th term $\left(n = 5 \right) = 2 \times 5^{5} = 2 \times 3125 = 6250$

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Geometric sequences and series
Exercise B, Question 3

Question:

The sixth term of a geometric sequence is 32 and the 3rd term is 4. Find the first term and the common ratio.

Solution:

Let the first term $= a$ and common ratio $= r$
6th term is 32
$\Rightarrow \quad ar^{6-1} = 32$
$\Rightarrow \quad ar^5 = 32$ ①
3rd term is 4
$\Rightarrow \quad ar^{3-1} = 4$
$\Rightarrow \quad ar^2 = 4$ ②
① ÷ ②:

$$\frac{\cancel{a}r^5}{\cancel{a}r^2} = \frac{32}{4}$$
$r^3 = 8$
$r = 2$
Common ratio is 2

Substitute $r = 2$ into equation ②
$a \times 2^2 = 4$
$a \times 4 = 4 \quad (\div 4)$
$a = 1$
First term is 1

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Geometric sequences and series

Exercise B, Question 4

Question:

Given that the first term of a geometric sequence is 4, and the third is 1, find possible values for the 6th term.

Solution:

First term is 4 $\Rightarrow$ $a = 4$ ①

Third term is 1 $\Rightarrow$ $ar^{3-1} = 1$ $\Rightarrow$ $ar^2 = 1$ ②

Substitute $a = 4$ into ②

$4r^2 = 1$ $(\div 4)$

$r^2 = \frac{1}{4}$ $\left(\sqrt{} \right)$

$r = \pm \frac{1}{2}$

The sixth term $= ar^{6-1} = ar^5$

If $r = \frac{1}{2}$ then sixth term $= 4 \times \left(\frac{1}{2} \right)^5 = \frac{1}{8}$

If $r = -\frac{1}{2}$ then sixth term $= 4 \times \left(-\frac{1}{2} \right)^5 = -\frac{1}{8}$

Possible values for sixth term are $\frac{1}{8}$ and $-\frac{1}{8}$.

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Geometric sequences and series
Exercise B, Question 5

Question:

The expressions $x - 6$, $2x$ and x^2 form the first three terms of a geometric progression. By calculating two different expressions for the common ratio, form and solve an equation in x to find possible values of the first term.

Solution:

If $x - 6$, $2x$ and x^2 are terms in a geometric progression then

$\frac{2x}{x-6} = \frac{x^2}{2x}$ (cancel first)

$\frac{2x}{x-6} = \frac{x}{2}$ (cross multiply)

$4x = x(x - 6)$
$4x = x^2 - 6x$
$0 = x^2 - 10x$
$0 = x(x - 10)$
$x = 0$ or 10
If $x = 0$ then first term $= 0 - 6 = -6$
If $x = 10$ then first term $= 10 - 6 = 4$

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Geometric sequences and series
Exercise C, Question 1

Question:

A population of ants is growing at a rate of 10% a year. If there were 200 ants in the initial population, write down the number after

(a) 1 year,

(b) 2 years,

(c) 3 years and

(d) 10 years.

Solution:

A growth of 10% a year gives a multiplication factor of 1.1.

(a) After 1 year number is $200 \times 1.1 = 220$

(b) After 2 years number is $200 \times 1.1^2 = 242$

(c) After 3 years number is
$200 \times 1.1^3 = 266.2 = 266$ (to nearest whole number)

(d) After 10 years number is
$200 \times 1.1^{10} = 518.748 \quad \ldots \quad = 519$ (to nearest whole number)

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Geometric sequences and series
Exercise C, Question 2

Question:

A motorcycle has four gears. The maximum speed in bottom gear is 40 km h^{-1} and the maximum speed in top gear is 120 km h^{-1}. Given that the maximum speeds in each successive gear form a geometric progression, calculate, in km h^{-1} to one decimal place, the maximum speeds in the two intermediate gears.

[E]

Solution:

Let maximum speed in bottom gear be a km h^{-1}
This gives maximum speeds in each successive gear to be
ar ar^2 ar^3
Where r is the common ratio.
We are given
$a = 40$ ①
$ar^3 = 120$ ②
Substitute ① into ②:
$40r^3 = 120$ (÷ 40)
$r^3 = 3$
$r = \sqrt[3]{3}$
$r = 1.442 \ldots$ (3 d.p.)

Maximum speed in 2nd gear is
ar = $40 \times 1.442 \ldots = 57.7$ km h^{-1}
Maximum speed in 3rd gear is
$ar^2 = 40 \times (1.442 \ldots)^2 = 83.2$ km h^{-1}

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Geometric sequences and series
Exercise C, Question 3

Question:

A car depreciates in value by 15% a year. If it is worth £11 054.25 after 3 years, what was its new price and when will it first be worth less than £5000?

Solution:

Let the car be worth £*A* when new.
If it depreciates by 15% each year the multiplication factor is 0.85 for every year.
We are given
price after 3 years is £11 054.25

$\Rightarrow \quad A \times (0.85)^3 = 11\,054.25$

$\Rightarrow \quad A = \frac{11\,054.25}{(0.85)^3} = 18\,000$

Its new price is £18 000

If its value is less than £5000
$18\,000 \times (0.85)^n < 5000$

$(0.85)^n < \frac{5000}{18\,000}$

$\log (0.85)^n < \log \left(\frac{5000}{18\,000} \right)$

$n \log \left(0.85 \right) < \log \left(\frac{5000}{18\,000} \right)$

$n > \frac{\log \left(\frac{5000}{18\,000} \right)}{\log (0.85)}$

Note: < changes to > because log (0.85) is negative.
So $n > 7.88$
n must be an integer.
So number of years is 8.

It is often easier to solve these problems using an equality rather than an inequality.
E.g. solve $18\,000 \times (0.85)^n = 5000$

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Geometric sequences and series
Exercise C, Question 4

Question:

The population decline in a school of whales can be modelled by a geometric progression. Initially there were 80 whales in the school. Four years later there were 40. Find out how many there will be at the end of the fifth year. (Round to the nearest whole number.)

Solution:

Let the common ratio be r—the multiplication factor.
Initially there are 80 whales
After 1 year there is $80r$
After 2 years there will be $80r^2$
After 3 years there will be $80r^3$
After 4 years there will be $80r^4$
We are told this number is 40

$80r^4 = 40 \quad (\div 80)$

$r^4 = \frac{40}{80}$

$r^4 = \frac{1}{2}$

$r = \sqrt[4]{\frac{1}{2}}$

$r = 0.840896 \quad \ldots$

After 5 years there will be
$40 \times 0.840896 \quad \ldots \quad = 33.635 \quad \ldots \quad = 34$ whales

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Geometric sequences and series
Exercise C, Question 5

Question:

Find which term in the progression 3, 12, 48, … is the first to exceed 1 000 000.

Solution:

$$3 \xrightarrow{\times 4} 12 \xrightarrow{\times 4} 48 \ \ldots$$

This is a geometric series with $a = 3$ and $r = 4$.
If the term exceeds 1 000 000 then
$ar^{n-1} > 1\,000\,000$
Substitute $a = 3$, $r = 4$
$3 \times 4^{n-1} > 1\,000\,000$

$$4^{n-1} > \frac{1\,000\,000}{3}$$

$$\log 4^{n-1} > \log\left(\frac{1\,000\,000}{3}\right)$$

$$\left(n - 1\right)\log 4 > \log\left(\frac{1\,000\,000}{3}\right)$$

$$\left(n - 1\right) > \frac{\log\left(\frac{1\,000\,000}{3}\right)}{\log 4}$$

$n - 1 > 9.173 \ldots$
$n > 10.173 \ldots$
So $n = 11$

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Geometric sequences and series
Exercise C, Question 6

Question:

A virus is spreading such that the number of people infected increases by 4% a day. Initially 100 people were diagnosed with the virus. How many days will it be before 1000 are infected?

Solution:

If the number of people infected increases by 4% the multiplication factor is 1.04.
After n days $100 \times (1.04)^n$ people will be infected.
If 1000 people are infected
$100 \times (1.04)^n = 1000$
$(1.04)^n = 10$
$\log (1.04)^n = \log 10$
$n \log (1.04) = 1$
$n = \frac{1}{\log (1.04)}$
$n = 58.708 \ldots$
It would take 59 days.

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Geometric sequences and series

Exercise C, Question 7

Question:

I invest £A in the bank at a rate of interest of 3.5% per annum. How long will it be before I double my money?

Solution:

If the increase is 3.5% per annum the multiplication factor is 1.035.
Therefore after n years I will have £ $A \times (1.035)^n$
If the money is doubled it will equal $2A$, therefore

$A \times (1.035)^n = 2A$

$(1.035)^n = 2$

$\log (1.035)^n = \log 2$

$n \log (1.035) = \log 2$

$n = \frac{\log 2}{\log (1.035)} = 20.14879 \ldots$

My money will double after 20.15 years.

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Geometric sequences and series
Exercise C, Question 8

Question:

The fish in a particular area of the North Sea are being reduced by 6% each year due to overfishing. How long would it be before the fish stocks are halved?

Solution:

The reduction is 6% which gives a multiplication factor of 0.94.
Let the number of fish now be F.
After n years there will be $F \times (0.94)^n$

When their number is halved the number will be $\frac{1}{2}F$

Set these equal to each other:

$$F \times (0.94)^n = \frac{1}{2}F$$

$$(0.94)^n = \frac{1}{2}$$

$$\log (0.94)^n = \log \left(\frac{1}{2} \right)$$

$$n \log \left(0.94 \right) = \log \left(\frac{1}{2} \right)$$

$$n = \frac{\log \left(\frac{1}{2} \right)}{\log (0.94)}$$

$n = 11.2$
The fish stocks will half in 11.2 years.

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Geometric sequences and series
Exercise D, Question 1

Question:

Find the sum of the following geometric series (to 3 d.p. if necessary):

(a) $1 + 2 + 4 + 8 + \ldots$ (8 terms)

(b) $32 + 16 + 8 + \ldots$ (10 terms)

(c) $4 - 12 + 36 - 108 \ldots$ (6 terms)

(d) $729 - 243 + 81 - \ldots - \frac{1}{3}$

(e) $\sum_{r=1}^{6} 4^r$

(f) $\sum_{r=1}^{8} 2 \times (3)^r$

(g) $\sum_{r=1}^{10} 6 \times \left(\frac{1}{2} \right)^r$

(h) $\sum_{r=0}^{5} 60 \times \left(-\frac{1}{3} \right)^r$

Solution:

(a) $1 + 2 + 4 + 8 + \ldots$ (8 terms)
In this series $a = 1$, $r = 2$, $n = 8$.

As $|r| > 1$ use $S_n = \frac{a(r^n - 1)}{r - 1}$.

$$S_8 = \frac{a(r^8 - 1)}{r - 1} = \frac{1 \times (2^8 - 1)}{2 - 1} = 256 - 1 = 255$$

(b) $32 + 16 + 8 + \ldots$ (10 terms)

In this series $a = 32$, $r = \frac{1}{2}$, $n = 10$.

As $|r| < 1$ use $S_n = \frac{a(1 - r^n)}{1 - r}$.

$$S_{10} = \frac{a(1-r^{10})}{1-r} = \frac{32\left[1-\left(\frac{1}{2}\right)^{10}\right]}{1-\frac{1}{2}} = 63.938 \text{ (3 d.p.)}$$

(c) $4 - 12 + 36 - 108 + \ldots$ (6 terms)
In this series $a = 4$, $r = -3$, $n = 6$.

As $|r| > 1$ use $S_n = \frac{a(r^n - 1)}{r-1}$.

$$S_6 = \frac{a(r^6-1)}{r-1} = \frac{4\left[(-3)^6 - 1\right]}{-3-1} = -728$$

(d) $729 - 243 + 81 - \ldots - \frac{1}{3}$

In this series $a = 729$, $r = \frac{-243}{729} = -\frac{1}{3}$ and the nth term is $-\frac{1}{3}$.

Using nth term $= ar^{n-1}$

$$-\frac{1}{3} = 729 \times \left(-\frac{1}{3}\right)^{n-1}$$

$$-\frac{1}{2187} = \left(-\frac{1}{3}\right)^{n-1}$$

$$\left(-\frac{1}{3}\right)^{7} = \left(-\frac{1}{3}\right)^{n-1}$$

So $n - 1 = 7$
$\Rightarrow \quad n = 8$
There are 8 terms in the series.

As $|r| < 1$ use $S_n = \frac{a(1-r^n)}{1-r}$ with $a = 729$, $r = -\frac{1}{3}$ and $n = 8$.

$$S_8 = \frac{729\left[1-\left(-\frac{1}{3}\right)^8\right]}{1-\left(-\frac{1}{3}\right)} = 546\frac{2}{3}$$

(e) $\sum_{r=1}^{6} 4^r = 4^1 + 4^2 + 4^3 + \ldots + 4^6$

A geometric series with $a = 4$, $r = 4$ and $n = 6$.

Use $S_n = \frac{a(r^n-1)}{r-1}$.

$$\sum_{r=1}^{6} 4^r = \frac{4(4^6-1)}{4-1} = 5460$$

(f) $\sum_{r=1}^{8} 2 \times (3)^r$

$= 2 \times 3^1 + 2 \times 3^2 + 2 \times 3^3 + \ldots + 2 \times 3^8$

$$= 2 \times \underbrace{(3^1 + 3^2 + 3^3 + \ldots + 3^8)}$$

A geometric series with $a = 3$, $r = 3$ and $n = 8$.

Use $S_n = \frac{a(r^n - 1)}{r - 1}$.

$$\sum_{r=1}^{8} 2 \times (3)^r = 2 \times \left[\frac{3(3^8 - 1)}{3 - 1} \right] = 19680$$

(g) $\sum_{r=1}^{10} 6 \times \left(\frac{1}{2} \right)^r$

$$= 6 \times \left(\frac{1}{2} \right)^1 + 6 \times \left(\frac{1}{2} \right)^2 + \ldots + 6 \times \left(\frac{1}{2} \right)^{10}$$

$$= 6 \times \underbrace{\left[\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \ldots + \left(\frac{1}{2}\right)^{10} \right]}$$

A geometric series with $a = \frac{1}{2}$, $r = \frac{1}{2}$ and $n = 10$.

Use $S_n = \frac{a(1 - r^n)}{1 - r}$

$$\sum_{r=1}^{10} 6 \times \left(\frac{1}{2} \right)^r = 6 \times \frac{\frac{1}{2}\left[1 - \left(\frac{1}{2}\right)^{10}\right]}{1 - \frac{1}{2}} = 5.994 \text{ (3 d.p.)}$$

(h) $\sum_{r=0}^{5} 60 \times \left(-\frac{1}{3} \right)^r$

$$= 60 \times \left(-\frac{1}{3} \right)^0 + 60 \times \left(-\frac{1}{3} \right)^1 + \ldots + 60 \times \left(-\frac{1}{3} \right)^5$$

$$= 60 \times \left[\left(-\frac{1}{3} \right)^0 + \left(-\frac{1}{3} \right)^1 + \ldots + \left(-\frac{1}{3} \right)^5 \right]$$

$$= 60 \times \underbrace{\left(1 - \frac{1}{3} + \frac{1}{9} \ldots - \frac{1}{243}\right)}$$

A geometric series with $a = 1$, $r = -\frac{1}{3}$ and $n = 6$.

Use $S_n = \frac{a(1 - r^n)}{1 - r}$

$$\sum_{r=0}^{5} 60 \times \left(-\frac{1}{3} \right)^r = 60 \times \frac{1\left[1 - \left(-\frac{1}{3}\right)^6\right]}{1 - \left(-\frac{1}{3}\right)} = 44.938 \text{ (3 d.p.)}$$

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Geometric sequences and series
Exercise D, Question 2

Question:

The sum of the first three terms of a geometric series is 30.5. If the first term is 8, find possible values of r.

Solution:

Let the common ratio be r
The first three terms are 8, $8r$ and $8r^2$.
Given that the first three terms add up to 30.5
$8 + 8r + 8r^2 = 30.5 \quad (\times 2)$
$16 + 16r + 16r^2 = 61$
$16r^2 + 16r - 45 = 0$
$(4r - 5) (4r + 9) = 0$
$r = \frac{5}{4}, \frac{-9}{4}$

Possible values of r are $\frac{5}{4}$ and $\frac{-9}{4}$.

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Geometric sequences and series
Exercise D, Question 3

Question:

The man who invented the game of chess was asked to name his reward. He asked for 1 grain of corn to be placed on the first square of his chessboard, 2 on the second, 4 on the third and so on until all 64 squares were covered. He then said he would like as many grains of corn as the chessboard carried. How many grains of corn did he claim as his prize?

Solution:

Number of grains = $\underbrace{1+2+4+8+\ldots}_{64\text{ terms}}$

This is a geometric series with $a = 1$, $r = 2$ and $n = 64$.

As $| r | > 1$ use $S_n = \frac{a(r^n - 1)}{r - 1}$.

Number of grains = $\frac{1(2^{64} - 1)}{2 - 1} = 2^{64} - 1$

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Geometric sequences and series
Exercise D, Question 4

Question:

Jane invests £4000 at the start of every year. She negotiates a rate of interest of 4% per annum, which is paid at the end of the year. How much is her investment worth at the end of (a) the 10th year and (b) the 20th year?

Solution:

Start of year 1 Jane has £4000
End of year 1 Jane has 4000×1.04
Start of year 2 Jane has $4000 \times 1.04 + 4000$
End of year 2 Jane has $(4000 \times 1.04 + 4000) \times 1.04$
$= 4000 \times 1.04^2 + 4000 \times 1.04$
$\vdots$
(a) End of year 10 Jane has
$4000 \times 1.04^{10} + 4000 \times 1.04^9 \quad \ldots \quad + \quad 4000 \times 1.04$
$= 4000 \times \underbrace{(1.04^{10}+1.04^9+ \ldots +1.04)}$

A geometric series with $a = 1.04$, $r = 1.04$ and $n = 10$.

$= 4000 \times \frac{1.04 (1.04^{10} - 1)}{1.04 - 1}$

= £49 945.41

(b) End of 20th year
$= 4000 \times \underbrace{(1.04^{20}+1.04^{19}+ \ldots +1.04)}$

A geometric series with $a = 1.04$, $r = 1.04$ and $n = 20$.

$= 4000 \times \frac{1.04 (1.04^{20} - 1)}{1.04 - 1}$

= £ 123 876.81

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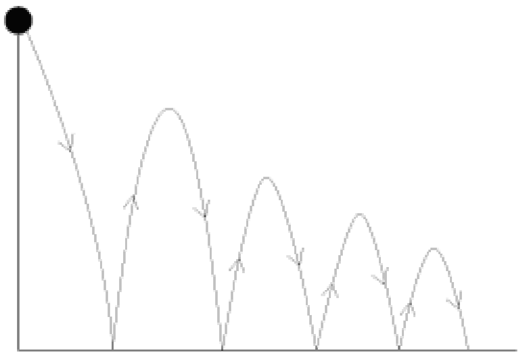
Geometric sequences and series
Exercise D, Question 5

Question:

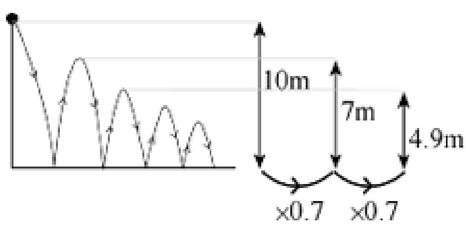
A ball is dropped from a height of 10 m. It bounces to a height of 7 m and continues to bounce. Subsequent heights to which it bounces follow a geometric sequence. Find out:



(a) How high it will bounce after the fourth bounce.

(b) The total distance travelled after it hits the ground for the sixth time.

Solution:



(a) After the first bounce it bounces to 7m
After the 2^{nd} bounce it bounces to 4.9m
After the 3^{rd} bounce it bounces to 3.43m
After the 4^{th} bounce it bounces to 2.401m

$$\left.\begin{matrix}7\\4.9\\3.43\\2.401\end{matrix}\right\} \times 0.7$$

(b) Total distance travelled

$$= \underset{1^{st}\text{ bounce}}{10} + \underset{2^{nd}\text{ bounce}}{7} + 7 + \underset{3^{rd}\text{ bounce}}{4.9} + 4.9 + \ldots$$

$$= 2 \times \underbrace{(10 + 7 + 4.9 + \ldots)}_{6\text{ terms}} - 10$$

$$a = 10,\ r = 0.7,\ n = 6$$

$$= 2 \times \frac{10(1 - 0.7^6)}{1 - 0.7} - 10$$

$= 48.8234$ m

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Geometric sequences and series
Exercise D, Question 6

Question:

Find the least value of n such that the sum $3 + 6 + 12 + 24 + \ldots$ to n terms would first exceed 1.5 million.

Solution:

$3 + 6 + 12 + 24 + \ldots$ is a geometric series with $a = 3$, $r = 2$.

So $S_n = \frac{a(r^n - 1)}{r - 1} = \frac{3(2^n - 1)}{2 - 1} = 3\left(2^n - 1\right)$

We want $S_n > 1.5$ million

$S_n > 1\ 500\ 000$

$3(2^n - 1) > 1\ 500\ 000$

$2^n - 1 > 500\ 000$

$2^n > 500\ 001$

$\log 2^n > \log 500\ 001$

$n \log 2 > \log 500\ 001$

$n > \frac{\log 500\ 001}{\log 2}$

$n > 18.9$

Least value of n is 19.

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Geometric sequences and series
Exercise D, Question 7

Question:

Find the least value of n such that the sum $5 + 4.5 + 4.05 + \ldots$ to n terms would first exceed 45.

Solution:

$5 + 4.5 + 4.05 + \ldots$ is a geometric series with $a = 5$ and $r = \frac{4.5}{5} = 0.9$.

Using $S_n = \frac{a(1 - r^n)}{1 - r} = \frac{5(1 - 0.9^n)}{1 - 0.9} = 50\left(1 - 0.9^n\right)$

We want $S_n > 45$

$50(1 - 0.9^n) > 45$

$\left(1 - 0.9^n\right) > \frac{45}{50}$

$1 - 0.9^n > 0.9$

$0.9^n < 0.1$

$\log(0.9)^n < \log(0.1)$

$n\log(0.9) < \log(0.1)$

$n > \frac{\log(0.1)}{\log(0.9)}$

$n > 21.85$

So $n = 22$

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Geometric sequences and series
Exercise D, Question 8

Question:

Richard is sponsored to cycle 1000 miles over a number of days. He cycles 10 miles on day 1, and increases this distance by 10% a day. How long will it take him to complete the challenge? What was the greatest number of miles he completed in a single day?

Solution:

Day one $= 10$ miles $\quad \times 1.1$
Day two $= 10 \times 1.1 = 11$ miles $\quad \times 1.1$
Day three $= 11 \times 1.1 = 12.1$ miles
$\vdots$

We want $\underbrace{10 + 11 + 12.1 + \dots}_{n \text{ days}} = 1000$

Use the sum formula $S_n = \frac{a(r^n - 1)}{r - 1}$ with $a = 10$, $r = 1.1$.

$$\frac{10(1.1^n - 1)}{1.1 - 1} = 1000$$

$$\frac{10(1.1^n - 1)}{0.1} = 1000$$

$1.1^n - 1 = 10$
$1.1^n = 11$
$\log 1.1^n = \log 11$
$n \log 1.1 = \log 11$
$n = \frac{\log 11}{\log 1.1}$
$n = 25.16$ days
It would take him 26 days to complete the challenge.

He would complete most miles on day 25
$= 10 \times 1.1^{24}$ (using ar^{n-1})
$= 98.5$ miles (3 s.f.)

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Geometric sequences and series
Exercise D, Question 9

Question:

A savings scheme is offering a rate of interest of 3.5% per annum for the lifetime of the plan. Alan wants to save up £20 000. He works out that he can afford to save £500 every year, which he will deposit on January 1st. If interest is paid on 31st of December, how many years will it be before he has saved up his £20 000?

Solution:

Jan. 1st year 1 = £500
Dec. 31st year 1 = 500×1.035
Jan. 1st year 2 = $500 \times 1.035 + 500$
Dec. 31st year 2 = $(500 \times 1.035 + 500) \times 1.035 = 500 \times 1.035^2 + 500 \times 1.035$
$\vdots$
Dec. 31st year n
$= 500 \times 1.035^n + \ldots + 500 \times 1.035^2 + 500 \times 1.035$
$= 500 \times \underbrace{(1.035^n + \ldots + 1.035^2 + 1.035)}$

A geometric series with $a = 1.035$, $r = 1.035$ and n.

Use $S_n = \frac{a (r^n - 1)}{r - 1}$.

Dec. 31st year n $= 500 \times \frac{1.035 (1.035^n - 1)}{1.035 - 1}$

Set this equal to £20 000

$20\,000 = 500 \times \frac{1.035 (1.035^n - 1)}{1.035 - 1}$

$\left(1.035^n - 1 \right) = \frac{20\,000 \times (1.035 - 1)}{500 \times 1.035}$

$1.035^n - 1 = 1.3526570 \ldots$
$1.035^n = 2.3526570 \ldots$
$\log (1.035^n) = \log 2.3526570 \ldots$
$n \log (1.035) = \log 2.3526570 \ldots$
$n = \frac{\log 2.3526570 \ldots}{\log 1.035}$

$n = 24.9$ years (3 s.f.)
It takes Alan 25 years to save £20 000.

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Geometric sequences and series
Exercise E, Question 1

Question:

Find the sum to infinity, if it exists, of the following series:

(a) $1 + 0.1 + 0.01 + 0.001 + \ldots$

(b) $1 + 2 + 4 + 8 + 16 + \ldots$

(c) $10 - 5 + 2.5 - 1.25 + \ldots$

(d) $2 + 6 + 10 + 14$

(e) $1 + 1 + 1 + 1 + 1 + \ldots$

(f) $3 + 1 + \frac{1}{3} + \frac{1}{9} + \ldots$

(g) $0.4 + 0.8 + 1.2 + 1.6 + \ldots$

(h) $9 + 8.1 + 7.29 + 6.561 + \ldots$

(i) $1 + r + r^2 + r^3 + \ldots$

(j) $1 - 2x + 4x^2 - 8x^3 + \ldots$

Solution:

(a) $1 + 0.1 + 0.01 + 0.001 + \ldots$
As $r = 0.1$, S_∞ exists.

$$S_\infty = \frac{a}{1-r} = \frac{1}{1-0.1} = \frac{1}{0.9} = \frac{10}{9}$$

(b) $1 + 2 + 4 + 8 + 16 + \ldots$
As $r = 2$, S_∞ does not exist.

(c) $10 - 5 + 2.5 - 1.25 + \ldots$
As $r = -\frac{1}{2}$, S_∞ exists.

$$S_\infty = \frac{a}{1-r} = \frac{10}{1-\left(-\frac{1}{2}\right)} = \frac{10}{\frac{3}{2}} = 10 \times \frac{2}{3} = \frac{20}{3} = 6\frac{2}{3}$$

(d) $2 + 6 + 10 + 14 + \ldots$
This is an arithmetic series.
S_∞ does not exist.

(e) $1 + 1 + 1 + 1 + 1 + \ldots$
As $r = 1$, S_∞ does not exist.

(f) $3 + 1 + \frac{1}{3} + \frac{1}{9} + \dots$

As $r = \frac{1}{3}$, S_∞ exists.

$$S_\infty = \frac{a}{1-r} = \frac{3}{1-\frac{1}{3}} = \frac{3}{\frac{2}{3}} = 3 \times \frac{3}{2} = \frac{9}{2} = 4\frac{1}{2}$$

(g) $0.4 + 0.8 + 1.2 + 1.6 + \dots$
This is an arithmetic series.

S_∞ does not exist.

(h) $9 + 8.1 + 7.29 + 6.561 + \dots$

As $r = \frac{8.1}{9} = 0.9$, S_∞ exists.

$$S_\infty = \frac{a}{1-r} = \frac{9}{1-0.9} = \frac{9}{0.1} = 90$$

(i) $1 + r + r^2 + r^3 + \dots$
S_∞ exists if $|r| < 1$.

$$S_\infty = \frac{1}{1-r} \text{ if } |r| < 1$$

(j) $1 - 2x + 4x^2 - 8x^3 + \dots$

As $r = -2x$, S_∞ exists if $\left| -2x \right| < 1 \Rightarrow \left| x \right| < \frac{1}{2}$.

$$S_\infty = \frac{1}{1-(-2x)} = \frac{1}{1+2x} \text{ if } \left| x \right| < \frac{1}{2}$$

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Geometric sequences and series
Exercise E, Question 2

Question:

Find the common ratio of a geometric series with a first term of 10 and a sum to infinity of 30.

Solution:

Substitute $a = 10$ and $S_{\infty} = 30$ into

$$S_{\infty} = \frac{a}{1 - r}$$

$$30 = \frac{10}{1 - r} \quad \times \left(1 - r \right)$$

$$30 \left(1 - r \right) = 10 \quad \left(\div 30 \right)$$

$$1 - r = \frac{10}{30}$$

$$1 - r = \frac{1}{3}$$

$$1 = \frac{1}{3} + r$$

$$\frac{2}{3} = r$$

The common ratio is $\frac{2}{3}$.

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Geometric sequences and series
Exercise E, Question 3

Question:

Find the common ratio of a geometric series with a first term of -5 and a sum to infinity of -3.

Solution:

Substitute $a = -5$ and $S_\infty = -3$ into

$$S_\infty = \frac{a}{1-r}$$

$$-3 = \frac{-5}{1-r}$$

$$-3(1-r) = -5$$

$$1 - r = \frac{-5}{-3}$$

$$1 - r = +\frac{5}{3}$$

$$1 = \frac{5}{3} + r$$

$$1 - \frac{5}{3} = r$$

$$-\frac{2}{3} = r$$

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Geometric sequences and series
Exercise E, Question 4

Question:

Find the first term of a geometric series with a common ratio of $\frac{2}{3}$ and a sum to infinity of 60.

Solution:

Substitute $r = \frac{2}{3}$ and $S_{\infty} = 60$ into

$$S_{\infty} = \frac{a}{1 - r}$$

$$60 = \frac{a}{1 - \frac{2}{3}}$$ (simplify denominator)

$$60 = \frac{a}{\frac{1}{3}}$$ (multiply by $\frac{1}{3}$)

$$60 \times \frac{1}{3} = a$$

$20 = a$
The first term is 20.

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Geometric sequences and series
Exercise E, Question 5

Question:

Find the first term of a geometric series with a common ratio of $-\frac{1}{3}$ and a sum to infinity of 10.

Solution:

Substitute $S_{\infty} = 10$ and $r = -\frac{1}{3}$ into

$S_{\infty} = \frac{a}{1-r}$

$10 = \frac{a}{1-\left(-\frac{1}{3}\right)}$

$10 = \frac{a}{\frac{4}{3}}$

$\frac{4}{3} \times 10 = a$

$a = \frac{40}{3}$

The first term is $\frac{40}{3}$.

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Geometric sequences and series
Exercise E, Question 6

Question:

Find the fraction equal to the recurring decimal 0.2323232323.

Solution:

$0.23232323 \ldots =$

$$\frac{23}{100} + \frac{23}{10000} + \frac{23}{1000000} + \ldots$$

$$\times\frac{1}{100} \qquad \times\frac{1}{100}$$

This is an infinite geometric series with $a = \frac{23}{100}$ and $r = \frac{1}{100}$.

Use $S_{\infty} = \frac{a}{1-r}$.

$$0.23232323 \ldots = \frac{\frac{23}{100}}{1-\frac{1}{100}} = \frac{\frac{23}{100}}{\frac{99}{100}} = \frac{23}{100} \times \frac{100}{99} = \frac{23}{99}$$

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Geometric sequences and series
Exercise E, Question 7

Question:

Find $\sum_{r=1}^{\infty} 4(0.5)^r$.

Solution:

$$\sum_{r=1}^{\infty} 4(0.5)^r$$

$= 4(0.5)^1 + 4(0.5)^2 + 4(0.5)^3 + \ldots$

$= 4 \times \underbrace{(0.5^1 + 0.5^2 + 0.5^3 + \ldots)}$

This is an infinite geometric series with $a = 0.5$ and $r = 0.5$.

Use $S_\infty = \frac{a}{1-r}$.

$$\sum_{r=1}^{\infty} 4(0.5)^r = 4 \times \frac{0.5}{1-0.5} = 4 \times \frac{0.5}{0.5} = 4$$

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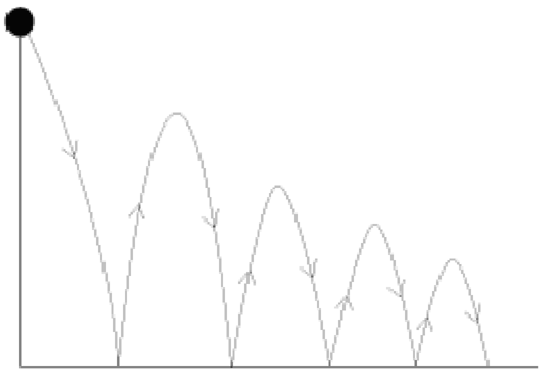
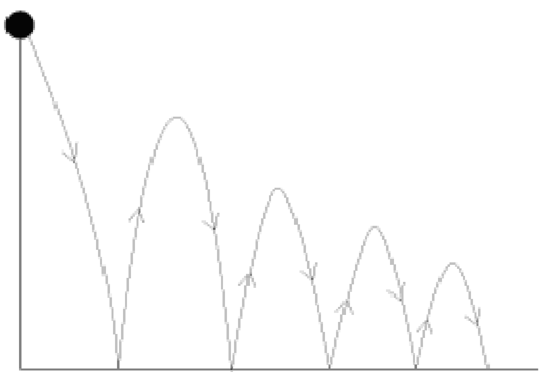
Geometric sequences and series
Exercise E, Question 8

Question:

A ball is dropped from a height of 10 m. It bounces to a height of 6 m, then 3.6, and so on following a geometric sequence.
Find the total distance travelled by the ball.



Solution:



Total distance

$$= 10+6+6+3.6+3.6+2.16+2.16+\ldots$$

$\times 0.6$ $\times 0.6$ $\times 0.6$

$$= 2\times \underbrace{(10+6+3.6+2.16+\ldots)} -10$$

This is an infinite geometric series with a = 10, r = 0.6.

Use $S_{\infty} = \frac{a}{1-r}$.

Total distance $= 2 \times \frac{10}{1-0.6} - 10 = 2 \times \frac{10}{0.4} - 10 = 50 - 10 = 40$ m

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Geometric sequences and series
Exercise E, Question 9

Question:

The sum to three terms of a geometric series is 9 and its sum to infinity is 8. What could you deduce about the common ratio? Why? Find the first term and common ratio.

Solution:

Let a = first term and r = common ratio.
If S_∞ exists then $|r| < 1$.
In fact as $S_\infty < S_3$ r must also be negative.

Using $S_3 = 9 \Rightarrow \frac{a(1 - r^3)}{1 - r} = 9$ ①

and $S_\infty = 8 \Rightarrow \frac{a}{1 - r} = 8$ ②

Substitute ② in ①:

$8(1 - r^3) = 9$

$1 - r^3 = \frac{9}{8}$

$r^3 = -\frac{1}{8}$

$r = -\frac{1}{2}$

Substitute $r = -\frac{1}{2}$ back into Equation ②:

$\frac{a}{1 - \left(-\frac{1}{2}\right)} = 8$

$a = 8 \times \frac{3}{2}$

$a = 12$

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Geometric sequences and series
Exercise E, Question 10

Question:

The sum to infinity of a geometric series is three times the sum to 2 terms. Find all possible values of the common ratio.

Solution:

Let a = first term and r = common ratio.
We are told $S_\infty = 3 \times S_2$

$\Rightarrow \quad \frac{\cancel{a}}{\cancel{1-r}} = 3 \times \frac{\cancel{a}(1-r^2)}{\cancel{1-r}}$

$\Rightarrow \quad 1 = 3(1 - r^2)$

$\Rightarrow \quad 1 = 3 - 3r^2$

$\Rightarrow \quad 3r^2 = 2$

$\Rightarrow \quad r^2 = \frac{2}{3}$

$\Rightarrow \quad r = \pm \sqrt{\frac{2}{3}}$

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Geometric sequences and series
Exercise F, Question 1

Question:

State which of the following series are geometric. For the ones that are, give the value of the common ratio *r*.

(a) $4 + 7 + 10 + 13 + 16 + \ldots$

(b) $4 + 6 + 9 + 13.5 + \ldots$

(c) $20 + 10 + 5 + 2.5 + \ldots$

(d) $4 - 8 + 16 - 32 + \ldots$

(e) $4 - 2 - 8 - 14 - \ldots$

(f) $1 + 1 + 1 + 1 + \ldots$

Solution:

(a) $$4 + 7 + 10 + 13 + 16 + \ldots$$
$$+3 \quad +3 \quad +3 \quad +3$$

Not geometric—you are adding 3 each time.

(b) $$4 + 6 + 9 + 13.5 + \ldots$$
$$\times 1.5 \quad \times 1.5 \quad \times 1.5$$

Geometric with $a = 4$ and $r = 1.5$.

(c) $$20 + 10 + 5 + 2.5 + \ldots$$
$$\times \frac{1}{2} \quad \times \frac{1}{2} \quad \times \frac{1}{2}$$

Geometric with $a = 20$ and $r = \frac{1}{2}$.

(d) $4 - 8 + 16 - 32 + \ldots$

$$= 4 + -8 + 16 + -32 + \ldots$$
$$\times -2 \quad \times -2 \quad \times -2$$

Geometric with $a = 4$ and $r = -2$.

(e) $4 - 2 - 8 - 14 + \ldots$

$$= 4 + -2 + -8 + -14 + \ldots$$
$$-6 \quad -6 \quad -6$$

Not geometric—you are subtracting 6 each time.

(f) $\underbrace{1 + 1}_{\times 1} \underbrace{+ 1}_{\times 1} \underbrace{+ 1}_{\times 1} + \ldots$

Geometric with $a = 1$ and $r = 1$.

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Geometric sequences and series
Exercise F, Question 2

Question:

Find the 8th and nth terms of the following geometric sequences:

(a) 10, 7, 4.9, …

(b) 5, 10, 20, …

(c) 4, − 4, 4, …

(d) 3, − 1.5, 0.75, …

Solution:

(a) 10, 7, 4.9, …

$a = 10, r = \frac{\text{2nd term}}{\text{1st term}} = \frac{7}{10} = 0.7$

8th term $= 10 \times (0.7)^{8-1} = 10 \times 0.7^7 = 0.823543$
nth term $= 10 \times (0.7)^{n-1}$

(b) 5, 10, 20, …

$a = 5, r = \frac{10}{5} = 2$

8th term $= 5 \times 2^{8-1} = 5 \times 2^7 = 640$
nth term $= 5 \times 2^{n-1}$

(c) 4, − 4, 4, …

$a = 4, r = \frac{-4}{4} = -1$

8th term $= 4 \times (-1)^{8-1} = 4 \times (-1)^7 = -4$
nth term $= 4 \times (-1)^{n-1}$

(d) 3, − 1.5, 0.75, …

$a = 3, r = \frac{-1.5}{3} = -0.5$

8th term $= 3 \times (-0.5)^{8-1} = 3 \times (-0.5)^7 = \frac{-3}{128} = -0.0234375$

nth term $= 3 \times (-0.5)^{n-1} = 3 \times \left(-\frac{1}{2}\right)^{n-1}$

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Geometric sequences and series
Exercise F, Question 3

Question:

Find the sum to 10 terms of the following geometric series:

(a) $4 + 8 + 16 + \ldots$

(b) $30 - 15 + 7.5 \ldots$

(c) $5 + 5 + 5 \ldots$

(d) $2 + 0.8 + 0.32 \ldots$

Solution:

(a) $4 + 8 + 16 + \ldots$
$a = 4$, $r = 2$

As $|r| > 1$ use $S_n = \frac{a(r^n - 1)}{r - 1}$

$S_{10} = \frac{4(2^{10} - 1)}{2 - 1} = 4092$

(b) $30 - 15 + 7.5 + \ldots$

$a = 30$, $r = -\frac{1}{2}$

As $|r| < 1$ use $S_n = \frac{a(1 - r^n)}{1 - r}$

$$S_{10} = \frac{30\left[1 - \left(-\frac{1}{2}\right)^{10}\right]}{1 - \left(-\frac{1}{2}\right)} = \frac{30\left[1 - \left(-\frac{1}{2}\right)^{10}\right]}{1 + \frac{1}{2}} = 19.98 \text{ (2 d.p.)}$$

(c) $5 + 5 + 5 + \ldots$
$a = 5$, $r = 1$
As $r = 1$ the sum formulae cannot be used.

$$S_{10} = \underbrace{5 + 5 + 5 + \ldots .5}_{10 \text{ terms}} = 50$$

(d) $2 + 0.8 + 0.32 + \ldots$

$a = 2$, $r = \frac{0.8}{2} = 0.4$

As $|r| < 1$ use $S_n = \frac{a(1 - r^n)}{1 - r}$

$S_{10} = \frac{2[1 - (0.4)^{10}]}{1 - 0.4} = \frac{2[1 - (0.4)^{10}]}{0.6} = 3.33 \text{ (2 d.p.)}$

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Geometric sequences and series
Exercise F, Question 4

Question:

Determine which of the following geometric series converge. For the ones that do, give the limiting value of this sum (i.e. S_{∞}).

(a) $6 + 2 + \frac{2}{3} + \ldots$

(b) $4 - 2 + 1 - \ldots$

(c) $5 + 10 + 20 + \ldots$

(d) $4 + 1 + 0.25 + \ldots$

Solution:

(a) $6 + 2 + \frac{2}{3} + \ldots$

$a = 6$ and $r = \frac{2}{6} = \frac{1}{3}$

As $|r| < 1$ series converges with limit

$$S_{\infty} = \frac{a}{1-r} = \frac{6}{1-\frac{1}{3}} = \frac{6}{\frac{2}{3}} = 9$$

(b) $4 - 2 + 1 - \ldots$
$= (4) + (-2) + (1) + \ldots$

$a = 4$ and $r = -\frac{2}{4} = -\frac{1}{2}$

As $|r| < 1$ series converges with limit

$$S_{\infty} = \frac{a}{1-r} = \frac{4}{1-\left(-\frac{1}{2}\right)} = \frac{4}{\frac{3}{2}} = \frac{8}{3}$$

(c) $5 + 10 + 20 + \ldots$
$a = 5$, $r = 2$
As $|r| > 1$ series does not converge.

(d) $4 + 1 + 0.25 + \ldots$

$a = 4$ and $r = \frac{1}{4}$

As $|r| < 1$ series converges with limit

$$S_{\infty} = \frac{a}{1-r} = \frac{4}{1-\frac{1}{4}} = \frac{4}{\frac{3}{4}} = \frac{16}{3}$$

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Geometric sequences and series
Exercise F, Question 5

Question:

A geometric series has third term 27 and sixth term 8:

(a) Show that the common ratio of the series is $\frac{2}{3}$.

(b) Find the first term of the series.

(c) Find the sum to infinity of the series.

(d) Find, to 3 significant figures, the difference between the sum of the first 10 terms of the series and the sum to infinity of the series.

[E]

Solution:

(a) Let a = first term and r = common ratio.
3rd term $= 27 \Rightarrow ar^2 = 27$ ①
6th term $= 8 \Rightarrow ar^5 = 8$ ②
Equation ② ÷ ①:

$$\frac{\cancel{a} r^5}{\cancel{a} r^2} = \frac{8}{27} \quad \left(\frac{r^5}{r^2} = r^{5-2} \right)$$

$$r^3 = \frac{8}{27}$$

$$r = \sqrt[3]{\frac{8}{27}} = \frac{2}{3}$$

The common ratio is $\frac{2}{3}$.

(b) Substitute $r = \frac{2}{3}$ back into Equation ①:

$$a \times \left(\frac{2}{3} \right)^2 = 27$$

$$a \times \frac{4}{9} = 27$$

$$a = \frac{27 \times 9}{4}$$

$a = 60.75$
The first term is 60.75

(c) Sum to infinity $= \frac{a}{1-r}$

$$\Rightarrow \quad S_{\infty} = \frac{60.75}{1 - \frac{2}{3}} = \frac{60.75}{\frac{1}{3}} = 182.25$$

Sum to infinity is 182.25

(d) Sum to ten terms $= \dfrac{a(1 - r^{10})}{1 - r}$

$$\text{So } S_{10} = \frac{60.75\left[1 - \left(\frac{2}{3}\right)^{10}\right]}{\left(1 - \frac{2}{3}\right)} = \frac{60.75\left[1 - \left(\frac{2}{3}\right)^{10}\right]}{\frac{1}{3}} = 179.0895 \quad \ldots$$

Difference between S_{10} and S_{∞} = 182.25 − 179.0895 = 3.16 (3 s.f.)

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Geometric sequences and series
Exercise F, Question 6

Question:

The second term of a geometric series is 80 and the fifth term of the series is 5.12:

(a) Show that the common ratio of the series is 0.4.
Calculate:

(b) The first term of the series.

(c) The sum to infinity of the series, giving your answer as an exact fraction.

(d) The difference between the sum to infinity of the series and the sum of the first 14 terms of the series, giving your answer in the form $a \times 10^n$, where $1 \leq a < 10$ and n is an integer.

[E]

Solution:

(a) 2nd term is 80 $\Rightarrow$ $ar^{2-1} = 80$ $\Rightarrow$ ar = 80 ①
5th term is 5.12 $\Rightarrow$ $ar^{5-1} = 5.12$ $\Rightarrow$ $ar^4 = 5.12$ ②
Equation ② ÷ Equation ①:

$$\frac{\cancel{a}r^4}{\cancel{a}r} = \frac{5.12}{80}$$

$r^3 = 0.064$ $\left(\sqrt[3]{} \right)$

$r = 0.4$
Hence common ratio = 0.4

(b) substitute $r = 0.4$ into Equation ①:
$a \times 0.4 = 80$ ($\div 0.4$)
$a = 200$
The first term in the series is 200.

(c) Sum to infinity $= \frac{a}{1-r} = \frac{200}{1-0.4} = \frac{200}{0.6} = 333\frac{1}{3}$

(d) Sum to n terms $= \frac{a(1-r^n)}{1-r}$

So $S_{14} = \frac{200(1-0.4^{14})}{(1-0.4)} = 333.3324385$

Required difference $S_{14} - S_{\infty} = 333.3324385 - 333\frac{1}{3} = 0.0008947 = 8.95 \times 10^{-4}$ (3 s.f.)

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Geometric sequences and series
Exercise F, Question 7

Question:

The nth term of a sequence is u_n, where $u_n = 95 \left(\frac{4}{5} \right)^n$, $n = 1, 2, 3, \ldots$

(a) Find the value of u_1 and u_2.
Giving your answers to 3 significant figures, calculate:

(b) The value of u_{21}.

(c) $\sum_{n=1}^{15} u_n$

(d) Find the sum to infinity of the series whose first term is u_1 and whose nth term is u_n.

[E]

Solution:

(a) $u_n = 95 \left(\frac{4}{5} \right)^n$

Replace n with 1 $\Rightarrow$ $u_1 = 95 \left(\frac{4}{5} \right)^1 = 76$

Replace n with 2 $\Rightarrow$ $u_2 = 95 \left(\frac{4}{5} \right)^2 = 60.8$

(b) Replace n with 21 $\Rightarrow$ $u_{21} = 95 \left(\frac{4}{5} \right)^{21} = 0.876$ (3 s.f.)

(c) $\sum_{n=1}^{15} u_n = \underbrace{76 + 60.8 + \ldots + 95\left(\frac{4}{5}\right)^{15}}_{15 \text{ terms}}$

A geometric series with $a = 76$ and $r = \frac{4}{5}$.

Use $S_n = \frac{a(1 - r^n)}{1 - r}$

$$\sum_{n=1}^{15} u_n = \frac{76\left[1-\left(\frac{4}{5}\right)^{15}\right]}{1-\frac{4}{5}} = \frac{76\left[1-\left(\frac{4}{5}\right)^{15}\right]}{\frac{1}{5}} \quad \left(\div\ \tfrac{1}{5}\text{ is equivalent to } \times 5\right)$$

$$\sum_{n=1}^{15} u_n = 76 \times 5 \times \left[1-\left(\frac{4}{5}\right)^{15}\right] = 366.63 = 367 \text{ (to 3 s.f.)}$$

(d) $S_\infty = \frac{a}{1-r} = \frac{76}{1-\frac{4}{5}} = \frac{76}{\frac{1}{5}} = 76 \times 5 = 380$

Sum to infinity is 380.

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Geometric sequences and series
Exercise F, Question 8

Question:

A sequence of numbers $u_1, u_2, \ldots, u_n, \ldots$ is given by the formula $u_n = 3\left(\frac{2}{3}\right)^n - 1$ where n is a positive integer.

(a) Find the values of u_1, u_2 and u_3.

(b) Show that $\sum_{n=1}^{15} u_n = -9.014$ to 4 significant figures.

(c) Prove that $u_{n+1} = 2\left(\frac{2}{3}\right)^n - 1$.

[E]

Solution:

(a) $u_n = 3\left(\frac{2}{3}\right)^n - 1$

Replace n with 1 $\Rightarrow$ $u_1 = 3 \times \left(\frac{2}{3}\right)^1 - 1 = 2 - 1 = 1$

Replace n with 2 $\Rightarrow$ $u_2 = 3 \times \left(\frac{2}{3}\right)^2 - 1 = 3 \times \frac{4}{9} - 1 = \frac{1}{3}$

Replace n with 3 $\Rightarrow$ $u_3 = 3 \times \left(\frac{2}{3}\right)^3 - 1 = 3 \times \frac{8}{27} - 1 = -\frac{1}{9}$

(b) $\sum_{n=1}^{15} u_n = \left[3 \times \left(\frac{2}{3}\right) - 1\right] + \left[3 \times \left(\frac{2}{3}\right)^2 - 1\right] + \left[3 \times \left(\frac{2}{3}\right)^3 - 1\right]$

$+ \ldots + \left[3 \times \left(\frac{2}{3}\right)^{15} - 1\right]$

$$= \underbrace{3\times\left(\tfrac{2}{3}\right)+3\times\left(\tfrac{2}{3}\right)^2+3\times\left(\tfrac{2}{3}\right)^3+\ldots+3\times\left(\tfrac{2}{3}\right)^{15}}_{\substack{\text{a geometric series with 15 terms,}\\ \text{where } a = 3\times\frac{2}{3} = 2 \text{ and } r = \frac{2}{3}}} \underbrace{-1-1-1-\ldots-1}_{\text{15 times}}$$

Use $S_n = \frac{a(1 - r^n)}{1 - r}$

$$\sum_{n=1}^{15} u_n = \frac{2\left[1 - \left(\frac{2}{3}\right)^{15}\right]}{1 - \frac{2}{3}} - 15 = 5.986 \ldots - 15 = -9.0137 \ldots = -9.014 \text{ (4 s.f.)}$$

(c) $u_{n+1} = 3 \times \left(\frac{2}{3}\right)^{n+1} - 1 = 3 \times \frac{2}{3} \times \left(\frac{2}{3}\right)^{n} - 1 = 2\left(\frac{2}{3}\right)^{n} - 1$

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Geometric sequences and series
Exercise F, Question 9

Question:

The third and fourth terms of a geometric series are 6.4 and 5.12 respectively. Find:

(a) The common ratio of the series.

(b) The first term of the series.

(c) The sum to infinity of the series.

(d) Calculate the difference between the sum to infinity of the series and the sum of the first 25 terms of the series.

[E]

Solution:

(a) Let a = first term and r = the common ratio of the series.
We are given
3rd term = 6.4 $\Rightarrow$ $ar^2 = 6.4$ ①
4th term = 5.12 $\Rightarrow$ $ar^3 = 5.12$ ②
Equation ② ÷ Equation ①:

$$\frac{\cancel{a}r^{\cancel{3}}}{\cancel{a}\cancel{r^2}} = \frac{5.12}{6.4}$$

$r = 0.8$
The common ratio is 0.8.

(b) Substitute $r = 0.8$ into Equation ①:
$a \times 0.8^2 = 6.4$

$$a = \frac{6.4}{0.8^2}$$

$a = 10$
The first term is 10.

(c) Use $S_\infty = \frac{a}{1-r}$ with $a = 10$ and $r = 0.8$.

$$S_\infty = \frac{10}{1-0.8} = \frac{10}{0.2} = 50$$

Sum to infinity is 50.

(d) $S_{25} = \frac{a(1-r^{25})}{1-r} = \frac{10(1-0.8^{25})}{1-0.8} = 49.8111 \ldots$

$S_\infty - S_{25} = 50 - 49.8111 \ldots$
$= 0.189$ (3 s.f.)

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Geometric sequences and series
Exercise F, Question 10

Question:

The price of a car depreciates by 15% per annum. If its new price is £20 000, find:

(a) A formula linking its value £V with its age a years.

(b) Its value after 5 years.

(c) The year in which it will be worth less than £4000.

Solution:

(a) If rate of depreciation is 15%, then car is worth 0.85 of its value at the start of the year.
New price = £20 000
After 1 year value = 20 000 × 0.85
After 2 years value = 20 000 × 0.85 × 0.85 = 20 000 × $(0.85)^2$
⋮
After a year value $V = 20\,000 \times (0.85)^a$

(b) Substitute $a = 5$:
$V = 20\,000 \times (0.85)^5 = 8874.10625$
Value of car after 5 years is £8874.11

(c) When value equals £4000
$4000 = 20\,000 \times (0.85)^a \quad (\div 20\,000)$
$0.2 = (0.85)^a$ (take logs both sides)
$\log(0.2) = \log(0.85)^a$ (use $\log a^n = n \log a$)
$\log(0.2) = a \log(0.85) \quad [\div \log(0.85)]$
$$a = \frac{\log(0.2)}{\log(0.85)}$$
$a = 9.90 \ldots$
It will be worth less than £4000 in the 10th year.

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Geometric sequences and series
Exercise F, Question 11

Question:

The first three terms of a geometric series are $p(3q+1)$, $p(2q+2)$ and $p(2q-1)$ respectively, where p and q are non-zero constants.

(a) Use algebra to show that one possible value of q is 5 and to find the other possible value of q.

(b) For each possible value of q, calculate the value of the common ratio of the series.
Given that $q = 5$ and that the sum to infinity of the geometric series is 896, calculate:

(c) The value of p.

(d) The sum, to 2 decimal places, of the first twelve terms of the series.

[E]

Solution:

(a) If $p(3q+1)$, $p(2q+2)$ and $p(2q-1)$ are consecutive terms in a geometric series then

$$\frac{p(2q+2)}{p(3q+1)} = \frac{p(2q-1)}{p(2q+2)}$$

$\frac{2q+2}{3q+1} = \frac{2q-1}{2q+2}$ (cross multiply)

$(2q+2)(2q+2) = (2q-1)(3q+1)$
$4q^2 + 8q + 4 = 6q^2 - 1q - 1$
$0 = 2q^2 - 9q - 5$
$0 = (2q+1)(q-5)$
$q = -\frac{1}{2}, 5$

(b) When $q = 5$ terms are $p(3\times5+1)$, $p(2\times5+2)$, $p(2\times5-1) = 16p$, $12p$ and $9p$

Common ratio $= \frac{12p}{16p} = \frac{3}{4}$

When $q = -\frac{1}{2}$ terms are $p\left(3\times-\frac{1}{2}+1\right)$, $p\left(2\times-\frac{1}{2}+2\right)$, $p\left(2\times-\frac{1}{2}-1\right) = -\frac{1}{2}p$, $1p$, $-2p$

Common ratio $= \frac{1p}{-\frac{1}{2}p} = -2$

(c) When $q = 5$ terms are $16p$, $12p$ and $9p$

Using $S_\infty = \frac{a}{1-r}$

$$896 = \frac{16p}{1 - \frac{3}{4}}$$

$$896 = \frac{16p}{\frac{1}{4}} \quad \left(\times \frac{1}{4} \right)$$

$224 = 16p$
$14 = p$
Therefore $p = 14$

(d) Using $S_n = \frac{a(1 - r^n)}{1 - r}$

$$S_{12} = \frac{16p\left[1 - \left(\frac{3}{4}\right)^{12}\right]}{1 - \frac{3}{4}}$$

$$p = 14 \quad \Rightarrow \quad S_{12} = \frac{16 \times 14\left[1 - \left(\frac{3}{4}\right)^{12}\right]}{\frac{1}{4}} = 867.617 \quad \ldots \quad = 867.62 \text{ (2 d.p.)}$$

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Geometric sequences and series
Exercise F, Question 12

Question:

A savings scheme pays 5% per annum compound interest. A deposit of £100 is invested in this scheme at the start of each year.

(a) Show that at the start of the third year, after the annual deposit has been made, the amount in the scheme is £315.25.

(b) Find the amount in the scheme at the start of the fortieth year, after the annual deposit has been made.

[E]

Solution:

(a) Start of year 1 = £100
End of year 1 $= 100 \times 1.05$
Start of year 2 $= (100 \times 1.05 + 100)$
End of year 2 $= (100 \times 1.05 + 100) \times 1.05 = 100 \times 1.05^2 + 100 \times 1.05$
Start of year 3 $= 100 \times 1.05^2 + 100 \times 1.05 + 100 = 110.25 + 105 + 100 =$ £ 315.25

(b) Amount at start of year 40
$= 100 \times 1.05^{39} + 100 \times 1.05^{38} + \ldots + 100 \times 1.05 + 100$
$= 100 \times \underbrace{(1.05^{39} + 1.05^{38} + \ldots + 1.05 + 1)}$

A geometric series with $a = 1$, $r = 1.05$ and $n = 40$.

Use $S_n = \frac{a (r^n - 1)}{r - 1}$

Amount at start of year 40

$= 100 \times \frac{1 (1.05^{40} - 1)}{1.05 - 1}$

$=$ £ 12 079.98

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Geometric sequences and series
Exercise F, Question 13

Question:

A competitor is running in a 25 km race. For the first 15 km, she runs at a steady rate of 12 km h^{-1}. After completing 15 km, she slows down and it is now observed that she takes 20% longer to complete each kilometre than she took to complete the previous kilometre.

(a) Find the time, in hours and minutes, the competitor takes to complete the first 16 km of the race.
The time taken to complete the rth kilometre is u_r hours.

(b) Show that, for $16 \le r \le 25$, $u_r = \frac{1}{12}(1.2)^{r-15}$.

(c) Using the answer to (b), or otherwise, find the time, to the nearest minute, that she takes to complete the race.

[E]

Solution:

(a) Using time $= \frac{\text{distance}}{\text{speed}} = \frac{15}{12} = 1.25$ hours $=$ 1 hour 15 mins.

The competitor takes 1 hour 15 mins for the first 15 km.

Time for each km is $\frac{1 \text{ hour } 15 \text{ mins}}{15} = \frac{75}{15} = 5$ mins

Time for the 16th km is $5 \times 1.2 = 6$ mins
Total time for first 16 km is 1 hour 15 mins + 6 mins = 1 hour 21 mins

(b) Time for the 17th km is $5 \times 1.2 \times 1.2 = 5 \times 1.2^2$ mins
Time for the 18th km is 5×1.2^3 mins

Time for the rth km is $5 \times (1.2)^{r-15}$ mins $= \frac{5 \times (1.2)^{r-15}}{60}$ hours

So $u_r = \frac{1}{12}(1.2)^{r-15}$

(c) Consider the 16th to the 25th kilometre.
Total time for this distance
$= 5 \times 1.2 + 5 \times 1.2^2 + 5 \times 1.2^3 + \ldots + 5 \times 1.2^{10}$
$= 5 \times \underbrace{(1.2 + 1.2^2 + 1.2^3 + \ldots 1.2^{10})}$

A geometric series with $a = 1.2$, $r = 1.2$ and $n = 10$.

$= 5 \times \frac{1.2(1.2^{10} - 1)}{1.2 - 1}$

$= 155.75$ mins
$= 156$ mins (to the nearest minute)
Total time for the race
$=$ time for 1st 15 km + time for last 10 km
$= 75 + 156$
$= 231$ mins
$= 3$ hours 51 mins

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Geometric sequences and series
Exercise F, Question 14

Question:

A liquid is kept in a barrel. At the start of a year the barrel is filled with 160 litres of the liquid. Due to evaporation, at the end of every year the amount of liquid in the barrel is reduced by 15% of its volume at the start of the year.

(a) Calculate the amount of liquid in the barrel at the end of the first year.

(b) Show that the amount of liquid in the barrel at the end of ten years is approximately 31.5 litres.
At the start of each year a new barrel is filled with 160 litres of liquid so that, at the end of 20 years, there are 20 barrels containing liquid.

(c) Calculate the total amount of liquid, to the nearest litre, in the barrels at the end of 20 years.

[E]

Solution:

(a) Liquid at start of year = 160 litres
Liquid at end of year $= 160 \times 0.85 = 136$ litres

(b) Liquid at end of year 2 $= 160 \times 0.85 \times 0.85 = 160 \times 0.85^2$
⋮
Liquid at end of year 10 $= 160 \times 0.85^{10} = 31.499 \ \ldots \ = 31.5$ litres

(c) Barrel 1 would have 20 years of evaporation. Amount $= 160 \times (0.85)^{20}$
Barrel 2 would have 19 years of evaporation. Amount $= 160 \times (0.85)^{19}$
⋮
Barrel 20 would have 1 year of evaporation. Amount $= 160 \times (0.85)^{1}$
Total amount of liquid
$= 160 \times 0.85^{20} + 160 \times 0.85^{19} + \ldots + 160 \times 0.85$
$= 160 \times \underbrace{(0.85^{20} + 0.85^{19} + \ldots + 0.85)}$

A geometric series with $a = 0.85$, $r = 0.85$ and $n = 20$.

Use $S_n = \frac{a (1 - r^n)}{1 - r}$

Total amount of liquid

$= 160 \times \frac{0.85 (1 - 0.85^{20})}{1 - 0.85}$

$= 871.52$
$= 872$ litres (to nearest litre)

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Geometric sequences and series
Exercise F, Question 15

Question:

At the beginning of the year 2000 a company bought a new machine for £15 000. Each year the value of the machine decreases by 20% of its value at the start of the year.

(a) Show that at the start of the year 2002, the value of the machine was £9600.

(b) When the value of the machine falls below £500, the company will replace it. Find the year in which the machine will be replaced.

(c) To plan for a replacement machine, the company pays £1000 at the start of each year into a savings account. The account pays interest of 5% per annum. The first payment was made when the machine was first bought and the last payment will be made at the start of the year in which the machine is replaced. Using your answer to part (b), find how much the savings account will be worth when the machine is replaced.

[E]

Solution:

(a) Beginning of 2000 value is £15 000
Beginning of 2001 value is 15000×0.8
Beginning of 2002 value is $15000 \times 0.8 \times 0.8 =$ £ 9600

(b) Beginning of 2003 value is $15000 \times \left(0.8 \right)^{3}$
After n years it will be worth $15000 \times \left(0.8 \right)^{n}$
Value falls below £500 when
$15000 \times \left(0.8 \right)^{n} < 500$

$$\left(0.8 \right)^{n} < \frac{500}{15000}$$

$$\left(0.8 \right)^{n} < \frac{1}{30}$$

$$\log \left(0.8 \right)^{n} < \log \left(\frac{1}{30} \right)$$

$$n \log \left(0.8 \right) < \log \left(\frac{1}{30} \right)$$

$$n > \frac{\log \left(\frac{1}{30} \right)}{\log \left(0.8 \right)}$$

$n > 15.24$
It will be replaced in 2015.

(c) Beginning of 2000 amount in account is £1000
End of 2000 amount in account is 1000×1.05
Beginning of 2001 amount in account is $1000 \times 1.05 + 1000$
End of 2001 amount in account is
$\left(1000 \times 1.05 + 1000 \right) \times 1.05 = 1000 \times 1.05^{2} + 1000 \times 1.05$
Beginning of 2002 amount in account is $1000 \times 1.05^{2} + 1000 \times 1.05 + 1000$
$\vdots$

Beginning of 2015 amount in account

$= 1000 \times 1.05^{15} + 1000 \times 1.05^{14} + \quad \ldots \quad + 1000 \times 1.05 + 1000$

$= 1000 \times \underbrace{(1.05^{15} + 1.05^{14} + \ldots. + 1.05 + 1)}$

A geometric series with $a = 1$, $r = 1.05$ and $n = 16$.

Use $S_n = \frac{a(r^n - 1)}{r - 1}$

Beginning of 2015 amount in account

$= 1000 \times \frac{1(1.05^{16} - 1)}{1.05 - 1}$

= £23 657.49

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Geometric sequences and series
Exercise F, Question 16

Question:

A mortgage is taken out for £80 000. It is to be paid by annual instalments of £5000 with the first payment being made at the end of the first year that the mortgage was taken out. Interest of 4% is then charged on any outstanding debt. Find the total time taken to pay off the mortgage.

Solution:

Mortgage $= £\,80\ 000$
Debt at end of year 1 $= (80\ 000 - 5000)$
Debt at start of year 2 $= (80\ 000 - 5000) \times 1.04$
Debt at end of year 2
$= (80\ 000 - 5000) \times 1.04 - 5000$
$= 80\ 000 \times 1.04 - 5000 \times 1.04 - 5000$
Debt at start of year 3
$= (80\ 000 \times 1.04 - 5000 \times 1.04 - 5000) \times 1.04$
$= 80\ 000 \times 1.04^2 - 5000 \times 1.04^2 - 5000 \times 1.04$
Debt at end of year 3 $= 80\ 000 \times 1.04^2 - 5000 \times 1.04^2 - 5000 \times 1.04 - 5000$
⋮
Debt at end of year n
$= 80\ 000 \times 1.04^{n-1} - 5000 \times 1.04^{n-1} - 5000 \times 1.04^{n-2} - \ldots - 5000 \times 1.04 - 5000$
Mortgage is paid off when this is zero.

$\Rightarrow\ 80\ 000 \times 1.04^{n-1} - 5000 \times 1.04^{n-1} - 5000 \times 1.04^{n-2} - \ldots - 5000 = 0$

$\Rightarrow\ 80\ 000 \times 1.04^{n-1} = 5000 \times 1.04^{n-1} + 5000 \times 1.04^{n-2} + \ldots + 5000$

$\Rightarrow\ 80\ 000 \times 1.04^{n-1} = 5000\ \underbrace{(1.04^{n-1} + 1.04^{n-2} + \ldots. + 1)}$

A geometric series with $a = 1$, $r = 1.04$ and n terms.

Use $S_n = \dfrac{a(r^n - 1)}{r - 1}$

$80\ 000 \times 1.04^{n-1} = 5000 \times \dfrac{1(1.04^n - 1)}{1.04 - 1}$

$80\ 000 \times 1.04^{n-1} = 125\ 000(1.04^n - 1)$
$80\ 000 \times 1.04^{n-1} = 125\ 000 \times 1.04^n - 125\ 000$
$80\ 000 \times 1.04^{n-1} = 125\ 000 \times 1.04 \times 1.04^{n-1} - 125\ 000$
$80\ 000 \times 1.04^{n-1} = 130\ 000 \times 1.04^{n-1} - 125\ 000$
$125\ 000 = 50\ 000 \times 1.04^{n-1}$

$\dfrac{125\ 000}{50\ 000} = 1.04^{n-1}$

$\dfrac{5}{2} = 1.04^{n-1}$

$\log\left(\dfrac{5}{2}\right) = \log(1.04)^{n-1}$

$\log\left(\dfrac{5}{2}\right) = (n - 1)\log 1.04$

$$\frac{\log\left(\frac{5}{2}\right)}{\log 1.04} = n - 1$$

$23.36 = n - 1$
$24.36 = n$
It takes 25 years to pay off the mortgage.