

Review exercise 1

- 1 Equation of circle with centre $(-3, 8)$ and radius r :

$$(x + 3)^2 + (y - 8)^2 = r^2$$

r = distance from $(-3, 8)$ to $(0, 9)$

$$r^2 = (0 + 3)^2 + (9 - 8)^2 = 9 + 1 = 10$$

The equation for C is:

$$(x + 3)^2 + (y - 8)^2 = 10$$

- 2 a Rearranging:

$$x^2 - 6x + y^2 + 2y = 10$$

Completing the square:

$$(x - 3)^2 - 9 + (y + 1)^2 - 1 = 10$$

$$(x - 3)^2 + (y + 1)^2 = 20$$

$$a = 3, b = -1, r = \sqrt{20}$$

- b The circle has centre $(3, -1)$ and radius $\sqrt{20}$.

- 3 a Rearranging $3x + y = 14$:

$$y = 14 - 3x$$

Solving simultaneously using substitution:

$$(x - 2)^2 + (14 - 3x - 3)^2 = 5$$

$$(x - 2)^2 + (-3x + 11)^2 = 5$$

$$x^2 - 4x + 4 + 9x^2 - 66x + 121 - 5 = 0$$

$$10x^2 - 70x + 120 = 0$$

$$x^2 - 7x + 12 = 0$$

$$(x - 3)(x - 4) = 0$$

$$\text{So } x = 3 \text{ and } x = 4$$

$$x = 3: y = 14 - 3 \times 3 = 5$$

$$x = 4: y = 14 - 3 \times 4 = 2$$

Point A is $(3, 5)$ and point B is $(4, 2)$.

- b Using Pythagoras' theorem:

$$\begin{aligned} \text{Length } AB &= \sqrt{(4 - 3)^2 + (2 - 5)^2} \\ &= \sqrt{10} \end{aligned}$$

- 4 The equation of the circle is $x^2 + y^2 = r^2$.

Solving simultaneously using substitution:

$$\begin{aligned}x^2 + (3x - 2)^2 &= r^2 \\x^2 + 9x^2 - 12x + 4 - r^2 &= 0 \\10x^2 - 12x + 4 - r^2 &= 0\end{aligned}$$

Using the discriminant for no solutions:

$$\begin{aligned}b^2 - 4ac &< 0 \\(-12)^2 - 4(10)(4 - r^2) &< 0 \\144 - 160 + 40r^2 &< 0\end{aligned}$$

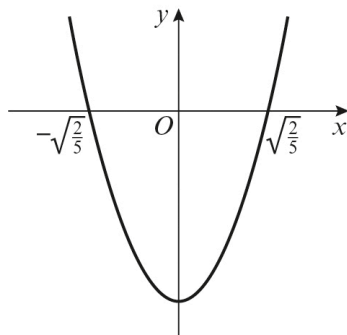
$$40r^2 - 16 < 0$$

$$\text{When } 40r^2 - 16 = 0$$

$$8(5r^2 - 2) = 0$$

$$r^2 = \frac{2}{5}$$

$$r = \pm \sqrt{\frac{2}{5}}$$



$$-\sqrt{\frac{2}{5}} < r < \sqrt{\frac{2}{5}}$$

However, the radius cannot be negative.

$$\text{So } 0 < r < \sqrt{\frac{2}{5}}$$

- 5 a Equation of circle with centre $(1, 5)$ and radius r :

$$(x - 1)^2 + (y - 5)^2 = r^2$$

$$r = \text{distance from } (1, 5) \text{ to } (4, -2)$$

$$\begin{aligned}r^2 &= (4 - 1)^2 + (-2 - 5)^2 \\&= 9 + 49 \\&= 58\end{aligned}$$

The equation for C is:

$$(x - 1)^2 + (y - 5)^2 = 58$$

5 b Gradient of the radius of the circle at P

$$= \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2 - 5}{4 - 1} = -\frac{7}{3}$$

$$\text{Gradient of the tangent} = \frac{3}{7}$$

Equation of the tangent at P :

$$y - y_1 = m(x - x_1)$$

$$y + 2 = \frac{3}{7}(x - 4)$$

$$3x - 7y - 26 = 0$$

6 a $AB^2 = (6 - 2)^2 + (5 - 1)^2$

$$= 4^2 + 4^2 = 32$$

$$BC^2 = (8 - 6)^2 + (3 - 5)^2$$

$$= 2^2 + 2^2 = 8$$

$$AC^2 = (8 - 2)^2 + (3 - 1)^2$$

$$= 6^2 + 2^2 = 40$$

Using Pythagoras' theorem:

$$AB^2 + BC^2 = 32 + 8 = 40 = AC^2$$

Therefore, $\angle ABC$ is 90° .

b As triangle ABC is a right-angled triangle,
 AC is a diameter of the circle.

c AC is a diameter of the circle, so the midpoint of AC is the centre.

$$\text{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

$$= \left(\frac{2 + 8}{2}, \frac{1 + 3}{2} \right)$$

$$= (5, 2)$$

$$\text{Radius} = \frac{1}{2} \times AC$$

$$= \frac{1}{2} \times \sqrt{40}$$

$$= \frac{1}{2} \times 2\sqrt{10}$$

$$= \sqrt{10}$$

The equation of the circle is:

$$(x - 5)^2 + (y - 2)^2 = 10$$

$$\begin{aligned}
 7 \quad \frac{2x^2 + 20x + 42}{224x + 4x^2 - 4x^3} &= \frac{x^2 + 10x + 21}{112x + 2x^2 - 2x^3} \\
 &= \frac{(x+3)(x+7)}{-2x(x^2 - x - 56)} \\
 &= \frac{(x+3)(x+7)}{-2x(x+7)(x-8)} \\
 &= \frac{(x+3)}{-2x(x-8)}
 \end{aligned}$$

$$a = 3, b = -2, c = -8$$

8 a Using the factor theorem:

$$\begin{aligned}
 f\left(\frac{1}{2}\right) &= 2\left(\frac{1}{2}\right)^3 - 7\left(\frac{1}{2}\right)^2 - 17\left(\frac{1}{2}\right) + 10 \\
 &= \frac{1}{4} - \frac{7}{4} - \frac{17}{2} + 10 \\
 &= 0
 \end{aligned}$$

So $(2x - 1)$ is a factor of $2x^3 - 7x^2 - 17x + 10$.

$$\begin{array}{r}
 \overline{) 2x^3 - 7x^2 - 17x + 10} \\
 \underline{2x^3 - x^2} \\
 -6x^2 - 17x \\
 \underline{-6x^2 + 3x} \\
 -20x + 10 \\
 \underline{-20x + 10} \\
 0
 \end{array}$$

$$\begin{aligned}
 2x^3 - 7x^2 - 17x + 10 &= (2x - 1)(x^2 - 3x - 10) \\
 &= (2x - 1)(x - 5)(x + 2)
 \end{aligned}$$

c $(2x - 1)(x - 5)(x + 2) = 0$

So $x = \frac{1}{2}, x = 5$ or $x = -2$

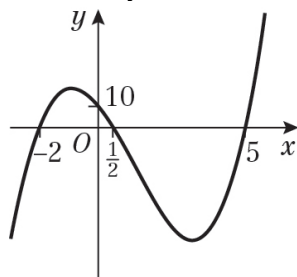
So the curve crosses the x -axis at $(\frac{1}{2}, 0)$, $(5, 0)$ and $(-2, 0)$.

When $x = 0, y = -1 \times -5 \times 2 = 10$

So the curve crosses the y -axis at $(0, 10)$.

$x \rightarrow \infty, y \rightarrow \infty$

$x \rightarrow -\infty, y \rightarrow -\infty$



$$9 \quad f(x) = 3x^3 + x^2 - 38x + c$$

$$a \quad f(3) = 0$$

$$3(3)^3 + (3)^2 - 38(3) + c = 0$$

$$3 \times 27 + 9 - 114 + c = 0$$

$$c = 24$$

$$b \quad f(x) = 3x^3 + x^2 - 38x + 24$$

$$f(3) = 0, \text{ so } (x - 3) \text{ is a factor of } 3x^3 + x^2 - 38x + 24.$$

$$\begin{array}{r}
 \overline{) 3x^3 + x^2 - 38x + 24} \\
 \underline{3x^3 + 9x^2} \\
 10x^2 - 38x \\
 \underline{10x^2 - 30x} \\
 -8x + 24 \\
 \underline{-8x + 24} \\
 0
 \end{array}$$

$$\begin{aligned}
 3x^3 + x^2 - 38x + 24 &= (x - 3)(3x^2 + 10x - 8) \\
 &= (x - 3)(3x - 2)(x + 4)
 \end{aligned}$$

$$10 \ a \quad g(x) = x^3 - 13x + 12$$

$$g(3) = (3)^3 - 13(3) + 12$$

$$= 27 - 39 + 12$$

$$= 0$$

So $(x - 3)$ is a factor of $g(x)$.

$$\begin{array}{r}
 \overline{) x^3 - 0x^2 - 13x + 12} \\
 \underline{x^3 - 3x^2} \\
 3x^2 - 13x \\
 \underline{3x^2 - 9x} \\
 -4x + 12 \\
 \underline{-4x + 12} \\
 0
 \end{array}$$

$$g(x) = x^3 - 13x + 12$$

$$= (x - 3)(x^2 + 3x - 4)$$

$$= (x - 3)(x + 4)(x - 1)$$

11 a $f(x) = x^3 + ax^2 + bx + 8$

$$f(3) = 2, \text{ so}$$

$$(3)^3 + a(3)^2 + b(3) + 8 = 2$$

$$27 + 9a + 3b + 8 = 2$$

$$9a + 3b = -33 \quad \text{(1)}$$

$$f(-1) = -2 \text{ so}$$

$$(-1)^3 + a(-1)^2 + b(-1) + 8 = -2$$

$$-1 + a - b + 8 = -2$$

$$a - b = -9 \quad \text{(2)}$$

Multiply equation (2) by 3 and add to equation (1)

$$9a + 3b = -33$$

$$\underline{3a - 3b = -27}$$

$$12a = -60$$

$$a = -5$$

When $a = -5$, $b = 4$

b $f(x) = x^3 - 5x^2 + 4x + 8$

$$f(2) = (2)^3 - 5(2)^2 + 4(2) + 8$$

$$= 8 - 20 + 8 + 8$$

$$= 4$$

12 a $f(x) = 2x^3 + ax^2 + bx + 6$

$$f(1) = 0, \text{ so}$$

$$2(1)^3 + a(1)^2 + b(1) + 6 = 0$$

$$2 + a + b + 6 = 0$$

$$a + b = -8 \quad \text{(1)}$$

$$f(-1) = 10, \text{ so}$$

$$2(-1)^3 + a(-1)^2 + b(-1) + 6 = 10$$

$$-2 + a - b + 6 = 10$$

$$a - b = 6 \quad \text{(2)}$$

Add equations (1) and (2)

$$a + b = -8$$

$$\underline{a - b = 6}$$

$$2a = -2$$

$$a = -1$$

When $a = -1$, $b = -7$

12 b $f(x) = 2x^3 - x^2 - 7x + 6$

$(x-1)$ is a factor, so

$$\begin{array}{r} 2x^2 + x - 6 \\ x-1 \overline{) 2x^3 - x^2 - 7x + 6} \end{array}$$

$$\underline{2x^3 - 2x^2}$$

$$x^2 - 7x$$

$$\underline{x^2 - x}$$

$$-6x + 6$$

$$\underline{-6x + 6}$$

$$0$$

$$2x^3 - x^2 - 7x + 6 = (x-1)(2x^2 + x - 6)$$

Now factorise the quadratic

$$(2x^2 + x - 6) = (2x - 3)(x + 2)$$

so

$$2x^3 - x^2 - 7x + 6 = (x-1)(2x-3)(x+2)$$

So $x = 1$, $x = \frac{3}{2}$, or $x = -2$

13 a $f(x) = x^4 + 5x^3 + ax + b$

$f(2) = f(-1)$, so

$$(2)^4 + 5(2)^3 + a(2) + b = (-1)^4 + 5(-1)^3 + a(-1) + b$$

$$16 + 40 + 2a = 1 - 5 - a$$

$$3a = -60$$

$$a = -20$$

b $f(x) = x^4 + 5x^3 - 20x + b$

$(x+3)$ is a factor so by the factor theorem $f(-3) = 0$

$$(-3)^4 + 5(-3)^3 - 20(-3) + b = 0$$

$$81 - 135 + 60 + b = 0$$

$$b = -6$$

14 a Example:

When $a = 0$ and $b = 0$, $0^2 + 0^2 = (0 + 0)^2$.

b $(a + b)^2 = a^2 + 2ab + b^2$

When $a > 0$ and $b > 0$, $2ab > 0$

Therefore $a^2 + b^2 < (a + b)^2$

When $a < 0$ and $b < 0$, $2ab > 0$

Therefore $a^2 + b^2 < (a + b)^2$

When $a > 0$ and $b < 0$, $2ab < 0$

Therefore $a^2 + b^2 > (a + b)^2$

When $a < 0$ and $b > 0$, $2ab < 0$

Therefore $a^2 + b^2 > (a + b)^2$

The conditions are $a > 0$ and $b > 0$ or $a < 0$ and $b < 0$.

$$\begin{aligned}
 \mathbf{15\ a} \quad p = 5: 5^2 &= 25 = 24 + 1 \\
 p = 7: 7^2 &= 49 = 2(24) + 1 \\
 p = 11: 11^2 &= 121 = 5(24) + 1 \\
 p = 13: 13^2 &= 169 = 7(24) + 1 \\
 p = 17: 17^2 &= 289 = 12(24) + 1 \\
 p = 19: 19^2 &= 361 = 15(24) + 1
 \end{aligned}$$

$$\mathbf{b} \quad 3(24) + 1 = 73 \text{ and } 73 \text{ is not a square number.}$$

16 a Rearranging:

$$x^2 - 10x + y^2 - 8y = -32$$

Completing the square:

$$(x - 5)^2 - 25 + (y - 4)^2 - 16 = -32$$

$$(x - 5)^2 + (y - 4)^2 = 9$$

$$(x - 5)^2 + (y - 4)^2 = 3^2$$

$$a = 5, b = 4, r = 3$$

b Centre of circle C is $(5, 4)$.

Centre of circle D is $(0, 0)$.

Using Pythagoras' theorem:

$$\text{Distance} = \sqrt{(5-0)^2 + (4-0)^2} = \sqrt{41}$$

c Radius of circle $C = 3$

Radius of circle $D = 3$

$$\text{Distance between the centres} = \sqrt{41}$$

$$3 + 3 < \sqrt{41}$$

Therefore, the circles C and D do not touch.

$$\mathbf{17\ a} \quad 5^x = 0.75$$

$$x \log 5 = \log 0.75$$

$$x = \frac{\log 0.75}{\log 5}$$

$$= -0.179 \text{ (3 s.f.)}$$

$$\mathbf{b} \quad 2 \log_5 x - \log_5 3x = 1$$

$$\log_5 x^2 - \log_5 3x = 1$$

$$\log_5 \left(\frac{x^2}{3x} \right) = 1$$

$$\frac{x^2}{3x} = 5^1$$

$$x^2 = 15x$$

$$x^2 - 15x = 0$$

$$x(x - 15) = 0$$

$$x = 0 \text{ or } x = 15$$

$$\text{since } x \neq 0, x = 15$$

$$\begin{aligned}
 \mathbf{18\ a} \quad & 3^{2x-1} = 10 \\
 & (2x-1)\log 3 = \log 10 \\
 & (2x-1) = \frac{\log 10}{\log 3} \\
 & x = \frac{1}{2} \left(\frac{\log 10}{\log 3} + 1 \right) \\
 & = 1.55 \text{ (3 s.f.)}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \log_2 x + \log_2 (9-2x) = 2 \\
 & \log_2 (x(9-2x)) = 2 \\
 & x(9-2x) = 2^2 \\
 & 2x^2 - 9x + 4 = 0 \\
 & (2x-1)(x-4) = 0 \\
 & x = \frac{1}{2} \text{ or } x = 4
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{19\ a} \quad & \log_p 12 - \left(\frac{1}{2} \log_p 9 + \frac{1}{3} \log_p 8 \right) \\
 & = \log_p 12 - \left(\log_p 9^{\frac{1}{2}} + \log_p 8^{\frac{1}{3}} \right) \\
 & = \log_p 12 - (\log_p 3 + \log_p 2) \\
 & = \log_p 12 - \log_p 6 \\
 & = \log_p \left(\frac{12}{6} \right) \\
 & = \log_p 2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \log_4 x = -1.5 \\
 & x = 4^{-1.5} \\
 & = \frac{1}{8}
 \end{aligned}$$

$$20 \log_x 64 + 3 \log_4 x - \log_x 4 = 5$$

$$\log_x 64 + \frac{3}{\log_x 4} - \log_x 4 = 5$$

$$\log_x 4^3 - \log_x 4 + \frac{3}{\log_x 4} = 5$$

$$\log_x \left(\frac{4^3}{4} \right) + \frac{3}{\log_x 4} = 5$$

$$\log_x 4^2 + \frac{3}{\log_x 4} = 5$$

$$2 \log_x 4 + \frac{3}{\log_x 4} = 5$$

Let $y = \log_x 4$, so

$$2y + \frac{3}{y} = 5$$

$$2y^2 + 3 = 5y$$

$$2y^2 - 5y + 3 = 0$$

$$(2y - 3)(y - 1) = 0$$

$$y = \frac{3}{2} \text{ or } y = 1$$

Therefore

$$\log_x 4 = \frac{3}{2} \text{ or } \log_x 4 = 1$$

$$\text{When } \log_x 4 = \frac{3}{2}$$

$$x^{\frac{3}{2}} = 4$$

$$x = 2.52 \text{ (3 s.f.)}$$

$$\text{When } \log_x 4 = 1$$

$$x^1 = 4$$

$$x = 4$$

$$21 \log_2 x + 6 \log_x 2 = 7$$

$$\log_2 x + \frac{6}{\log_2 x} = 7$$

Let $y = \log_2 x$, so

$$y + \frac{6}{y} = 7$$

$$y^2 - 7y + 6 = 0$$

$$(y-1)(y-6) = 0$$

$$y = 1 \text{ or } y = 6$$

Therefore

$$\log_2 x = 1 \text{ or } \log_2 x = 6$$

When $\log_2 x = 1$

$$x = 2$$

When $\log_2 x = 6$

$$x = 2^6 = 64$$

$$22 \log_3 9t = \log_9 \left(\frac{12}{t} \right)^2 + 2$$

$$\frac{\log 9t}{\log 3} = \frac{\log \left(\frac{12}{t} \right)^2}{\log 9} + 2$$

$$\frac{\log 9t}{\log 3} = \frac{2 \log \left(\frac{12}{t} \right)}{2 \log 3} + 2$$

$$\log 9t = \log \left(\frac{12}{t} \right) + 2 \log 3$$

$$\log 9t = \log \left(\frac{108}{t} \right)$$

$$9t = \frac{108}{t}$$

$$t^2 = 12$$

$$t = 2\sqrt{3}$$

$$t \neq -2\sqrt{3}, \text{ since it's not valid.}$$

$$\begin{aligned}
 23 \text{ a } (1 - 2x)^{10} &= 1^{10} + \binom{10}{1} 1^9 (-2x) + \binom{10}{2} 1^8 (-2x)^2 \\
 &\quad + \binom{10}{3} 1^7 (-2x)^3 + \dots \\
 &= 1 + 10(-2x) + \frac{10(9)}{2} (-2x)^2 \\
 &\quad + \frac{10(9)(8)}{6} (-2x)^3 + \dots \\
 &= 1 - 20x + 180x^2 - 960x^3 + \dots
 \end{aligned}$$

$$\begin{aligned}
 23 \text{ b } (0.98)^{10} &= (1 - 2(0.01))^{10} \\
 &= 1 - 20(0.01) + 180(0.01)^2 - 960(0.01)^3 + \dots \\
 &= 0.817 \text{ (3 d.p.)}
 \end{aligned}$$

$$\begin{aligned}
 24 \quad (1 + 2x)^5 &= 1^5 + \binom{5}{1} 1^4 (2x) + \binom{5}{2} 1^3 (2x)^2 + \dots \\
 &= 1 + 5(2x) + \frac{5(4)}{2} (2x)^2 + \dots \\
 &= 1 + 10x + 40x^2 + \dots
 \end{aligned}$$

$$\begin{aligned}
 (2 - x)(1 + 2x)^5 &= (2 - x)(1 + 10x + 40x^2 + \dots) \\
 &= 2 + 20x + 80x^2 + \dots - x - 10x^2 + \dots \\
 &= 2 + 19x + 70x^2 + \dots \\
 &\approx 2 + 19x + 70x^2
 \end{aligned}$$

$$a = 2, b = 19, c = 70$$

$$\begin{aligned}
 25 \quad (2 - 4x)^q & \\
 x \text{ term} &= \binom{q}{q-1} 2^{q-1} (-4x)^1 \\
 &= q \times 2^{q-1} \times -4x \\
 &= -4 \times 2^{q-1} qx \\
 -4 \times 2^{q-1} q &= -32q \\
 2^{q-1} &= 8 \\
 q - 1 &= 3 \\
 q &= 4
 \end{aligned}$$

26 a Using the binomial expansion

$$\begin{aligned}(1+4x)^{\frac{3}{2}} &= 1 + \binom{\frac{3}{2}}{1}(4x) + \frac{\binom{\frac{3}{2}}{2}\binom{1}{2}}{2!}(4x)^2 + \frac{\binom{\frac{3}{2}}{3}\binom{1}{2}\binom{-1}{2}}{3!}(4x)^3 + \dots \\ &= 1 + 6x + 6x^2 - 4x^3 + \dots\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad \left(1 + 4\left(\frac{3}{100}\right)\right)^{\frac{3}{2}} &= \left(\frac{112}{100}\right)^{\frac{3}{2}} \\ &= \left(\sqrt{\frac{112}{100}}\right)^3 \\ &= \frac{112\sqrt{112}}{1000}\end{aligned}$$

$$\mathbf{c} \quad 1 + 6\left(\frac{3}{100}\right) + 6\left(\frac{3}{100}\right)^2 - 4\left(\frac{3}{100}\right)^3 = 1.185292$$

$$\text{So } \frac{112\sqrt{112}}{1000} \approx 1.185292$$

$$\Rightarrow \sqrt{112} \approx \frac{1185.292}{112} = 10.582962857\dots = 10.58296 \text{ (5 d.p.)}$$

d Using a calculator $\sqrt{112} = 10.5830052$ (7 d.p.)

$$\text{Percentage error} = \frac{10.5830052 - 10.5829643}{10.5830052} \times 100 = 0.00039\% \text{ (5 d.p.)}$$

Note, you will get different answers if you use values rounded to 5 d.p. in calculating the percentage error.

Challenge

1 a $f(x) = 2x^4 + ax^3 - 23x^2 + bx + 24$

$x^2 + x - 6$ is a factor so

$$\begin{array}{r}
 2x^2 + (a-2)x - (9+a) \\
 x^2 + x - 6 \overline{) 2x^4 + ax^3 - 23x^2 + bx + 24} \\
 \underline{2x^4 + 2x^3 - 12x^2} \\
 (a-2)x^3 - 11x^2 + bx \\
 \underline{(a-2)x^3 + (a-2)x^2 - 6(a-2)x} \\
 (-9-a)x^2 + (6a+b-12)x + 24 \\
 \underline{(-9-a)x^2 + (-9-a)x - 6(-9-a)} \\
 (7a+b-3)x + (-30-6a)
 \end{array}$$

Since $x^2 + x - 6$ is a factor

$$-30 - 6a = 0 \Rightarrow a = -5$$

$$7a + b - 3 = 0 \Rightarrow b = 38$$

Alternative solution: by factorisation:

$$2x^4 + ax^3 - 23x^2 + bx + 24$$

$$(x^2 + x - 6)(2x^2 - 7x - 4)$$

$$x^3 \text{ coefficient: } a = 2 - 7 = -5$$

$$x \text{ coefficient: } b = -6 \times -7 - 4 = 38$$

b Substitute for a and b into

$$2x^2 + (a-2)x - (9+a) = 0$$

$$2x^2 - 7x - 4 = 0$$

$$(2x+1)(x-4) = 0$$

Therefore

$$\begin{aligned}
 2x^4 + ax^3 - 23x^2 + bx + 24 &= (x^2 + x - 6)(2x+1)(x-4) \\
 &= (x+3)(x-2)(2x+1)(x-4)
 \end{aligned}$$

2 ~~a~~ Rearranging $x^2 + y^2 + 8x - 10y = 59$:

$$x^2 + 8x + y^2 - 10y = 59$$

Completing the square:

$$(x + 4)^2 - 16 + (y - 5)^2 - 25 = 59$$

$$(x + 4)^2 + (y - 5)^2 = 100$$

Both circles have the same centre at $(-4, 5)$. The radius of one circle is 8 and the other is 10, so

$(x + 4)^2 + (y - 5)^2 = 8^2$ lies completely inside $x^2 + y^2 + 8x - 10y = 59$.

3

$$8^{2y+1} = 4^{2x-2}$$

$$(2y+1)\log 8 = (2x-2)\log 4$$

$$3(2y+1)\log 2 = 4(x-1)\log 2$$

$$6y+3 = 4x-4$$

$$y = \frac{4x-7}{6}$$

$$\log_2 y = 1 + \log_4 x$$

$$\log_2 \left(\frac{4x-7}{6} \right) = 1 + \frac{\log_2 x}{\log_2 4}$$

$$\log_2 \left(\frac{4x-7}{6} \right) = 1 + \frac{1}{2} \log_2 x$$

$$\log_2 \left(\frac{4x-7}{6} \right) = 1 + \log_2 x^{\frac{1}{2}}$$

$$\log_2 \left(\frac{4x-7}{6x^{\frac{1}{2}}} \right) = 1$$

$$\frac{4x-7}{6x^{\frac{1}{2}}} = 2$$

$$4x - 12x^{\frac{1}{2}} - 7 = 0$$

$$\text{Let } z = x^{\frac{1}{2}}$$

$$4z^2 - 12z - 7 = 0$$

$$(2z+1)(2z-7) = 0$$

$$z = -\frac{1}{2} \text{ or } z = \frac{7}{2}$$

So

$$x^{\frac{1}{2}} = -\frac{1}{2} \text{ or } x^{\frac{1}{2}} = \frac{7}{2}$$

$$x = \frac{1}{4} \text{ or } x = \frac{49}{4}$$

3 When $x = \frac{1}{4}$

$$y = \frac{4\left(\frac{1}{4}\right) - 7}{6} = -1$$

When $x = \frac{49}{4}$

$$y = \frac{4\left(\frac{49}{4}\right) - 7}{6} = 7$$

4 LHS = $\binom{n}{k} + \binom{n}{k+1}$

$$= \frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-k-1)!}$$

$$= \frac{n!(k+1)}{(k+1)!(n-k)!} + \frac{n!(n-k)}{(k+1)!(n-k)!}$$

$$= \frac{n!((k+1) + (n-k))}{(k+1)!(n-k)!}$$

$$= \frac{n!(n+1)}{(k+1)!(n-k)!}$$

$$= \frac{(n+1)!}{(k+1)!(n-k)!}$$

$$= \binom{n+1}{k+1}$$

$$= \text{RHS}$$