## Chapter review 8

**1 a** $2 = 5 + 2x - x^2$
$\Rightarrow x^2 - 2x - 3 = 0$
$\Rightarrow (x-3)(x+1) = 0$
$\Rightarrow x = -1(A), 3(B)$

**b** Area of $R = \int_{-1}^{3}(5 + 2x - x^2 - 2)\,dx$
$= \int_{-1}^{3}(3 + 2x - x^2)\,dx$
$= \left(3x + x^2 - \frac{1}{3}x^3\right)_{-1}^{3}$
$= \left(9 + 9 - \frac{27}{3}\right) - \left(-3 + 1 + \frac{1}{3}\right)$
$= 9 + 2 - \frac{1}{3}$
$= 10\frac{2}{3}$

**2 a** $\left(x^{\frac{1}{2}} - 4\right)\left(x^{-\frac{1}{2}} - 1\right)$
$= 1 - 4x^{-\frac{1}{2}} - x^{\frac{1}{2}} + 4 = 5 - 4x^{-\frac{1}{2}} - x^{\frac{1}{2}}$
$\int\left(x^{\frac{1}{2}} - 4\right)\left(x^{-\frac{1}{2}} - 1\right)\,dx = 5x - \frac{4x^{\frac{1}{2}}}{\frac{1}{2}} - \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$
$= 5x - 8x^{\frac{1}{2}} - \frac{2}{3}x^{\frac{3}{2}} + c$

**b** $\int_{1}^{4}\left(x^{\frac{1}{2}} - 4\right)\left(x^{-\frac{1}{2}} - 1\right)\,dx = \left(5x - 8x^{\frac{1}{2}} - \frac{2}{3}x^{\frac{3}{2}}\right)_{1}^{4}$
$= \left(20 - 8\times 2 - \frac{2}{3}\times 2^3\right) - \left(5 - 8 - \frac{2}{3}\right)$
$= 4 - \frac{16}{3} + 3 + \frac{2}{3}$
$= 7 - \frac{14}{3}$
$= \frac{7}{3}$ or $2\frac{1}{3}$

**3 a** $(x-3)^2 = x^2 - 6x + 9$
So $x(x-3)^2 = x^3 - 6x^2 + 9x$
$y = 0 \Rightarrow x = 0$ or $3$ (twice)
So $A$ is the point $(3, 0)$.

**3 b** $\frac{dy}{dx} = 0 \Rightarrow 0 = 3x^2 - 12x + 9$
$\Rightarrow 0 = 3(x^2 - 4x + 3)$
$\Rightarrow 0 = 3(x-3)(x-1)$
$\Rightarrow 0 = 1$ or $3$
$x = 3$ at $A$, the minimum, so $B$ is $(1, 4)$
(Found by substituting $x = 1$ into original equation.)

**c** Area of $R = \int_{0}^{3}(x^3 - 6x^2 + 9x)\,dx$
$= \left(\frac{1}{4}x^4 - 2x^3 + \frac{9}{2}x^2\right)_{0}^{3}$
$= \left(\frac{81}{4} - 2\times 27 + \frac{9}{2}\times 9\right) - (0)$
$= 6\frac{3}{4}$

**4 a** $y = 3x^{\frac{1}{2}} - 4x^{-\frac{1}{2}}$
$\frac{dy}{dx} = \frac{3}{2}x^{-\frac{1}{2}} + \frac{1}{2}\times 4x^{-\frac{3}{2}}$
$\frac{dy}{dx} = \frac{3}{2}x^{-\frac{1}{2}} + 2x^{-\frac{3}{2}}$

**b** $\int y\,dx = \int\left(3x^{\frac{1}{2}} - 4x^{-\frac{1}{2}}\right)dx$
$= \frac{3x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{4x^{\frac{1}{2}}}{\frac{1}{2}} + c$
$= 2x^{\frac{3}{2}} - 8x^{\frac{1}{2}} + c$

**c** $\int_{1}^{3} y\,dx = \left(2x^{\frac{3}{2}} - 8x^{\frac{1}{2}}\right)_{1}^{3}$
$= \left(2\times 3\sqrt{3} - 8\sqrt{3}\right) - (2 - 8)$
$= -2\sqrt{3} + 6$
$= 6 - 2\sqrt{3}$
So $A = 6$ and $B = -2$

**5 a** $y = 12x^{\frac{1}{2}} - x^{\frac{3}{2}}$
$\frac{dy}{dx} = 6x^{-\frac{1}{2}} - \frac{3}{2}x^{\frac{1}{2}}$
$= \frac{3}{2}x^{-\frac{1}{2}}(4 - x)$

**b** $\frac{dy}{dx} = 0 \Rightarrow x = 4, y = 12\times 2 - 2^3 = 16$
So $B$ is the point $(4, 16)$.

**5 c** Area $= \int_0^{12}\left(12x^{\frac{1}{2}} - x^{\frac{3}{2}}\right)\mathrm{d}x$

$= \left(\dfrac{12x^{\frac{3}{2}}}{\frac{3}{2}} - \dfrac{x^{\frac{5}{2}}}{\frac{5}{2}}\right)_0^{12}$

$= \left(8x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}}\right)_0^{12}$

$= \left(8\times\sqrt{12^3} - \frac{2}{5}\sqrt{12^5}\right) - (0)$

$= 133\ (3\text{ s.f.})$

**6 a** $x(8-x) = 12$

$\Rightarrow 8x - x^2 = 12$

$\Rightarrow 0 = x^2 - 8x + 12$

$\Rightarrow 0 = (x-6)(x-2)$

$\Rightarrow x = 2$ or $x = 6$

$M$ is on the same line as $L$.

So $M$ is the point $(6, 12)$.

**b** Area $= \int_6^8 \left(8x - x^2\right)\mathrm{d}x$

$= \left(4x^2 - \dfrac{x^3}{3}\right)_6^8$

$= \left(4\times 64 - \frac{512}{3}\right) - \left(4\times 36 - \frac{216}{3}\right)$

$= 256 - 170\frac{2}{3} - 144 + 72$

$= 13\frac{1}{3}$

**7 a** $A$ is the point $(1, 0)$, $B$ is the point $(5, 0)$.

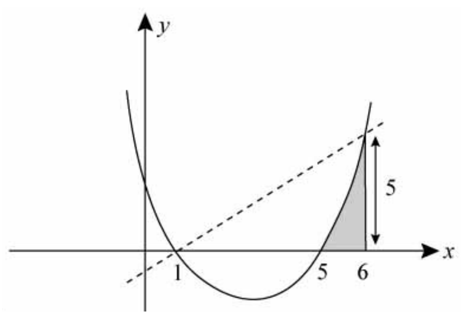
$x - 1 = (x-1)(x-5)$

$\Rightarrow 0 = (x-1)(x-5-1)$

$\Rightarrow 0 = (x-1)(x-6)$

$\Rightarrow x = 1,\ x = 6$

So $C$ is the point $(6, 5)$.

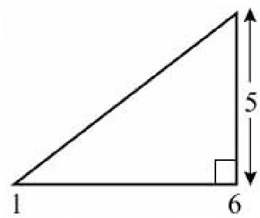



**7 b** Drop a perpendicular from $C$ to the $x$-axis to a point $D$.

The area of the shaded region is

Area of triangle $ABD$ $\int_5^6 (x-1)(x-5)\ \mathrm{d}x$

$=$ Area of $ABD$ $- \int_5^6 \left(x^2 - 6x + 5\right)\ \mathrm{d}x$




Area $= \left(\frac{1}{2}\times 5\times 5\right) - \int_5^6 \left(x^2 - 6x + 5\right)\ \mathrm{d}x$

$= 12\frac{1}{2} - \left[\frac{1}{3}x^3 - 3x^2 + 5x\right]_5^6$

$= 12\frac{1}{2} - \left[(72 - 108 + 30) - \left(41\frac{2}{3} - 75 + 25\right)\right]$

$= 12\frac{1}{2} - (-6) - \left(-8\frac{1}{3}\right)$

$= 12\frac{1}{2} + 6 + 8\frac{1}{3}$

$= 26\frac{5}{6}$

**8 a** For the point $A$, which lies on the line and the curve

$4q + 25 = p + 40 - 16$

$\Rightarrow\ 4q = p - 1$ **(1)**

For the point $B$, which lies on the line and the curve

$8q + 25 = p + 80 - 64$

$\Rightarrow\ 8q = p - 9$ **(2)**

Subtracting **(2)** − **(1)**

$\Rightarrow 4q = -8$

$\Rightarrow q = -2$

Substituting into **(1)**

$\Rightarrow p = 1 + 4q$

$\Rightarrow p = -7$

**8 b** At $A$, $y = 4q + 25 = 17$

So $C$ is given by

$$17 = -7 + 10x - x^2$$

$$x^2 - 10x + 24 = 0$$

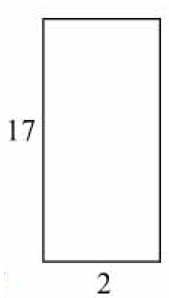
$$(x-6)(x-4) = 0$$

$$x = 4,\ x = 6$$

So $C$ is the point $(6, 17)$

**c** The required area is

$$\int_4^6 (-7 + 10x - x^2)\,\mathrm{d}x - \text{area of rectangle}$$




$$\text{Area } = \left(-7x + 5x^2 - \tfrac{1}{3}x^3\right)_4^6 - 34$$

$$= \left(-42 + 180 - 72\right) - \left(-28 + 80 - \tfrac{64}{3}\right) - 34$$

$$= \tfrac{4}{3} \text{ or } 1\tfrac{1}{3}$$

**9** $$A^2 = \int_4^9 \left(\frac{3}{\sqrt{x}} - A\right)\mathrm{d}x$$

$$= \int_4^9 \left(3x^{-\frac{1}{2}} - A\right)\mathrm{d}x$$

$$= \left[\frac{3x^{\frac{1}{2}}}{\frac{1}{2}} - Ax\right]_4^9$$

$$= \left[6x^{\frac{1}{2}} - Ax\right]_4^9$$

$$= \left(6(9)^{\frac{1}{2}} - A(9)\right) - \left(6(4)^{\frac{1}{2}} - A(4)\right)$$

$$= (18 - 9A) - (12 - 4A)$$

$$0 = (A + 6)(A - 1)$$

$$A = -6 \text{ or } A = 1$$

**10 a** $$\mathrm{f}'(x) = \frac{(2-x^2)^3}{x^2}$$

$$= \frac{(2-x^2)(2-x^2)(2-x^2)}{x^2}$$

$$= \frac{(4-4x^2+x^4)(2-x^2)}{x^2}$$

$$= x^{-2}(8 - 12x^2 + 6x^4 - x^6)$$

$$= 8x^{-2} - 12 + 6x^2 - x^4$$

So $A = 6$ and $B = -1$

**b** $\mathrm{f}''(x) = -16x^{-3} + 12x - 4x^3$

**c** $$\mathrm{f}(x) = \int (8x^{-2} - 12 + 6x^2 - x^4)\,\mathrm{d}x$$

$$= \frac{8x^{-1}}{-1} - 12x + \frac{6x^3}{3} - \frac{x^5}{5} + c$$

$$= -\frac{8}{x} - 12x + 2x^3 - \frac{x^5}{5} + c$$

When $x = -2$ and $y = 9$

$$-\frac{8}{-2} - 12(-2) + 2(-2)^3 - \frac{(-2)^5}{5} + c = 9$$

$$4 + 24 - 16 + \tfrac{32}{5} + c = 9$$

$$c = -\tfrac{47}{5}$$

$$\mathrm{f}(x) = -\frac{8}{x} - 12x + 2x^3 - \frac{x^5}{5} - \frac{47}{5}$$

**11 a** $y = 3 - 5x - 2x^2$

When $y = 0$, $3 - 5x - 2x^2 = 0$

$(3 + x)(1 - 2x) = 0$

$x = -3$ or $x = \tfrac{1}{2}$

The points are $A(-3, 0)$ and $B(\tfrac{1}{2}, 0)$.

**b** $$\int_{-3}^{\frac{1}{2}} (3 - 5x - 2x^2)\,\mathrm{d}x$$

$$= \left[3x - \frac{5x^2}{2} - \frac{2x^3}{3}\right]_{-3}^{\frac{1}{2}}$$

$$= \left(3\left(\tfrac{1}{2}\right) - \frac{5\left(\tfrac{1}{2}\right)^2}{2} - \frac{2\left(\tfrac{1}{2}\right)^3}{3}\right)$$

$$- \left(3(-3) - \frac{5(-3)^2}{2} - \frac{2(-3)^3}{3}\right)$$

$$= \left(\tfrac{3}{2} - \tfrac{5}{8} - \tfrac{1}{12}\right) - \left(-9 - \tfrac{45}{2} + \tfrac{54}{3}\right)$$

$$= 14\tfrac{7}{24}$$

**12 a** $(x-4)(2x+3)=0$
$x=4$ or $x=-\frac{3}{2}$
The points are $A(-\frac{3}{2}, 0)$ and $B(4, 0)$.

**b** $R=\int_{-\frac{3}{2}}^{4}(x-4)(2x+3)\,dx$

$=\int_{-\frac{3}{2}}^{4}(2x^2-5x-12)\,dx$

$=\left[\frac{2x^3}{3}-\frac{5x^2}{2}-12x\right]_{-\frac{3}{2}}^{4}$

$=\left(\frac{2(4)^3}{3}-\frac{5(4)^2}{2}-12(4)\right)-\left(\frac{2(-\frac{3}{2})^3}{3}-\frac{5(-\frac{3}{2})^2}{2}-12(-\frac{3}{2})\right)$

$=\left(\frac{128}{3}-40-48\right)-\left(-\frac{9}{4}-\frac{45}{8}+18\right)$

$=-55\frac{11}{24}$

Area $=55\frac{11}{24}$

**13 a** $x(x-3)(x+2)=0$
$x=0$, $x=3$ or $x=-2$
The points are $A(-2, 0)$ and $B(3, 0)$.

**b** $\int_{-2}^{0}x(x-3)(x+2)\,dx-\int_{0}^{3}x(x-3)(x+2)\,dx=\int_{-2}^{0}(x^3-x^2-6x)\,dx-\int_{0}^{3}(x^3-x^2-6x)\,dx$

$\int_{-2}^{0}(x^3-x^2-6x)\,dx=\left[\frac{x^4}{4}-\frac{x^3}{3}-\frac{6x^2}{2}\right]_{-2}^{0}$

$=\left[\frac{x^4}{4}-\frac{x^3}{3}-3x^2\right]_{-2}^{0}$

$=\left(\frac{0^4}{4}-\frac{0^3}{3}-3(0)^2\right)-\left(\frac{(-2)^4}{4}-\frac{(-2)^3}{3}-3(-2)^2\right)$

$=0-\left(4+\frac{8}{3}-12\right)$

$=5\frac{1}{3}$

$\int_{0}^{3}(x^3-x^2-6x)\,dx=\left[\frac{x^4}{4}-\frac{x^3}{3}-3x^2\right]_{0}^{3}$

$=\left(\frac{3^4}{4}-\frac{3^3}{3}-3(3)^2\right)-\left(\frac{0^4}{4}-\frac{0^3}{3}-3(0)^2\right)$

$=\left(\frac{81}{4}-9-27\right)$

$=-15\frac{3}{4}$

Total area is $5\frac{1}{3}-(-15\frac{3}{4})=21\frac{1}{12}$

**14 a** $y = \dfrac{5}{x^2+1}$

| $x$ | 0 | 0.5 | 1 | 1.5 | 2 | 2.5 | 3 |
|---|---|---|---|---|---|---|---|
| $\dfrac{5}{x^2+1}$ | 5 | 4 | 2.5 | 1.538 | 1 | 0.690 | 0.5 |

**b** $A = \int_0^3 \left(\dfrac{5}{x^2+1}\right) dx$

$\int_a^b y\, dx = \frac{1}{2}h\left(y_0 + 2\left(y_1 + y_2 + \ldots + y_{n-1}\right) + y_n\right)$

$\int_0^3 \left(\dfrac{5}{x^2+1}\right) dx = \frac{1}{2}(0.5)\left(5 + 2\left(4 + 2.5 + 1.538 + 1 + 0.69\right) + 0.5\right)$

$= 6.24$ (3 s.f.)

**c** $\int_0^3 \left(4 + \dfrac{5}{x^2+1}\right) dx$

$\int_0^3 \left(4 + \dfrac{5}{x^2+1}\right) dx = \int_0^3 4\, dx + \int_0^3 \left(\dfrac{5}{x^2+1}\right) dx$

$= 12 + 6.24$

$= 18.24$

**15 a** $y = \sqrt{3^x + x}$

| $x$ | 0 | 0.25 | 0.5 | 0.75 | 1 |
|---|---|---|---|---|---|
| $\sqrt{3^x + x}$ | 1 | 1.251 | 1.494 | 1.741 | 2 |

**b** $A = \int_0^1 \sqrt{3^x + x}\, dx$

$\int_a^b y\, dx = \frac{1}{2}h\left(y_0 + 2\left(y_1 + y_2 + \ldots + y_{n-1}\right) + y_n\right)$

$\int_0^1 \sqrt{3^x + x}\, dx = \frac{1}{2}(0.25)\left(1 + 2\left(1.251 + 1.494 + 1.741\right) + 2\right)$

$= 1.50$ (3 s.f.)

**16 a** $y = 8 + 4x - x^2$ and $y = x^2 - 4x + 8$

$8 + 4x - x^2 = x^2 - 4x + 8$

$2x^2 - 8x = 0$

$2x(x-4) = 0$

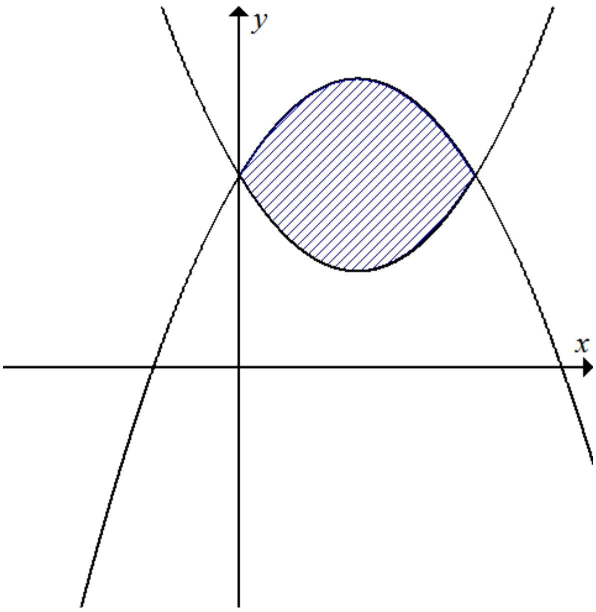
$x = 0$ or $x = 4$

When $x = 0$, $y = 8$

When $x = 4$, $y = 8$

So the curves intersect at (0, 8) and (4, 8)

**16 b**




The area of the shaded region is given by

$$A = \int_0^4 \left(8+4x-x^2\right) \mathrm{d}x - \int_0^4 \left(x^2-4x+8\right) \mathrm{d}x$$

$$= \int_0^4 \left(8+4x-x^2\right) \mathrm{d}x - \int_0^4 \left(x^2-4x+8\right) \mathrm{d}x$$

$$= \left[8x+2x^2-\frac{1}{3}x^3\right]_0^4 - \left[\frac{1}{3}x^3-2x^2+8x\right]_0^4$$

$$= \left[\left(8(4)+2(4)^2-\frac{1}{3}(4)^3\right)-\left(8(0)+2(0)^2-\frac{1}{3}(0)^3\right)\right]$$

$$-\left[\left(\frac{1}{3}(4)^3-2(4)^2+8(4)\right)-\left(\frac{1}{3}(0)^3-2(0)^2+8(0)\right)\right]$$

$$= \frac{128}{3} - \frac{64}{3}$$

$$= \frac{64}{3}$$

Alternatively, because the limits are the same

$$A = \int_0^4 \left(8+4x-x^2\right)-\left(x^2-4x+8\right) \mathrm{d}x$$

$$= \int_0^4 \left(8x-2x^2\right) \mathrm{d}x$$

$$= \left[4x^2-\frac{2}{3}x^3\right]_0^4$$

$$= \left[\left(4(4)^2-\frac{2}{3}(4)^3\right)-\left(4(0)^2-\frac{2}{3}(0)^3\right)\right]$$

$$= \left[\left(\frac{64}{3}\right)-(0)\right]$$

$$= \frac{64}{3}$$

**Challenge**

**a** The shaded area beneath the $x$-axis is given by

$$A = \int_0^1 x(x-1)(x+2)\,dx$$
$$= \int_0^1 (x^3 + x^2 - 2x)\,dx$$
$$= \left[\frac{1}{4}x^4 + \frac{1}{3}x^3 - x^2\right]_0^1$$
$$= \left(\frac{1}{4}(1)^4 + \frac{1}{3}(1)^3 - (1)^2\right) - \left(\frac{1}{4}(0)^4 + \frac{1}{3}(0)^3 - (0)^2\right)$$
$$= -\frac{5}{12}$$

So the area beneath the $x$-axis is $\frac{5}{12}$

The shaded area above the $x$-axis is given by

$$A = \int_x^0 (x^3 + x^2 - 2x)\,dx$$
$$= \left[\frac{1}{4}x^4 + \frac{1}{3}x^3 - x^2\right]_x^0$$
$$= \left(\frac{1}{4}(0)^4 + \frac{1}{3}(0)^3 - (0)^2\right) - \left(\frac{1}{4}(x)^4 + \frac{1}{3}(x)^3 - (x)^2\right)$$
$$= -\frac{1}{4}x^4 - \frac{1}{3}x^3 + x^2$$

Since the areas are equal

$$-\frac{1}{4}x^4 - \frac{1}{3}x^3 + x^2 = \frac{5}{12}$$
$$-3x^4 - 4x^3 + 12x^2 = 5$$
$$3x^4 + 4x^3 - 12x^2 + 5 = 0$$

Let $f(x) = 3x^4 + 4x^3 - 12x^2 + 5$

By the factor theorem if $(x-1)$ is a factor then $f(1) = 0$

$$f(1) = 3(1)^4 + 4(1)^3 - 12(1)^2 + 5 = 0$$

Therefore $(x-1)$ is a factor

$$
\begin{array}{r}
3x^3+7x^2-5x-5 \\
x-1 \overline{) 3x^4+4x^3-12x^2+5} \\
\underline{3x^4-3x^3} \\
7x^3-12x^2 \\
\underline{7x^3-7x^2} \\
-5x^2+5 \\
\underline{-5x^2+5x} \\
5x-5 \\
\underline{5x-5} \\
0
\end{array}
$$

$3x^4+4x^3-12x^2+5=(x-1)(3x^3+7x^2-5x-5)$

Let $g(x)=3x^3+7x^2-5x-5$

By the factor theorem if $(x-1)$ is a factor then $g(1)=0$

$g(1)=3(1)^3+7(1)^2-5(1)-5=0$

Therefore $(x-1)$ is a factor

$$
\begin{array}{r}
3x^2+10x+5 \\
x-1 \overline{) 3x^3+7x^2-5x-5} \\
\underline{3x^3-3x^2} \\
10x^2-5x \\
\underline{10x^2-10x} \\
5x-5 \\
\underline{5x-5} \\
0
\end{array}
$$

$3x^3+7x^2-5x-5=(x-1)(3x^2+10x+5)$

and therefore

$3x^4+4x^3-12x^2+5=(x-1)^2(3x^2+10x+5)$ as required

**b** $(x-1)^2(3x^2+10x+5)=0$

$(x-1)^2=0$ has solutions at $x = 1$

$3x^2+10x+5=0$ has solutions at

$$x=\frac{-10\pm\sqrt{(10)^2-4(3)(5)}}{2(3)}$$

$$=\frac{-10\pm\sqrt{40}}{2(3)}$$

$x=\frac{-5+\sqrt{10}}{3}$ or $x=\frac{-5-\sqrt{10}}{3}$

Since the $x$-coordinate of $A$ lies between $x = 0$ and $x = -2$

$A$ has coordinates $\left(\frac{-5+\sqrt{10}}{3},0\right)$

$x=1$ and $x=\frac{-5-\sqrt{10}}{3}$ are the $x$-values where the curve $(x-1)^2(3x^2+10x+5)=0$ cuts the $x$-axis.