Exercise 8D

1 a A, B are given by $6 = x^2 + 2$

$$x^2 = 4$$
$$x = \pm 2$$

So A is (−2, 6) and B is (2, 6).

b $\text{Area} = \int_{-2}^{2} \left(6 - \left(x^2 + 2\right)\right) \text{d}x$

$$= \int_{-2}^{2} \left(4 - x^2\right) \text{d}x$$
$$= \left(4x - \frac{x^3}{3}\right)_{-2}^{2}$$
$$= \left(8 - \tfrac{8}{3}\right) - \left(-8 + \tfrac{8}{3}\right)$$
$$= 16 - 2 \times \tfrac{16}{3}$$
$$= 5\tfrac{1}{3}$$

2 a A, B are given by $3 = 4x - x^2$

$$x^2 - 4x + 3 = 0$$
$$(x-3)(x-1) = 0$$
$$x = 1, 3$$

So A is (1, 3) and B is (3, 3).

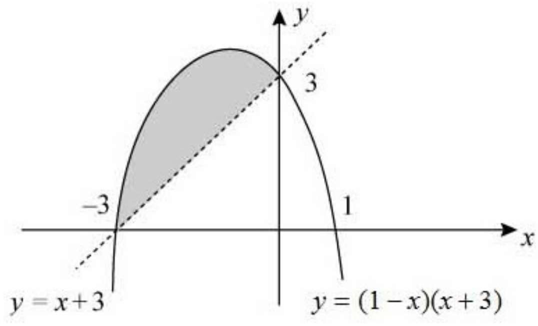
b $\text{Area} = \int_{1}^{3} \left(\left(4x - x^2\right) - 3\right) \text{d}x$

$$= \int_{1}^{3} \left(4x - x^2 - 3\right) \text{d}x$$
$$= \left(2x^2 - \frac{x^3}{3} - 3x\right)_{1}^{3}$$
$$= (18 - 9 - 9) - \left(2 - \tfrac{1}{3} - 3\right)$$
$$= 1\tfrac{1}{3}$$

3 $\text{Area} = \int_{-1}^{1} (\text{curve} - \text{line})\, \text{d}x$

$$= \int_{-1}^{1} \left(9 - 3x - 5x^2 - x^3 - (4 - 4x)\right) \text{d}x$$
$$= \int_{-1}^{1} \left(5 + x - 5x^2 - x^3\right) \text{d}x$$
$$= \left(5x + \frac{x^2}{2} - \frac{5}{3}x^3 - \frac{x^4}{4}\right)_{-1}^{1}$$
$$= \left(5 + \tfrac{1}{2} - \tfrac{5}{3} - \tfrac{1}{4}\right) - \left(-5 + \tfrac{1}{2} + \tfrac{5}{3} - \tfrac{1}{4}\right)$$
$$= 10 - \tfrac{10}{3}$$
$$= \tfrac{20}{3} \text{ or } 6\tfrac{2}{3}$$

4 $y = (1-x)(x+3)$ is ∩ shaped and crosses the x-axis at (1, 0) and (−3, 0).
$y = x + 3$ is a straight line passing through (−3, 0) and (0, 3)



Intersections occur when

$$x + 3 = (1-x)(x+3)$$

$$0 = (x+3)(1 - x - 1)$$
$$0 = -x(x+3)$$
$$x = -3 \text{ or } x = 0$$

$$\text{Area} = \int_{-3}^{0} \left((1-x)(x+3) - (x+3)\right) \text{d}x$$
$$= \int_{-3}^{0} \left(-x^2 - 3x\right) \text{d}x$$
$$= \left(-\frac{x^3}{3} - \frac{3x^2}{2}\right)_{-3}^{0}$$
$$= (0) - \left(\tfrac{27}{3} - \tfrac{27}{2}\right)$$
$$= \tfrac{27}{6}$$
$$= \tfrac{9}{2} \text{ or } 4\tfrac{1}{2}$$

5 a A is given by $x(4+x) = 12$

$$x^2 + 4x - 12 = 0$$
$$(x+6)(x-2) = 0$$
$$x = 2 \text{ or } x = -6$$

So A is (2, 12)

5 b R is found by $\int_0^2 x(4+x)\ \mathrm{d}x$ away from a rectangle of area 12 × 2 = 24.

So area of $R = 24 - \int_0^2 (x^2+4x)\ \mathrm{d}x$

$$= 24 - \left(\frac{x^3}{3} + 2x^2\right)_0^2$$

Area of $R = 24 - \left\{\left(\frac{8}{3}+8\right) - (0)\right\}$

$$= 24 - \frac{32}{3}$$

$$= \frac{40}{3} \text{ or } 13\frac{1}{3}$$

6 a Intersections occur when $7 - x = x^2 + 1$

$0 = x^2 + x - 6$

$0 = (x+3)(x-2)$

$x = 2$ or -3

Area of R_1, is given by

$$\int_{-3}^{2} \left(7 - x - (x^2+1)\right)\ \mathrm{d}x$$

$$= \int_{-3}^{2} (6 - x - x^2)\ \mathrm{d}x$$

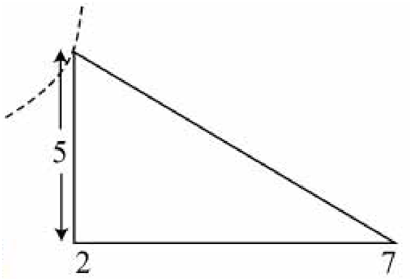
$$= \left(6x - \frac{x^2}{2} - \frac{x^3}{3}\right)_{-3}^{2}$$

$$= \left(12 - \frac{4}{2} - \frac{8}{3}\right) - \left(-18 - \frac{9}{2} + \frac{27}{3}\right)$$

$$= 20\frac{5}{6}$$

b Area of R_2, is given by

$\int_0^2 (x^2+1)\ \mathrm{d}x$ + area of the triangle.



Area of $R_2 = \left(\frac{x^3}{3} + x\right)_0^2 + \frac{1}{2} \times 5 \times 5$

$$= \left(\frac{8}{3} + 2\right) - (0) + \frac{25}{2}$$

$$= 17\frac{1}{6}$$

7 a When $x = 1$, $y = 1 - \frac{2}{1} + 1$

$$= 0$$

So $(1, 0)$ lies on C.

7 b When $x = 8$, $y = 8^{\frac{2}{3}} - \dfrac{2}{8^{\frac{1}{3}}} + 1$

$$= 2^2 - \frac{2}{2} + 1$$

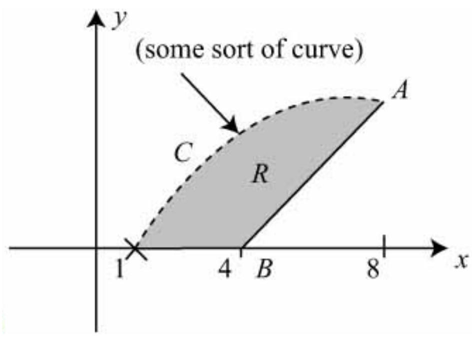
$$= 4$$

So $(8, 4)$ lies on C.

c A is the point (8, 4) and B is the point (4, 0).

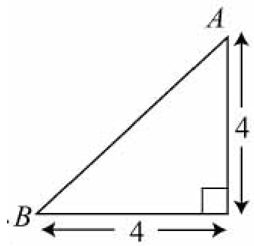
Gradient of line through AB is $\dfrac{4-0}{8-4} = 1$.

So the equation is $y - 0 = x - 4$ or $y = x - 4$



d Area of R is given by

$\int_1^8 (\text{curve})\ \mathrm{d}x$ − area of the triangle.



Area $R = \int_1^8 \left(x^{\frac{2}{3}} - \dfrac{2}{x^{\frac{1}{3}}} + 1\right) \mathrm{d}x - \frac{1}{2} \times 4 \times 4$

$$= \left(\frac{3}{5}x^{\frac{5}{3}} - \frac{2x^{\frac{2}{3}}}{\frac{2}{3}} + x\right)_1^8 - 8$$

$$= \left(\frac{3}{5} \times 32 - 3 \times 4 + 8\right)$$

$$-\left(\frac{3}{5} - 3 + 1\right) - 8$$

$$= \frac{96}{5} - 4 - \frac{7}{5} - 8$$

$$= \frac{29}{5}$$

$$= 5\frac{4}{5}$$

8 Area $= \int_{\frac{1}{2}}^{2}\left(\text{line } AB - \left(\frac{2}{x^2}+x\right)\right)\,dx$

Substitute $\frac{1}{2}$ and 2 for x into the equation to find

A is $\left(\frac{1}{2}, 8\frac{1}{2}\right)$ and B is $\left(2, 2\frac{1}{2}\right)$.

The gradient of $AB = \frac{6}{-1\frac{1}{2}} = -4$

So the equation is $y - 2\frac{1}{2} = -4(x-2)$

$y = 10\frac{1}{2} - 4x$

Area $= \int_{\frac{1}{2}}^{2}\left(10\frac{1}{2} - 5x - 2x^{-2}\right)\,dx$

$= \left[\frac{21}{2}x - \frac{5}{2}x^2 - \frac{2x^{-1}}{-1}\right]_{\frac{1}{2}}^{2}$

$= \left[\frac{21}{2}x - \frac{5}{2}x^2 + \frac{2}{x}\right]_{\frac{1}{2}}^{2}$

$= (21 - 10 + 1) - \left(\frac{21}{4} - \frac{5}{8} + 4\right)$

$= 12 - 8\frac{5}{8}$

$= 3\frac{3}{8}$ or 3.375

$= 3.38$ (3 s.f.)

9 a On the line, when $x = 4$, $y = 4 - \frac{1}{2}\times 4$

$= 2$

On the curve, when $x = 4$,

$y = 3\times\sqrt{4} - \sqrt{64} + 4$

$= 6 - 8 + 4$

$= 2$

So the point (4, 2) lies on the line and the curve.

9 b Area $= \int_0^4\left(3x^{\frac{1}{2}} - x^{\frac{3}{2}} + 4 - \left(4 - \frac{1}{2}x\right)\right)\,dx$

$= \int_0^4\left(3x^{\frac{1}{2}} - x^{\frac{3}{2}} + \frac{1}{2}x\right)\,dx$

$= \left[\frac{3x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + \frac{x^2}{4}\right]_0^4$

$= \left[2x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}} + \frac{1}{4}x^2\right]_0^4$

$= \left(2\times 8 - \frac{2}{5}\times 32 + 4\right) - (0)$

$= 20 - \frac{64}{5}$

$= \frac{36}{5}$ or 7.2

10 a $y = x^2(x+4)$

$y = 0 \Rightarrow x = 0\ (\text{twice}), -4$

Area of R_1 is

$\int_{-4}^{0}\left(x^3 + 4x^2\right)\,dx = \left[\frac{x^4}{4} + \frac{4}{3}x^3\right]_{-4}^{0}$

$= (0) - \left(\frac{4^4}{4} - \frac{4^4}{3}\right)$

$= \frac{4^4}{12}$

$= \frac{4^3}{3}$

$= \frac{64}{3}$ or $21\frac{1}{3}$

b Area of R_2 is $\int_0^2\left(x^3 + 4x^2\right)\,dx +$ area of the triangle.

Area of $R_2 = \left[\frac{x^4}{4} + \frac{4}{3}x^3\right]_0^2 + 12(b-2)$

$= \left(\frac{16}{4} + \frac{32}{3}\right) - (0) + 12(b-2)$

$= 14\frac{2}{3} + 12b - 24$

$= -9\frac{1}{3} + 12b$

Area of $R_2 =$ area of R_1

$\Rightarrow -9\frac{1}{3} + 12b = 21\frac{1}{3}$

So $12b = 30\frac{2}{3}$

$\Rightarrow b = 2\frac{5}{9}$ or 2.56 (3 s.f.)

11 a The intersections occur when

$$10-x=2x^2-5x+4$$
$$0=2x^2-4x-6$$
$$0=2(x+1)(x-3)$$
$$x=-1 \text{ or } x=3$$

When $x=-1$, $y=11$, A is $(-1, 11)$.
When $x=3$, $y=7$, B is $(3, 7)$.

b Area $= \int_{-1}^{3}[(10-x)-\left(2x^2-5x+4\right)]\ \mathrm{d}x$

$$=\int_{-1}^{3}(10-x-2x^2+5x-4)\ \mathrm{d}x$$
$$=\int_{-1}^{3}(6+4x-2x^2)\ \mathrm{d}x$$
$$=\left[6x+2x^2-\tfrac{2}{3}x^3\right]_{-1}^{3}$$
$$=\left(18+18-18\right)-\left(-6+2+\tfrac{2}{3}\right)$$
$$=18+3\tfrac{1}{3}$$
$$=21\tfrac{1}{3}$$