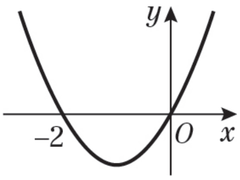
Exercise 8C

1 a $y = x(x+2)$ is $\cup$ shaped

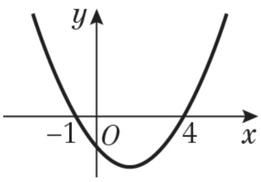
$y = 0 \Rightarrow x = 0, -2$



$$\text{Area} = \int_{-2}^{0} x(x+2)\,dx$$
$$= -\int_{-2}^{0} (x^2 + 2x)\,dx$$
$$= -\left(\frac{x^3}{3} + x^2\right)_{-2}^{0}$$
$$= \left\{(0) - \left(-\frac{8}{3} + 4\right)\right\}$$
$$= -\left(-\frac{4}{3}\right)$$
$$= \tfrac{4}{3} \text{ or } 1\tfrac{1}{3}$$

b $y = (x+1)(x-4)$ is $\cup$ shaped

$y = 0 \Rightarrow x = -1, 4$



$$\int_{-1}^{4} (x+1)(x-4)\,dx = \int_{-1}^{4} (x^2 - 3x - 4)\,dx$$
$$= \left(\frac{x^3}{3} - \frac{3x^2}{2} - 4x\right)_{-1}^{4}$$
$$= \left(\frac{64}{3} - \frac{3}{2} \times 16 - 16\right)$$
$$- \left(-\frac{1}{3} - \frac{3}{2} + 4\right)$$
$$= \frac{64}{3} - 40 + \frac{11}{6} - 4$$
$$= -20\tfrac{5}{6}$$

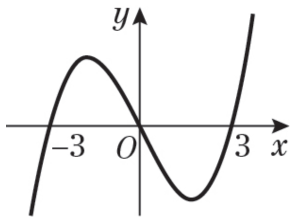
So area $= 20\tfrac{5}{6}$

1 c $y = (x+3)x(x-3)$

$y = 0 \Rightarrow x = -3, 0, 3$

$x \to \infty, y \to \infty$

$x \to -\infty, y \to -\infty$



$$\int y\,dx = \int (x^3 - 9x)\,dx$$
$$= \left(\frac{x^4}{4} - \frac{9}{2}x^2\right)$$
$$\int_{-3}^{0} y\,dx = (0) - \left(\frac{81}{4} - \frac{9}{2} \times 9\right)$$
$$= +\frac{81}{4}$$
$$\int_{0}^{3} y\,dx = \left(\frac{81}{4} - \frac{9}{2} \times 9\right) - (0)$$
$$= -\frac{81}{4}$$
$$\text{So area} = \frac{81}{4} + \frac{81}{4}$$
$$= \frac{81}{2} \text{ or } 40\tfrac{1}{2}$$

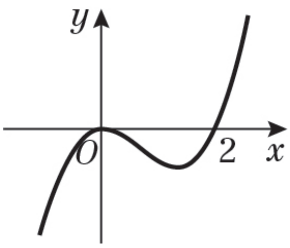
1 d $y = x^2(x-2)$

$y = 0 \Rightarrow x = 0\,(\text{twice}), 2$

There is a turning point at (0, 0).

$x \to \infty, y \to \infty$

$x \to -\infty, y \to -\infty$



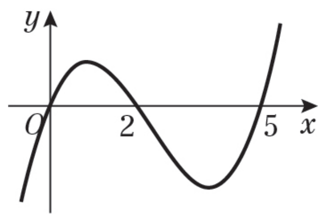
$$\text{Area} = -\int_0^2 x^2(x-2)\,dx$$
$$= -\int_0^2 (x^3 - 2x^2)\,dx$$
$$= -\left(\frac{x^4}{4} - \frac{2}{3}x^3\right)_0^2$$
$$= -\left\{\left(\frac{16}{4} - \frac{2}{3}\times 8\right) - (0)\right\}$$
$$= -\left(4 - \frac{16}{3}\right)$$
$$= \frac{4}{3} \text{ or } 1\tfrac{1}{3}$$

1 e $y = x(x-2)(x-5)$

$y = 0 \Rightarrow x = 0, 2, 5$

$x \to \infty, y \to \infty$

$x \to -\infty, y \to -\infty$



$$\int y\,dx = \int x(x^2 - 7x + 10)\,dx$$
$$= \int (x^3 - 7x^2 + 10x)\,dx$$
$$= \left(\frac{x^4}{4} - \frac{7}{3}x^3 + 5x^2\right)$$
$$\int_0^2 y\,dx = \left(\frac{16}{4} - \frac{7}{3}\times 8 + 20\right) - (0)$$
$$= 24 - \frac{56}{3}$$
$$= 5\tfrac{1}{3}$$
$$\int_2^5 y\,dx = \left(\frac{625}{4} - \frac{7}{3}\times 125 + 125\right) - \left(5\tfrac{1}{3}\right)$$
$$= -15\tfrac{3}{4}$$

So area $= 5\tfrac{1}{3} + 15\tfrac{3}{4}$
$= 21\tfrac{1}{12}$

2 a $x(x+3)(2-x) = 0$

$x = 0, x = -3$ or $x = 2$

$A(-3, 0), B(2, 0)$

2 b $\int_0^2 x(x+3)(2-x)\,dx - \int_{-3}^0 x(x+3)(2-x)\,dx$

$= \int_0^2 (-x^3 - x^2 + 6x)\,dx$

$- \int_{-3}^0 (-x^3 - x^2 + 6x)\,dx$

$\int_0^2 (-x^3 - x^2 + 6x)\,dx$

$= \left[-\frac{x^4}{4} - \frac{x^3}{3} + \frac{6x^2}{2}\right]_0^2$

$= \left[-\frac{x^4}{4} - \frac{x^3}{3} + 3x^2\right]_0^2$

$= \left(-\frac{2^4}{4} - \frac{2^3}{3} + 3(2)^2\right) - \left(-\frac{0^4}{4} - \frac{0^3}{3} + 3(0)^2\right)$

$= \left(-4 - \frac{8}{3} + 12\right)$

$= 5\frac{1}{3}$

$\int_{-3}^0 (-x^3 - x^2 + 6x)\,dx$

$= \left[-\frac{x^4}{4} - \frac{x^3}{3} + 3x^2\right]_{-3}^0$

$= \left(-\frac{0^4}{4} - \frac{0^3}{3} + 3(0)^2\right)$

$- \left(-\frac{(-3)^4}{4} - \frac{(-3)^3}{3} + 3(-3)^2\right)$

$= -\left(-\frac{81}{4} + 9 + 27\right)$

$= -15\frac{3}{4}$

Total area $= 5\frac{1}{3} + 15\frac{3}{4}$

$= 21\frac{1}{12}$

3 a $f(-3) = -(-3)^3 + 4(-3)^2 + 11(-3) - 30$

$= 27 + 36 - 33 - 30 = 0$

3 b

$$
\begin{array}{r}
-x^2 + 7x - 10 \\
x+3\overline{)\,-x^3 + 4x^2 + 11x - 30} \\
\underline{-x^3 - 3x^2 \qquad\qquad\quad} \\
7x^2 + 11x \qquad \\
\underline{7x^2 + 21x \qquad} \\
-10x - 30 \\
\underline{-10x - 30} \\
0
\end{array}
$$

$f(x) = (x + 3)(-x^2 + 7x - 10)$

c $f(x) = (x + 3)(-x + 2)(x - 5)$

d $x = -3$, $x = 2$ or $x = 5$

$(-3, 0)$, $(2, 0)$ and $(5, 0)$

3 e Total area is:

$$\int_2^5 (-x^3+4x^2+11x-30)\,dx - \int_{-3}^2 (-x^3+4x^2+11x-30)\,dx$$

$$\int_2^5 (-x^3+4x^2+11x-30)\,dx$$

$$=\left[-\frac{x^4}{4}+\frac{4x^3}{3}+\frac{11x^2}{2}-30x\right]_2^5$$

$$=\left(-\frac{5^4}{4}+\frac{4(5)^3}{3}+\frac{11(5)^2}{2}-30(5)\right) - \left(-\frac{2^4}{4}+\frac{4(2)^3}{3}+\frac{11(2)^2}{2}-30(2)\right)$$

$$=\left(-\frac{625}{4}+\frac{500}{3}+\frac{275}{2}-150\right) - \left(-4+\frac{32}{3}+22-60\right)$$

$$=29\tfrac{1}{4}$$

$$\int_{-3}^2 (-x^3+4x^2+11x-30)\,dx$$

$$=\left[-\frac{x^4}{4}+\frac{4x^3}{3}+\frac{11x^2}{2}-30x\right]_{-3}^2$$

$$=\left(-\frac{2^4}{4}+\frac{4(2)^3}{3}+\frac{11(2)^2}{2}-30(2)\right) - \left(-\frac{(-3)^4}{4}+\frac{4(-3)^3}{3}+\frac{11(-3)^2}{2}-30(-3)\right)$$

$$\int_{-3}^2 (-x^3+4x^2+11x-30)\,dx$$

$$=\left(-4+\frac{32}{3}+22-60\right) - \left(-\frac{81}{4}-\frac{108}{3}+\frac{99}{2}+90\right)$$

$$=-114\tfrac{7}{12}$$

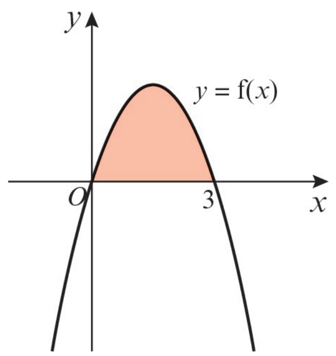
Total area $= 29\tfrac{1}{4} + 114\tfrac{7}{12}$

$= 143\tfrac{5}{6}$

Challenge

1 a $x(3-x)=0$

$x=0$ or $x=3$



$f(x)=3x-x^2$

$$\text{Area} = \int_0^3 (3x-x^2)\,dx$$

$$=\left[\frac{3x^2}{2}-\frac{x^3}{3}\right]_0^3$$

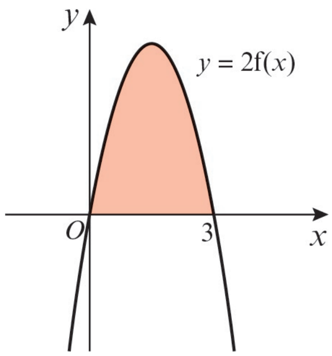
$$=\left(\frac{3(3)^2}{2}-\frac{3^3}{3}\right)-\left(\frac{3(0)^2}{2}-\frac{0^3}{3}\right)$$

$$=\left(\tfrac{27}{2}-9\right)$$

$$=4\tfrac{1}{2}$$

Challenge

1 b



$f(x) = 6x - 2x^2$

$$\text{Area} = \int_0^3 (6x - 2x^2)\,dx$$
$$= \left[\frac{6x^2}{2} - \frac{2x^3}{3}\right]_0^3$$
$$= \left[3x^2 - \frac{2x^3}{3}\right]_0^3$$
$$= \left(3(3)^2 - \frac{2(3)^3}{3}\right) - \left(3(0)^2 - \frac{2(0)^3}{3}\right)$$
$$= (27 - 18)$$
$$= 9$$

c $f(x) = a(3x - x^2)$

Area = a × area of f(x)

$= a \times 4\frac{1}{2}$

$$= \frac{9a}{2}$$

d $y = f(x + a)$ is a translation of $y = f(x)$ by $\begin{pmatrix} -a \\ 0 \end{pmatrix}$.

Therefore, the area of $y = f(x + a)$ is equal to the area of $y = f(x)$.

The area of $y = f(x + a)$ is $4\frac{1}{2}$

1 e $f(ax) = 3ax - a^2x^2$

$$\text{Area} = \int_0^{\frac{3}{a}} (3ax - a^2x^2)\,dx$$
$$= \left[\frac{3ax^2}{2} - \frac{a^2x^3}{3}\right]_0^{\frac{3}{a}}$$
$$= \left(\frac{3a\left(\frac{3}{a}\right)^2}{2} - \frac{a^2\left(\frac{3}{a}\right)^3}{3}\right) - \left(\frac{3(0)^2}{2} - \frac{0^3}{3}\right)$$
$$= \left(\frac{27}{2a} - \frac{9}{a}\right)$$
$$= \frac{9}{2a}$$