Chapter review 7

1 a $y = x^{\frac{3}{2}} + \frac{48}{x} \quad (x > 0)$

$\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} - \frac{48}{x^2}$

Putting $\frac{dy}{dx} = 0$:

$\frac{3}{2}x^{\frac{1}{2}} = \frac{48}{x^2}$

$x^{\frac{5}{2}} = 32$

$x = 4$

Substituting $x = 4$ into $y = x^{\frac{3}{2}} + \frac{48}{x}$ gives:

$y = 8 + 12 = 20$

So $x = 4$ and $y = 20$ when $\frac{dy}{dx} = 0$.

b $\frac{d^2y}{dx^2} = \frac{3}{4}x^{-\frac{1}{2}} + \frac{96}{x^3}$

When $x = 4$, $\frac{d^2y}{dx^2} = \frac{3}{8} + \frac{96}{64} = \frac{15}{8} > 0$

$\therefore$ minimum

2 $y = x^3 - 5x^2 + 7x - 14$

$\frac{dy}{dx} = 3x^2 - 10x + 7$

Putting $3x^2 - 10x + 7 = 0$

$(3x - 7)(x - 1) = 0$

So $x = \frac{7}{3}$ or $x = 1$

When $x = \frac{7}{3}$,

$y = \left(\frac{7}{3}\right)^3 - 5\left(\frac{7}{3}\right)^2 + 7\left(\frac{7}{3}\right) - 14$

$= -\frac{329}{27}$

$y = -12\frac{5}{27}$

When $x = 1$,

$y = 1^3 - 5(1)^2 + 7(1) - 14$

$= -11$

So $\left(\frac{7}{3}, -12\frac{5}{27}\right)$ and $(1, -11)$ are stationary points.

3 a $f'(x) = x^2 - 2 + \frac{1}{x^2} \quad (x > 0)$

$f''(x) = 2x - \frac{2}{x^3}$

When $x = 4$, $f''(x) = 8 - \frac{2}{64}$

$= 7\frac{31}{32}$

b For an increasing function, $f'(x) \geq 0$

$x^2 - 2 + \frac{1}{x^2} \geq 0$

$\left(x - \frac{1}{x}\right)^2 \geq 0$

This is true for all x, except $x = 1$ (where $f'(1) = 0$).

So the function is an increasing function.

4 $y = x^3 - 6x^2 + 9x$

$\frac{dy}{dx} = 3x^2 - 12x + 9$

Putting $3x^2 - 12x + 9 = 0$

$3(x^2 - 4x + 3) = 0$

$3(x - 1)(x - 3) = 0$

So $x = 1$ or $x = 3$

So there are stationary points when $x = 1$ and $x = 3$.

$\frac{d^2y}{dx^2} = 6x - 12$

When $x = 1$, $\frac{d^2y}{dx^2} = 6 - 12 = -6 < 0$, so maximum point

When $x = 3$, $\frac{d^2y}{dx^2} = 18 - 12 = 6 > 0$, so minimum point

When $x = 1$, $y = 1 - 6 + 9 = 4$

So $(1, 4)$ is a maximum point.

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5 a $f(x) = 3x^4 - 8x^3 - 6x^2 + 24x + 20$

$f'(x) = 12x^3 - 24x^2 - 12x + 24$

$= 12(x^3 - 2x^2 - x + 2)$

$= 12(x - 1)(x^2 - x - 2)$

$= 12(x - 1)(x - 2)(x + 1)$

So $x = 1$, $x = 2$ or $x = -1$

$f(1) = 3 - 8 - 6 + 24 + 20$

$= 33$

$f(2) = 3(2)^4 - 8(2)^3 - 6(2)^2 + 24(2) + 20$

$= 28$

$f(-1) = 3 + 8 - 6 - 24 + 20$

$= 1$

So (1, 33), (2, 28) and (−1, 1) are stationary points.

$f''(x) = 36x^2 - 48x - 12$

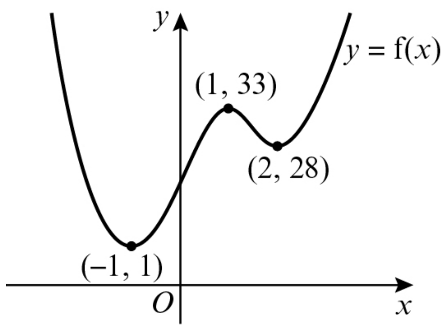
$f''(1) = 36 - 48 - 12 = -24 < 0$, so maximum

$f''(2) = 36(2)^2 - 48(2) - 12 = 36 > 0$, so minimum

$f''(-1)$, $y = 36 + 48 - 12 = 72 > 0$, so minimum

So (1, 33) is a maximum point and (2, 28) and (−1, 1) are minimum points.

b



6 a $f(x) = 200 - \dfrac{250}{x} - x^2$

$f'(x) = \dfrac{250}{x^2} - 2x$

6 b At the maximum point, B, $f'(x) = 0$

$\dfrac{250}{x^2} - 2x = 0$

$\dfrac{250}{x^2} = 2x$

$250 = 2x^3$

$x^3 = 125$

$x = 5$

When $x = 5$, $y = f(5) = 200 - \dfrac{250}{5} - 5^2$

$= 125$

The coordinates of B are (5, 125).

7 a P has coordinates m, $\left(x,\ 5 - \frac{1}{2}x^2\right)$.

$OP^2 = (x - 0)^2 + \left(5 - \frac{1}{2}x^2 - 0\right)^2$

$= x^2 + 25 - 5x^2 + \dfrac{1}{4}x^4$

$= \dfrac{1}{4}x^4 - 4x^2 + 25$

b Given $f(x) = \dfrac{1}{4}x^4 - 4x^2 + 25$

$f'(x) = x^3 - 8x$

When $f'(x) = 0$,

$x^3 - 8x = 0$

$x(x^2 - 8) = 0$

$x = 0$ or $x^2 = 8$

$x = 0$ or $x = \pm 2\sqrt{2}$

c $f''(x) = 3x^2 - 8$

When $x = 0$, $f''(x) = -8 < 0$, so maximum

When $x^2 = 8$, $f''(x) = 3 \times 8 - 8 = 16 > 0$, so minimum

Substituting $x^2 = 8$ into $f(x)$:

$OP^2 = \dfrac{1}{4} \times 8^2 - 4 \times 8 + 25 = 9$

So $OP = 3$ when $x = \pm 2\sqrt{2}$

8 a $y = 3 + 5x + x^2 - x^3$

Let $y = 0$, then

$3 + 5x + x^2 - x^3 = 0$

$(3 - x)(1 + 2x + x^2) = 0$

$(3 - x)(1 + x)^2 = 0$

$x = 3$ or $x = -1$ when $y = 0$

The curve touches the x-axis at $x = -1$ (A) and cuts the axis at $x = 3$ (C).

C has coordinates (3, 0)

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8 b $\frac{dy}{dx} = 5 + 2x - 3x^2$

Putting $\frac{dy}{dx} = 0$

$5 + 2x - 3x^2 = 0$

$(5 - 3x)(1 + x) = 0$

So $x = \frac{5}{3}$ or $x = -1$

When $x = \frac{5}{3}$,

$y = 3 + 5\left(\frac{5}{3}\right) + \left(\frac{5}{3}\right)^2 - \left(\frac{5}{3}\right)^3 = 9\frac{13}{27}$

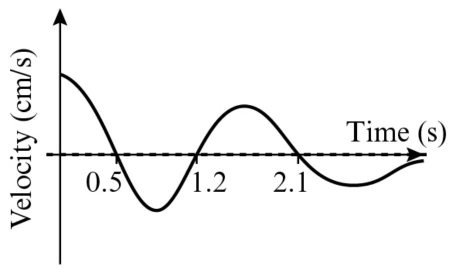
So B is $\left(\frac{5}{3}, 9\frac{13}{27}\right)$.

When $x = -1$, $y = 0$

So A is $(-1, 0)$.

9

x	$y = f(x)$	$y = f'(x)$
$0 < x < 0.5$	Positive gradient	Above x-axis
$x = 0.5$	Maximum	Cuts x-axis
$0.5 < x < 1.2$	Negative gradient	Below x-axis
$x = 1.2$	Minimum	Cuts x-axis
$1.2 < x < 2.1$	Positive gradient	Above x-axis
$x = 2.1$	Maximum	Cuts x-axis
$x > 2.1$	Negative gradient	Below x-axis with asymptote at $y = 0$



10 $V = \pi(40r - r^2 - r^3)$

$\frac{dV}{dr} = 40\pi - 2\pi r - 3\pi r^2$

Putting $\frac{dV}{dr} = 0$

$\pi(40 - 2r - 3r^2) = 0$

$(4 + r)(10 - 3r) = 0$

$r = \frac{10}{3}$ or $r = -4$

As r is positive, $r = \frac{10}{3}$

Substituting into the given expression for V:

$V = \pi\left(40 \times \frac{10}{3} - \frac{100}{9} - \frac{1000}{27}\right) = \frac{2300}{27}\pi$

11 $A = 2\pi x^2 + \frac{2000}{x} = 2\pi x^2 + 2000x^{-1}$

$\frac{dA}{dx} = 4\pi x - 2000x^{-2} = 4\pi x - \frac{2000}{x^2}$

Putting $\frac{dA}{dx} = 0$

$4\pi x = \frac{2000}{x^2}$

$x^3 = \frac{2000}{4\pi} = \frac{500}{\pi}$

12 a The total length of wire is

$\left(2y + x + \frac{\pi x}{2}\right)$ m

As total length is 2 m,

$2y + x\left(1 + \frac{\pi}{2}\right) = 2$

$y = 1 - \frac{1}{2}x\left(1 + \frac{\pi}{2}\right)$

b Area, $R = xy + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2$

Substituting $y = 1 - \frac{1}{2}x\left(1 + \frac{\pi}{2}\right)$ gives:

$R = x\left(1 - \frac{1}{2}x - \frac{\pi}{4}x\right) + \frac{\pi}{8}x^2$

$= \frac{x}{8}(8 - 4x - 2\pi x + \pi x)$

$= \frac{x}{8}(8 - 4x - \pi x)$

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12 c For maximum R, $\frac{dR}{dx} = 0$

$R = x - \frac{1}{2}x^2 - \frac{\pi}{8}x^2$

$\frac{dR}{dx} = 1 - x - \frac{\pi}{4}x$

Putting $\frac{dR}{dx} = 0$

$x = \frac{1}{1+\frac{\pi}{4}}$

$= \frac{4}{4+\pi}$

Substituting $x = \frac{4}{4+\pi}$ into R:

$R = \frac{1}{2(4+\pi)}\left(8 - \frac{16}{4+\pi} - \frac{4\pi}{4+\pi}\right)$

$R = \frac{1}{2(4+\pi)} \times \frac{32 + 8\pi - 16 - 4\pi}{4+\pi}$

$= \frac{1}{2(4+\pi)} \times \frac{16+4\pi}{4+\pi}$

$= \frac{4(4+\pi)}{2(4+\pi)^2}$

$= \frac{2}{4+\pi}$

13 a Let the height of the tin be h cm.
The area of the curved surface of the tin $= 2\pi xh$ cm^2
The area of the base of the tin $= \pi x^2$ cm^2
The area of the curved surface of the lid $= 2\pi x$ cm^2
The area of the top of the lid $= \pi x^2$ cm^2
Total area of sheet metal is 80π cm^2.
So $2\pi x^2 + 2\pi x + 2\pi xh = 80\pi$

$$h = \frac{40 - x - x^2}{x}$$

The volume, V, of the tin is given by
$V = \pi x^2 h$

$= \frac{\pi x^2(40 - x - x^2)}{x}$

$= \pi(40x - x^2 - x^3)$

13 b $\frac{dV}{dx} = \pi(40 - 2x - 3x^2)$

Putting $\frac{dV}{dx} = 0$

$40 - 2x - 3x^2 = 0$

$(10 - 3x)(4 + x) = 0$

So $x = \frac{10}{3}$ or $x = -4$

But x is positive, so $x = \frac{10}{3}$

c $\frac{d^2V}{dx^2} = \pi(-2 - 6x)$

When $x = \frac{10}{3}$, $\frac{d^2V}{dx^2} = \pi(-2 - 20) < 0$

So V is a maximum.

d $V = \pi\left(40 \times \frac{10}{3} - \left(\frac{10}{3}\right)^2 - \left(\frac{10}{3}\right)^3\right)$

$= \pi\left(\frac{400}{3} - \frac{100}{9} - \frac{1000}{27}\right)$

$= \frac{2300}{27}\pi$

e Lid has surface area $\pi x^2 + 2\pi x$

When $x = \frac{10}{3}$,

this is $\pi\left(\frac{100}{9} + \frac{20}{3}\right) = \frac{160}{9}\pi$

Percentage of total surface area =

$\frac{\frac{160}{9}\pi}{80\pi} \times 100 = \frac{200}{9} = 22.2...\%$

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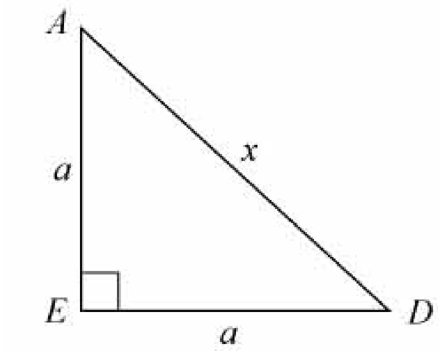
14 a Let the equal sides of ΔADE be a metres.



Using Pythagoras' theorem,

$a^2 + a^2 = x^2$

$2a^2 = x^2$

$a^2 = \frac{x^2}{2}$

Area of $\Delta ADE = \frac{1}{2} \times \text{base} \times \text{height}$

$= \frac{1}{2} \times a \times a$

$= \frac{x^2}{4}\ \text{m}^2$

b Area of two triangular ends

$= 2 \times \frac{x^2}{4} = \frac{x^2}{2}$

Let the length $AB = CD = y$ metres

Area of two rectangular sides

$= 2 \times ay = 2ay = 2\sqrt{\frac{x^2}{2}}\,y$

So $S = \frac{x^2}{2} + 2\sqrt{\frac{x^2}{2}}\,y = \frac{x^2}{2} + xy\sqrt{2}$

But capacity of storage tank $= \frac{1}{4}x^2 \times y$

So $\frac{1}{4}x^2y = 4000$

$y = \frac{16\,000}{x^2}$

Substituting for y in equation for S gives:

$S = \frac{x^2}{2} + \frac{16\,000\sqrt{2}}{x}$

14 c $\frac{dS}{dx} = x - \frac{16\,000\sqrt{2}}{x^2}$

Putting $\frac{dS}{dx} = 0$

$x = \frac{16\,000\sqrt{2}}{x^2}$

$x^3 = 16\,000\sqrt{2}$

$x = 20\sqrt{2} = 28.28$ (4 s.f.)

When $x = 20\sqrt{2}$,

$S = 400 + 800 = 1200$

d $\frac{d^2S}{dx^2} = 1 + \frac{32\,000\sqrt{2}}{x^3}$

When $x = 20\sqrt{2}$, $\frac{d^2S}{dx^2} = 3 > 0$, so value is a minimum.

Challenge

a Any constant function of the form $y = k$, where k is any real number

b For example, $f(x) = x$ for $0 < x < 1$, $f(x) = -x$ for $1 < x < 2$ or any suitably defined piecewise function.

c For example $f(x) = \tan\left(\pi\left(x - \frac{1}{2}\right)\right)$ or the piecewise function $f(x) = x$ for $0 < x < 1$, and 0 otherwise, or any other suitably defined piecewise function.

d For example the piecewise function $f(x) = x$ if x is rational and $-x$ if x is irrational.
Note that even though this function is not differentiable, $f(x)$ is **increasing in value** as x increases for the rational values of x and $f(x)$ is **decreasing in value** as x increases for the irrational values of x

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