Exercise 6F

1 a $4\cos^2\theta = 1 \Rightarrow \cos^2\theta = \frac{1}{4}$

So $\cos\theta = \pm\frac{1}{2}$

Solutions are $60°, 120°, 240°, 300°$.

b $2\sin^2\theta - 1 = 0 \Rightarrow \sin^2\theta = \frac{1}{2}$

So $\sin\theta = \pm\frac{1}{\sqrt{2}}$

Solutions are in all four quadrants, at 45° to the horizontal.

So $\theta = 45°, 135°, 225°, 315°$

c Factorising, $\sin\theta(3\sin\theta + 1) = 0$

So $\sin\theta = 0$ or $\sin\theta = -\frac{1}{3}$

Solutions of $\sin\theta = 0$ are

$\theta = 0°\ 180°, 360°$ (from graph)

Solutions of $\sin\theta = -\frac{1}{3}$ are

$\theta = 199°,\ 341°$ (3 s.f.)

These are in the third and fourth quadrants.

d $\tan^2\theta - 2\tan\theta - 10 = 0$

So $\tan\theta = \frac{2 \pm \sqrt{4+40}}{2}$

$= \frac{2 \pm \sqrt{44}}{2}$

$(= -2.3166\ldots \text{ or } 4.3166\ldots)$

Solutions of $\tan\theta = \frac{2-\sqrt{44}}{2}$ are in the second and fourth quadrants.

So $\theta = 113.35°, 293.3°$

Solutions of $\tan\theta = \frac{2+\sqrt{44}}{2}$ are in the first and third quadrants.

So $\theta = 76.95\ldots°, 256.95\ldots°$

Solution set: $77.0°, 113°, 257°, 293°$

1 e Factorising LHS of

$2\cos^2\theta - 5\cos\theta + 2 = 0$

$(2\cos\theta - 1)(\cos\theta - 2) = 0$

So $2\cos\theta - 1 = 0$ or $\cos\theta - 2 = 0$

As $\cos\theta \le 1, \cos\theta = 2$ has no solutions.

Solutions of $\cos\theta = \frac{1}{2}$ are $\theta = 60°,\ 300°$

f $\sin^2\theta - 2\sin\theta - 1 = 0$

So $\sin\theta = \frac{2 \pm \sqrt{8}}{2}$

Solve $\sin\theta = \frac{2-\sqrt{8}}{2}$ as $\frac{2+\sqrt{8}}{2} > 1$

$\theta = 204°, 336°$

The solutions are in the third and fourth quadrants as $\frac{2-\sqrt{8}}{2} < 0$.

g $\tan^2 2\theta = 3 \Rightarrow \tan 2\theta = \pm\sqrt{3}$

Solve $\tan X = +\sqrt{3}$ and $\tan X = -\sqrt{3}$, where $X = 2\theta$

The interval for X is $0 \le X \le 720°$.

For $\tan X = \sqrt{3}$,

$X = 60°, 240°, 420°, 600°$

So $\theta = \frac{X}{2} = 30°, 120°, 210°, 300°$

For $\tan X = -\sqrt{3}$,

$X = 120°, 300°, 480°, 660°$

So $\theta = 60°, 150°, 240°, 330°$

Solution set:

$\theta = 30°, 60°, 120°, 150°, 210°, 240°, 300°, 330°$

2 a $\sin^2 2\theta = 1,\ -\pi \le \theta \le \pi$

Let $X = 2\theta$

$\sin^2 X = 1,\ -2\pi \le \theta \le 2\pi$

$\sin X = \pm 1$

$X = \pm \dfrac{k\pi}{2}$

$X = -\dfrac{3\pi}{2}, X = -\dfrac{\pi}{2},\ X = \dfrac{\pi}{2}$ and $X = \dfrac{3\pi}{2}$

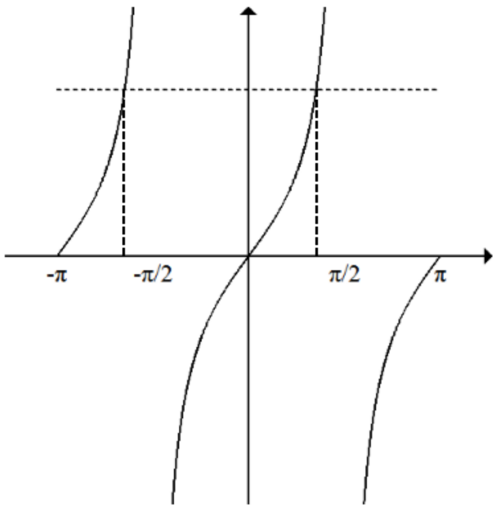
Since $X = 2\theta$

$\theta = -\dfrac{3\pi}{4}, \theta = -\dfrac{\pi}{4},\ \theta = \dfrac{\pi}{4}$ and $\theta = \dfrac{3\pi}{4}$

b $\tan^2 \theta = 2\tan\theta,\ -\pi \le \theta \le \pi$

$\tan\theta(\tan\theta - 2) = 0$

$\tan\theta = 0$ or $\tan\theta = 2$



When $\tan\theta = 0$

$\theta = -\pi,\ \theta = 0$ and $\theta = \pi$

When $\tan\theta = 2$

$\theta = -\pi + 1.11 = -2.03$ and $\theta = 1.11$

c $\cos\theta(\cos\theta - 2) = 1,\ -\pi \le \theta \le \pi$

$\cos^2\theta - 2\cos\theta - 1 = 0$

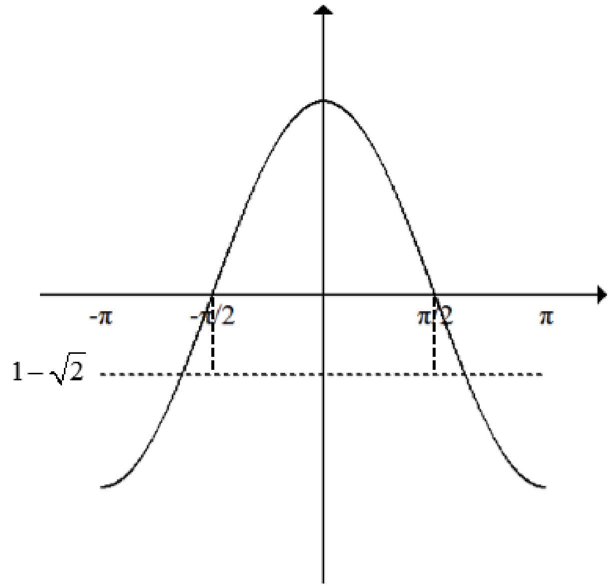
$(\cos\theta - 1)^2 - 2 = 0$

$\cos\theta - 1 = \pm\sqrt{2}$

$\cos\theta = 1 \pm \sqrt{2}$

Since $\cos\theta \ne 1 + \sqrt{2}$

$\cos\theta = 1 - \sqrt{2}$



$\theta = -2.00$ and $\theta = 2.00$

d $4\sin\theta = \tan\theta,\ -\pi \le \theta \le \pi$

$4\sin\theta = \dfrac{\sin\theta}{\cos\theta}$

$4\sin\theta\cos\theta = \sin\theta$

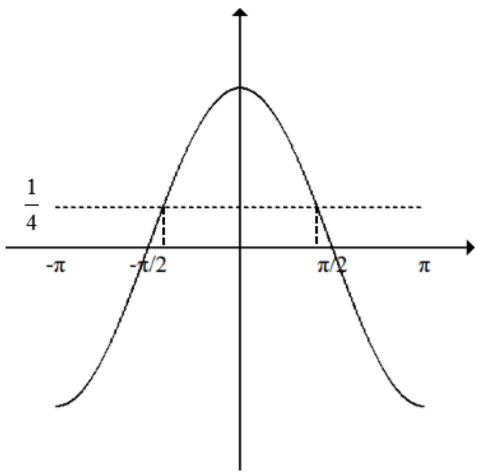
$4\sin\theta\cos\theta - \sin\theta = 0$

$\sin\theta(4\cos\theta - 1) = 0$

$\sin\theta = 0$ and $\cos\theta = \dfrac{1}{4}$

When $\sin\theta = 0$

$\theta = -\pi,\ \theta = 0$ and $\theta = \pi$



When $\cos\theta = \dfrac{1}{4}$

$\theta = -1.32$ and $\theta = 1.32$

3 a $4(\sin^2\theta - \cos\theta) = 3 - 2\cos\theta,\ 0 \le \theta \le \pi$

$4(1 - \cos^2\theta - \cos\theta) = 3 - 2\cos\theta$

$4 - 4\cos^2\theta - 4\cos\theta = 3 - 2\cos\theta$

$4\cos^2\theta + 2\cos\theta - 1 = 0$

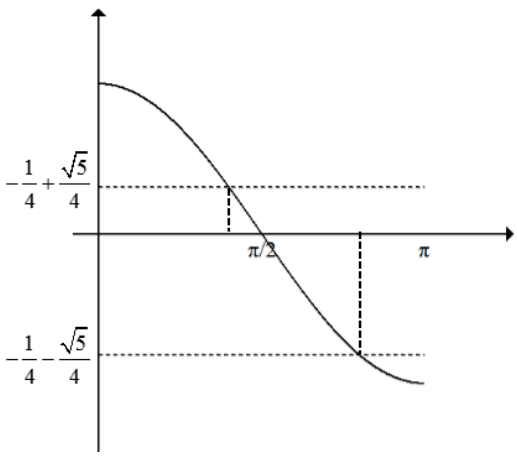
$4\left(\cos^2\theta + \frac{1}{2}\cos\theta - \frac{1}{4}\right) = 0$

$\left(\cos\theta + \frac{1}{4}\right)^2 - \frac{1}{16} - \frac{1}{4} = 0$

$\left(\cos\theta + \frac{1}{4}\right)^2 = \frac{5}{16}$

$\cos\theta + \frac{1}{4} = \pm\sqrt{\frac{5}{16}}$

$\cos\theta = -\frac{1}{4} \pm \frac{\sqrt{5}}{4}$



$\theta = \frac{2}{5}\pi$ and $\theta = \frac{4}{5}\pi$

b $2\sin^2\theta = 3(1 - \cos\theta),\ 0 \le \theta \le \pi$

$2(1 - \cos^2\theta) = 3(1 - \cos\theta)$

$2 - 2\cos^2\theta = 3 - 3\cos\theta$

$2\cos^2\theta - 3\cos\theta + 1 = 0$

$(2\cos\theta - 1)(\cos\theta - 1) = 0$

$\cos\theta = 1$ and $\cos\theta = \frac{1}{2}$

When $\cos\theta = 1$, $\theta = 0$

$\cos\theta = \frac{1}{2}$, $\theta = \frac{\pi}{3}$

3 c $4\cos^2\theta - 5\sin\theta - 5 = 0,\ 0 \le \theta \le \pi$

$4\cos^2\theta - 5\sin\theta - 5 = 0$

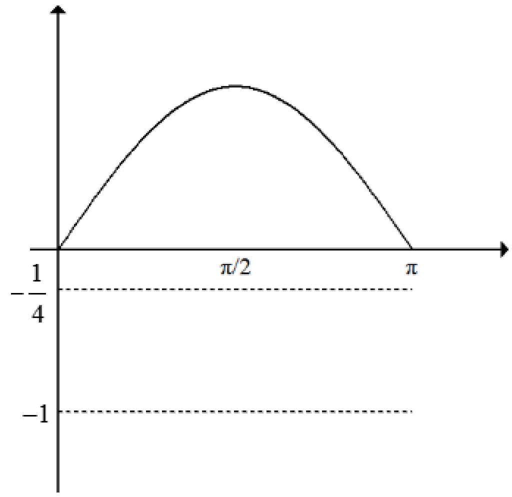
$4(1 - \sin^2\theta) - 5\sin\theta - 5 = 0$

$4 - 4\sin^2\theta - 5\sin\theta - 5 = 0$

$4\sin^2\theta + 5\sin\theta + 1 = 0$

$(4\sin\theta + 1)(\sin\theta + 1) = 0$

$\sin\theta = -1$ and $\sin\theta = -\frac{1}{4}$



So no solutions in the range.

4 a $5\sin^2\theta = 4\cos^2\theta$

$\Rightarrow \tan^2\theta = \frac{4}{5}$ as $\tan\theta = \frac{\sin\theta}{\cos\theta}$

So $\tan\theta = \pm\sqrt{\frac{4}{5}}$

There are solutions from each of the quadrants

(angle to horizontal is 41.8°).

$\theta = \pm 138°, \pm 41.8°$

4 **b** $\tan\theta = \cos\theta$

$\Rightarrow \dfrac{\sin\theta}{\cos\theta} = \cos\theta$

$\Rightarrow \sin\theta = \cos^2\theta$

$\Rightarrow \sin\theta = 1 - \sin^2\theta$

$\Rightarrow \sin^2\theta + \sin\theta - 1 = 0$

So $\sin\theta = \dfrac{-1 \pm \sqrt{5}}{2}$

There are only solutions from

$\sin\theta = \dfrac{-1+\sqrt{5}}{2} \quad \left(\text{as } \dfrac{-1-\sqrt{5}}{2} < -1\right)$

Solutions are $\theta = 38.2°, 142°$

(first and second quadrants).

5 $8 \sin^2 x + 6 \cos x - 9 = 0$ can be written as
$8(1 - \cos^2 x) + 6 \cos x - 9 = 0$
which reduces to
$8 \cos^2 x - 6 \cos x + 1 = 0$
So $(4 \cos x - 1)(2 \cos x - 1) = 0$

$\cos x = \dfrac{1}{4}$ or $\cos x = \dfrac{1}{2}$

So $x = 75.5°, 284.5°, 60°, 300°$

The solutions are
$x = 60°, 75.5°, 284.5°, 300°$

6 $1 + \sin^2 x = \dfrac{7}{2}\cos^2 x$

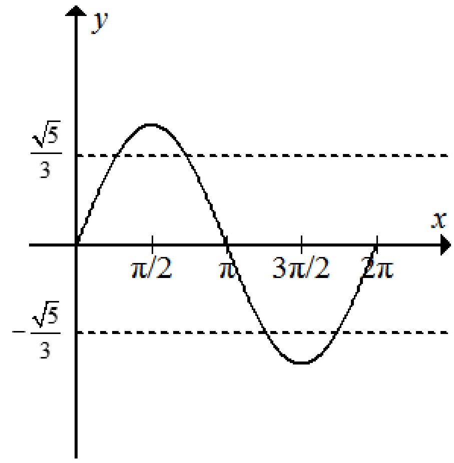
$1 + \sin^2 x = \dfrac{7}{2}\left(1 - \sin^2 x\right)$

$2 + 2\sin^2 x = 7 - 7\sin^2 x$

$9\sin^2 x = 5$

$\sin^2 x = \dfrac{5}{9}$

$\sin x = \pm\dfrac{\sqrt{5}}{3}$



$x = 0.841, \pi - 0.841, \pi + 0.841, 2\pi - 0.841$
$x = 0.841, 2.30, 3.98, 5.44$

7 $2 \cos^2 x + \cos x - 6 = 0$

$(2 \cos x - 3)(\cos x + 2) = 0$

$\cos x = \dfrac{3}{2}$ or $\cos x = -2$

There are no solutions to $\cos x = \dfrac{3}{2}$ or

$\cos x = -2$, so the equation has no solutions.

8 **a** $\cos^2 x = 2 - \sin x$ can be written as
$(1 - \sin^2 x) = 2 - \sin x$
$\sin^2 x - \sin x + 1 = 0$

b $\sin^2 x - \sin x + 1 = 0$
Using the discriminant
$b^2 - 4ac = (-1)^2 - 4 \times 1 \times 1$
$= -3$

As $b^2 - 4ac < 0$, therefore there are no real roots.

9 **a** $\tan^2 x - 2\tan x - 4 = 0$

$$\tan x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-4)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{20}}{2}$$

$$= \frac{2 \pm 2\sqrt{5}}{2}$$

$$= 1 \pm \sqrt{5}$$

$p = 1$, $q = 5$

b $\tan x = 1 \pm \sqrt{5}$

$x = 72.8°, 252.8°, 432.8°, -51.0°, 129.0°, 309.0°, 489.0°$

So the solutions are

$x = 72.8°, 129.0°, 252.8°, 309.0°, 432.8°, 489.0°$

Challenge

a Let $X = 3\theta$

So $\cos^2 X - \cos X - 2 = 0$

$(\cos X + 1)(\cos X - 2) = 0$

$\cos X = -1$ or $\cos X = 2$

$\cos X = 2$ has no solutions so $\cos X = -1$

As $X = 3\theta$, then as $-180° \leq \theta \leq 180°$

So $3 \times -180° \leq X \leq 3 \times 180°$

So the interval for X is $-540° \leq X \leq 540°$.

$X = -540°, -180°, 180°, 540°$

I.e. $3\theta = -540°, -180°, 180°, 540°$

So $\theta = -180°, -60°, 60°, 180°$

b Let $X = \theta - 45°$

So $\tan^2 X = 1$

$\tan X = \pm 1$

As $X = \theta - 45°$, then as $0 \leq \theta \leq 360°$

So $0 - 45° \leq X \leq 360° - 45°$

So the interval for X is $-45° \leq X \leq 315°$.

$X = -45°, 135°, 315°, 45°, 225°$

I.e. $\theta - 45° = -45°, 45°, 135°, 225°, 315°$

So $\theta = 0°, 90°, 180°, 270°, 360°$