## Chapter review 5

**1 a** $ar^2 = 27$ and $ar^5 = 8$

$$\frac{ar^5}{ar^2} = \frac{8}{27}$$

$$r^3 = \frac{8}{27}$$

$$r = \frac{2}{3}$$

**b** $ar^2 = 27$

$$a\left(\frac{2}{3}\right)^2 = 27$$

$$a = \frac{243}{4}$$

**c** $S_\infty = \frac{a}{1-r}$

$$= \frac{\frac{243}{4}}{1-\left(\frac{2}{3}\right)}$$

$$= \frac{729}{4}$$

**d** $S_n = \frac{a(1-r^n)}{1-r}$

$$S_{10} = \frac{\frac{243}{4}\left(1-\left(\frac{2}{3}\right)^{10}\right)}{1-\frac{2}{3}}$$

$$= \frac{58025}{324}$$

$$S_\infty - S_{10} = \frac{729}{4} - \frac{58025}{324}$$

$$= \frac{256}{81}$$

$$= 3.16 \text{ (3 s.f.)}$$

**2 a** $ar = 80$ and $ar^4 = 5.12$

$$\frac{ar^4}{ar} = \frac{5.12}{80}$$

$$r^3 = \frac{8}{125}$$

$$r = \frac{2}{5}$$

**b** $ar = 80$

$$a\left(\frac{2}{5}\right) = 80$$

$$a = 200$$

**c** $S_\infty = \frac{a}{1-r}$

$$= \frac{200}{1-\left(\frac{2}{5}\right)}$$

$$= \frac{1000}{3}$$

**d** $S_n = \frac{a(1-r^n)}{1-r}$

$$S_{14} = \frac{200\left(1-\left(\frac{2}{5}\right)^{14}\right)}{1-\frac{2}{5}}$$

$$= 333.3324...$$

$$S_\infty - S_{14} = \frac{1000}{3} - 333.3324...$$

$$= 8.95 \times 10^{-4} \text{ (3 s.f.)}$$

**3 a** $u_n = 95\left(\frac{4}{5}\right)^n$

$$u_1 = 95\left(\frac{4}{5}\right)^1 = 76$$

$$u_2 = 95\left(\frac{4}{5}\right)^2 = \frac{304}{5}$$

**b** $u_{21} = 95\left(\frac{4}{5}\right)^{21} = 0.876 \text{ (3 s.f.)}$

**3 c** $S_n = \dfrac{a(1-r^n)}{1-r}$

$S_{15} = \dfrac{76\left(1-\left(\frac{4}{5}\right)^{15}\right)}{1-\frac{4}{5}}$

$= 366.62...$

$= 367$ (3 s.f.)

**d** $S_\infty = \dfrac{a}{1-r}$

$S_\infty = \dfrac{76}{1-\frac{4}{5}}$

$= 380$

**4 a** $u_n = 3\left(\frac{2}{3}\right)^n - 1$

$u_1 = 3\left(\frac{2}{3}\right)^1 - 1 = 1$

$u_2 = 3\left(\frac{2}{3}\right)^2 - 1 = \frac{1}{3}$

$u_3 = 3\left(\frac{2}{3}\right)^3 - 1 = -\frac{1}{9}$

**b** $\sum_{n=1}^{15} 3\left(\frac{2}{3}\right)^n - 1 = \sum_{n=1}^{15} 3\left(\frac{2}{3}\right)^n - \sum_{n=1}^{15} 1$

$= \sum_{n=1}^{15} 3\left(\frac{2}{3}\right)^n - 15$

$S_{15} = \dfrac{2\left(1-\left(\frac{2}{3}\right)^{15}\right)}{1-\frac{2}{3}}$

$= 5.986...$

$S_{15} - 15 = -9.014$ (4 s.f.)

**c** $u_{n+1} = 3\left(\frac{2}{3}\right)^{n+1} - 1$

$= 3\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)^n - 1$

$= \frac{1}{3}\left(2\times 3\left(\frac{2}{3}\right)^n - 3\right)$

$= \frac{1}{3}\left(2\times 3\left(\frac{2}{3}\right)^n - 2 - 1\right)$

$= \frac{1}{3}\left(2\times\left(3\left(\frac{2}{3}\right)^n - 1\right) - 1\right)$

$= \dfrac{2u_n - 1}{3}$

**5 a** $ar^2 = 6.4$ and $ar^3 = 5.12$

$\dfrac{ar^3}{ar^2} = \dfrac{5.12}{6.4}$

$r = 0.8$

**b** $ar^2 = 6.4$

$a(0.8)^2 = 6.4$

$a = 10$

**c** $S_\infty = \dfrac{a}{1-r}$

$S_\infty = \dfrac{10}{1-0.8}$

$= 50$

**d** $S_n = \dfrac{a(1-r^n)}{1-r}$

$S_{25} = \dfrac{10(1-0.8^{25})}{1-0.8}$

$= 49.811...$

$S_\infty - S_{25} = 50 - 49.811...$

$= 0.1888...$

$= 0.189$ (3 s.f.)

**6 a** Let price of the car be $P$.

$P = 20000 \times 0.85^n$

$P = 20000 \times 0.85^5$

$= 8874.10...$

$= \$8874$ (to the nearest \$)

**b** The car will be worth \$4000 after

$20000 \times 0.85^n = 4000$

$0.85^n = 0.2$

$n \log 0.85 = \log 0.2$

$n = \dfrac{\log 0.2}{\log 0.85}$

$n = 9.903...$

So the car will be worth less than \$4000 when $n > 9.90$ years (3 s.f.).

**7 a** $p(3q+1),\ p(2q+2),\ p(2q-1)$

$\dfrac{p(3q+1)}{p(2q+2)} = \dfrac{p(2q+2)}{p(2q-1)}$

$(3q+1)(2q-1) = (2q+2)(2q+2)$

$6q^2 - q - 1 = 4q^2 + 8q + 4$

$2q^2 - 9q - 5 = 0$

$(q-5)(2q+1) = 0$

$q = 5$ or $q = -\dfrac{1}{2}$

**7 b** When $q = 5$ the first three terms are $16p,\ 12p,\ 9p$.

$S_\infty = \dfrac{a}{1-r}$

$\dfrac{16p}{1-0.75} = 896$

$p = 14$

$S_n = \dfrac{a(1-r^n)}{1-r}$

$S_{12} = \dfrac{224(1-0.75^{12})}{1-0.75}$

$= 867.617...$

$= 867.62$ (2 d.p.)

**8 a** $S_n = a + (a+d) + (a+2d) + ... + (a+(n-2)d) + (a+(n-1)d)$ (1)

$S_n = (a+(n-1)d) + (a+(n-2)d) + ... + (a+d) + a$ (2)

Adding (1) and (2) gives

$2S_n = n(2a+(n-1)d)$

$S_n = \frac{n}{2}(2a+(n-1)d)$

**b** $S_{100} = \frac{100}{2}(2(1)+(100-1)1)$

$= 5050$

**9** $\sum_{r=1}^{n}(4r-3) > 2000$

$a = 1$ and $d = 4$

$\frac{n}{2}(2(1)+4(n-1)) > 2000$

$n(2+4(n-1)) > 4000$

$n(2n-1) > 2000$

$2n^2 - n - 2000 > 0$

$n = \frac{1 \pm \sqrt{(-1)^2 - 4(2)(-2000)}}{2(2)}$

$= 31.87...$

So $n = 32$.

**10 a** $2a + d = 47$ (1)

$a + 29d = -62$ (2)

Subtracting (1) from 2 × (2)

$2a + 58d - (2a+d) = -124 - 47$

$57d = -171$

$d = -3$

When $d = -3$, $a = 25$.

**10 b** $S_n = \frac{n}{2}(2a+(n-1)d)$

$S_{60} = \frac{60}{2}(2(25)+(60-1)(-3))$

$= -3810$

**11 a** The series is 3, 6, 9, … 399

$a = 3$, $d = 3$, $n = 133$

$$S_n = \frac{n}{2}\left(2a + (n-1)d\right)$$

$$S_{133} = \frac{133}{2}\left(2(3) + (133-1)3\right)$$

$$= 26733$$

**b** $S_n = \frac{1}{2}n(n+1)$

$$S_{400} = \frac{1}{2}(400)(401)$$

$$= 80200$$

$$80200 - 26733 = 53467$$

**12** $S_n = \frac{n}{2}\left(2a + (n-1)d\right)$

$$S_{10} = \frac{10}{2}\left(2a + (10-1)d\right) = 675$$

$$2a + 9d = 135$$

Since the longest side is twice the length of the shortest side

$a + 9d = 2a \Rightarrow a = 9d$

$3a = 135$

$a = 45$ cm

**13** The series is 4, 8, 12, … with $2n$ terms

$$S_n = \frac{n}{2}\left(2a + (n-1)d\right)$$

$$S_{2n} = \frac{2n}{2}\left(2(4) + 4(2n-1)\right)$$

$$= 4n(2n+1)$$

**14 a** $u_{n+1} = ku_n - 4,\ u_1 = 2$

$u_2 = 2k - 4$

$u_3 = k(2k-4) - 4$

$= 2(k^2 - 2k - 2)$

**b** $u_3 = 26$

$2(k^2 - 2k - 2) = 26$

$k^2 - 2k - 15 = 0$

$(k+3)(k-5) = 0$

$k = -3$ or $k = 5$

**15 a** $a + 4d = 14$ and $S_3 = -3$

$$S_n = \frac{n}{2}\left(2a + (n-1)d\right)$$

$$S_3 = \frac{3}{2}\left(2a + (3-1)d\right) = -3$$

$a + d = -1$

$a + 4d - (a+d) = 14 - (-1)$

$3d = 15$

$d = 5$

$a = -6$

**b** $T(n) = -11 + 5n$

$-11 + 5n > 282$

$5n > 293$

$n > \frac{293}{5}$

$n = 59$

**16 a** $a + 3d = 3k$ and $S_6 = 7k + 9$

$$S_n = \frac{n}{2}\left(2a + (n-1)d\right)$$

$$S_6 = \frac{6}{2}\left(2a + (6-1)d\right) = 7k + 9$$

$6a + 15d = 7k + 9$ (1)

$a + 3d = 3k$ (2)

Subtracting 5 × (2) from (1)

$6a + 15d - 5(a + 3d) = 7k + 9 - 15k$

$a = 9 - 8k$

**b** $a = 9 - 8k$ and $a + 3d = 3k$

$9 - 8k + 3d = 3k$

$d = \frac{11k - 9}{3}$

**c** $a + 6d = 12$

where $a = 9 - 8k$ and $d = \frac{11k-9}{3}$

$$9 - 8k + 6\left(\frac{11k-9}{3}\right) = 12$$

$9 - 8k + 22k - 18 = 12$

$14k = 21$

$k = \frac{3}{2}$

**16 d** $a = 9 - 8k$, $d = \dfrac{11k - 9}{3}$ and $k = \dfrac{3}{2}$

$a = 9 - 8\left(\dfrac{3}{2}\right) = -3$

$d = \dfrac{11\left(\dfrac{3}{2}\right) - 9}{3} = \dfrac{5}{2}$

$S_n = \dfrac{n}{2}\left(2a + (n-1)d\right)$

$S_{20} = \dfrac{20}{2}\left(2(-3) + \dfrac{5}{2}(20-1)\right)$

$= 415$

**17 a** $a_{n+1} = \dfrac{1}{a_n}$, $a_1 = p$

$a_2 = \dfrac{1}{p}$

$a_3 = \dfrac{1}{\left(\dfrac{1}{p}\right)} = p$

$a_1 = a_3$

Therefore the sequence is periodic with order 2.

**b** $\displaystyle\sum_{n=1}^{1000} a_n = 500\left(p + \frac{1}{p}\right)$

**18 a** $a_1 = k$

$a_{n+1} = 2a_n + 6,\ n \geq 1$

$a_2 = 2k + 6$

$a_3 = 2(2k+6) + 6 = 4k + 18$

$a_1 < a_2 < a_3 \Rightarrow k < 2k + 6 < 4k + 18$

$0 < k + 6 < 3k + 18$

$-2k < 12$

$k > -6$

**b** $a_4 = 2(4k + 18) + 6 = 8k + 42$

**c** $\displaystyle\sum_{r=1}^{4} a_r = k + 2k + 6 + 4k + 18 + 8k + 42$

$= 15k + 66$

$= 3(5k + 22)$

So since this is a multiple of 3 it is divisible by 3.

**19 a** $a = 130$ and $S_\infty = 650$

$S_\infty = \dfrac{a}{1-r}$

$\dfrac{130}{1-r} = 650$

$r = \dfrac{4}{5}$

**b** $ar^6 = 130\left(\dfrac{4}{5}\right)^6 = 34.078...$

$ar^7 = 130\left(\dfrac{4}{5}\right)^7 = 27.262...$

$ar^7 - ar^6 = 6.82$ (3 s.f.)

**c** $S_n = \dfrac{a(1-r^n)}{1-r}$

$S_7 = \dfrac{130\left(1 - \left(\dfrac{4}{5}\right)^7\right)}{1 - \dfrac{4}{5}}$

$= 513.685...$

$= 513.69$ (2 d.p.)

**d** $S_n = \dfrac{a(1-r^n)}{1-r}$

$\dfrac{130\left(1 - \left(\dfrac{4}{5}\right)^n\right)}{1 - \left(\dfrac{4}{5}\right)} > 600$

$5\left(1 - \left(\dfrac{4}{5}\right)^n\right) > \dfrac{60}{13}$

$1 - \left(\dfrac{4}{5}\right)^n > \dfrac{12}{13}$

$\left(\dfrac{4}{5}\right)^n < \dfrac{1}{13}$

$n\log\left(\dfrac{4}{5}\right) < -\log 13$

$n > -\dfrac{\log 13}{\log\left(\dfrac{4}{5}\right)}$

**20 a** The model is $25000\times1.02^n$ where $n$ is the time in years since 2018.
$25000\times1.02^2=26010$

**b** $25000\times1.02^n>50000$
$1.02^n>2$
$n\log1.02>\log2$
$n>\frac{\log2}{\log1.02}$

**c** $n>\frac{\log2}{\log1.02}$
$n>35.002...$
2053

**d** $S_n=\frac{a(1-r^n)}{1-r}$
$S_8=\frac{25000(1.02^8-1)}{1.02-1}$
$=214574.22...$
$=214574$ appointments

**e** People may visit the doctor more frequently than once a year, some may not visit at all, depending on state of health.

**21 a** $2n+1$

**b** $2k+1=301$
$k=150$

**c i** $S_n=\frac{n}{2}(2a+(n-1)d)$
$S_q=\frac{q}{2}(2(3)+2(q-1))$
$=q(2+q)$
Since $S_q=p$
$p=q(2+q)$
$q^2+2q-p=0$

**21 c ii** $q^2+2q-1520>0$
$q^2+2q-1520>0$
$q=\frac{-2\pm\sqrt{2^2-4(1)(-1520)}}{2}$
$=\frac{-2\pm78}{2}$
$q=-40$ or $q=38$
$q>38$, so $q=39$ rows

**22 a** $ar=-3$ and $S_\infty=6.75$
$S_\infty=\frac{a}{1-r}$
$\frac{a}{1-r}=6.75$
$a=6.75(1-r)$
Substituting for $a$ into $ar=-3$ gives
$6.75r(1-r)=-3$
$6.75r-6.75r^2=-3$
$6.75r^2-6.75r-3=0$
$27r^2-27r-12=0$

**b** $27r^2-27r-12=0$
$r=\frac{27\pm\sqrt{(-27)^2-4(27)(-12)}}{2(27)}$
$=\frac{27\pm45}{54}$
$r=\frac{4}{3}$ or $r=-\frac{1}{3}$
Since the series is convergent $|r|<1$ so
$r=-\frac{1}{3}$

**c** $S_n=\frac{a(1-r^n)}{1-r}$
$S_5=\frac{9\left(1-\left(-\frac{1}{3}\right)^5\right)}{1-\left(-\frac{1}{3}\right)}$
$=\frac{61}{9}$
$=6.78$ (2 d.p.)

## Challenge

**a** $u_{n+2} = 5u_{n+1} - 6u_n$

$u_n = p\times 3^n + q\times 2^n$

$u_{n+2} = 5\left(p\times 3^{n+1} + q\times 2^{n+1}\right) - 6\left(p\times 3^n + q\times 2^n\right)$

$= 5\left(\frac{1}{3}p\times 3^{n+2} + \frac{1}{2}q\times 2^{n+2}\right) - 6\left(\frac{1}{9}p\times 3^{n+2} + \frac{1}{4}q\times 2^{n+2}\right)$

$= \frac{5}{3}p\times 3^{n+2} + \frac{5}{2}q\times 2^{n+2} - \frac{6}{9}p\times 3^{n+2} - \frac{6}{4}q\times 2^{n+2}$

$= p\times 3^{n+2} + q\times 2^{n+2}$ (as required)

**b** $u_1 = 5$ and $u_2 = 12$

$u_n = p\times 3^n + q\times 2^n$

$u_1 = p\times 3^1 + q\times 2^1 \Rightarrow 3p + 2q = 5$ **(1)**

$u_2 = p\times 3^2 + q\times 2^2 \Rightarrow 9p + 4q = 12$ **(2)**

Subtracting 2 × **(1)** from **(2)** gives

$9p + 4q - 2(3p + 2q) = 12 - 2(5)$

$3p = 2$

$p = \frac{2}{3}$

$q = \frac{3}{2}$

$u_n = \frac{2}{3}\times 3^n + \frac{3}{2}\times 2^n$

**c** $u_{100} = \frac{2}{3}\times 3^{100} + \frac{3}{2}\times 2^{100}$

$= 3.436\times 10^{47}$ (3 s.f.)

So 48 digits.