Exercise 5I

1 a Initial amount = \$4000
(start of month 1)
Start of month 2 = \$(4000 + 200)
Start of month 3 = \$(4000 + 200 + 200)
= \$(4000 + 2 × 200)
Start of month 10 = \$(4000 + 9 ×200)
= \$(4000 + 1800)
= \$5800

b Start of *m*th month
= \$(4000 + (*m* − 1) × 200)
= \$(4000 + 200*m* − 200)
= \$(3800 + 200*m*)

2

$$\underset{\text{Year 1}}{20\,000} \xrightarrow[\text{1st increment}]{+} \underset{\text{Year 2}}{20\,500} \xrightarrow[\text{2nd increment}]{+} \underset{\text{Year 3}}{21\,000} \xrightarrow[\text{3rd increment}]{+} \underset{\text{Year 4}}{21\,500} + \ldots$$

Nour will reach her maximum salary after
$$\frac{25\,000 - 20\,000}{500} = 10 \text{ increments}$$
This will be after 11 years.

a Total amount after 10 years
$$= \underbrace{20\,000 + 20\,500 + 21\,000 + \ldots}$$

This is an arithmetic series with
a = 20 000, d = 500 and n = 10. Use

$$S_n = \frac{n}{2}\left(2a + (n-1)d\right).$$
$$= \frac{10}{2}\left(40\,000 + 9 \times 500\right)$$
$$= 5 \times 44\,500$$
$$= €222\,500$$

b From year 11 to year 15 she will continue to earn €25 000.
Total in this time = 5 × 25 000
= €125 000.
Total amount in the first 15 years is
€222 500 + €125 000 = €347 500

c It is unlikely her salary will rise by the same amount each year.

3 Amount saved by James
$$= \underbrace{1 + 2 + 3 + \ldots + 42}$$

This is an arithmetic series with a = 1, d = 1, n = 42 and L = 42.

a Use $S_n = \frac{n}{2}(a + L)$
$$= \frac{42}{2}(1 + 42)$$
$$= 21 \times 43$$
$$= 903\text{c}$$
$$= \$9.03$$

b To save \$100 we need
$$\underbrace{1 + 2 + 3 + \ldots}_{\text{Sum to } n \text{ terms}} = 10\,000$$
$$\frac{n}{2}\left(2 \times 1 + (n-1) \times 1\right) = 10\,000$$
$$\frac{n}{2}(n+1) = 10\,000$$
$$n(n+1) = 20\,000$$
$$n^2 + n - 20\,000 = 0$$
$$n = \frac{-1 \pm \sqrt{(1)^2 - 4 \times 1 \times (-20\,000)}}{2}$$
$$n = 140.9 \text{ or } -141.9$$
It takes James 141 days to save \$100.

4 A growth of 10% a year gives a multiplication factor of 1.1.

a After 1 year number is 200 × 1.1 = 220

b After 2 years number is 200 × 1.1^2 = 242

c After 3 years number is
200 × 1.1^3 = 266.2 = 266
(to nearest whole number)

d After 10 years number is
200 × 1.1^{10} = 518.748… = 519
(to nearest whole number)

5 Let maximum speed in bottom gear be a km h^{-1}
This gives maximum speeds in each successive gear of ar, ar^2, ar^3, where r is the common ratio.
We are given
$a = 40$ **(1)**
$ar^3 = 120$ **(2)**
Substitute (**1**) into (**2**):

$$40r^3 = 120 \quad (\div 40)$$
$$r^3 = 3$$
$$r = \sqrt[3]{3}$$
$$r = 1.442\ldots \ (3\text{ d.p.})$$

Maximum speed in 2nd gear is
$ar = 40 \times 1.442\ldots = 57.7$ km h^{-1}
Maximum speed in 3rd gear is
$ar^2 = 40 \times (1.442\ldots)^2 = 83.2$ km h^{-1}

6 a $r = 0.85$
$a \times 0.85^3 = 11\,054.25$
$a = €18\,000$

b $18\,000 \times 0.85^n > 5000$

$$0.85^n > \frac{5}{18}$$
$$n > \frac{\log\left(\frac{5}{18}\right)}{\log(0.85)}$$
$$n > 7.88$$

7 a Total commission

$$= \underbrace{10 + 20 + 30 + \ldots + 520}$$

Arithmetic series with $a = 10$, $d = 10$, $n = 52$

$$= \frac{52}{2}\left(2\times10+(52-1)\times10\right) \text{ using}$$
$$S_n = \frac{n}{2}\left(2a+(n-1)d\right)$$
$$= 26(20+51\times10)$$
$$= 26(20+510)$$
$$= 26\times530$$
$$= \$13780$$

b Commission = policies for year 1 + policies for 2nd week of year 2
= 520 + 22 = $542

c Total commission for year 2
= Commission for year 1 policies + Commission for year 2 policies
= 520 × 52 + (11 + 22 + 33 + ... 52 × 11)

Use $S_n = \frac{n}{2} = \left(2a+(n-1)d\right)$

with $n = 52$, $a = 11$, $d = 11$

$$= 27\,040 + \frac{52}{2}\left(2\times11+(52-1)\times11\right)$$
$$= 27\,040 + 26\times(22+51\times11)$$
$$= 27\,040 + \$15158$$
$$= \$42198$$

8 a Cost of drilling to 500 m

$$= \underset{\substack{\uparrow\\ \text{1st}\\ 50\text{ m}}}{500} + \underset{\substack{\uparrow\\ \text{2nd}\\ 50\text{ m}}}{640} + \underset{\substack{\uparrow\\ \text{3rd}\\ 50\text{ m}}}{780} + \ldots$$

There would be 10 terms because there are 10 lots of 50 m in 500 m.
Arithmetic series with $a = 500$, $d = 140$ and $n = 10$

Using $S_n = \frac{n}{2}\left(2a+(n-1)d\right)$

$$= \frac{10}{2}\left(2\times500+(10-1)\times140\right)$$
$$= 5(1000+9\times140)$$
$$= 5\times2260$$
$$= \$11300$$

8 b This time we are given S = 76 000. The first term will still be 500 and d remains 140

Use $S = \frac{n}{2}(2a+(n-1)d)$ with S = 76 000, a = 500, d = 140, and solve for n.

$$76000 = \frac{n}{2}(2\times 500+(n-1)\times 140)$$
$$76000 = \frac{n}{2}(1000+140(n-1))$$
$$76000 = n(500+70(n-1))$$
$$76000 = n(500+70n-70)$$
$$76000 = n(70n+430n) \ \text{(multiply out)}$$
$$76000 = 70n^2+430n \ (\div 10)$$
$$7600 = 7n^2+43n$$
$$0 = 7n^2+43n-7600$$
$$n = \frac{-43\pm\sqrt{(43)^2-4\times 7\times(-7600)}}{2\times 7}$$
$$n = 30.02, \ (-36.16)$$

Only accept the positive answer, so there are 30 terms (to the nearest term).
So the greatest depth that can be drilled is 30 × 50 = 1500 m (to the nearest 50 m).

c Max's amount

$$= \underbrace{890 + (890+d) + (890+2d) + \ldots}_{40\text{ years}}$$

Use $S_n = \frac{n}{2}(2a+(n-1)d)$ with n = 40, a = 890 and d.

$$S_{40} = \frac{40}{2}(2\times 890+(40-1)d)$$
$$= 20(1780+39d)$$

Use the fact that
Max's saving = Sara's savings

$$20(1780+39d) = 59000 \quad (\div 20)$$
$$1780+39d = 2950 \quad (-1780)$$
$$39d = 1170 \quad (\div 39)$$
$$d = 30$$

9 a 1st year = 500
2nd year = 550 = 500 + 1 × 50
3rd year = 600 = 500 + 2 × 50
⋮
40th year = 500 + 39 × 50 = €2450

b Total amount paid in

$$= \underbrace{€500 + €550 + €600 + \ldots + €2450}$$

This is an arithmetic series with a = 500, d = 50, L = 2450 and n = 40.

$$S_n = \frac{n}{2}(a+L)$$
$$S_{40} = \frac{40}{2}(500+2450)$$
$$= 20\times 2950$$
$$= €59000$$

10 If the number of people infected increases by 4% the multiplication factor is 1.04.
After n days 100 × $(1.04)^n$ people will be infected.
If 1000 people are infected

$$100\times(1.04)^n = 1000$$
$$(1.04)^n = 10$$
$$\log(1.04)^n = \log 10$$
$$n\log(1.04) = 1$$
$$n = \frac{1}{\log(1.04)}$$
$$n = 58.708\ldots$$

It would take 59 days.

11 If the increase is 3.5% per annum the multiplication factor is 1.035.
Therefore after n years I will have £A × $(1.035)^n$.
If the money is doubled it will equal 2A, therefore

$$A\times(1.035)^n = 2A$$
$$(1.035)^n = 2$$
$$\log(1.035)^n = \log 2$$
$$n\log(1.035) = \log 2$$
$$n = \frac{\log 2}{\log(1.035)} = 20.14879\ldots$$

My money will double after 20.15 years.

12 The reduction is 6% which gives a multiplication factor of 0.94.
Let the number of fish now be F.
After n years there will be $F \times (0.94)^n$.
When their number is halved the number will be $\frac{1}{2}F$.
Set these equal to each other:

$$F \times (0.94)^n = \frac{1}{2}F$$
$$(0.94)^n = \frac{1}{2}$$
$$\log(0.94)^n = \log\left(\frac{1}{2}\right)$$
$$n\log(0.94) = \log\left(\frac{1}{2}\right)$$
$$n = \frac{\log\left(\frac{1}{2}\right)}{\log(0.94)}$$
$$n = 11.2$$

The fish stocks will halve in 11.2 years.

13

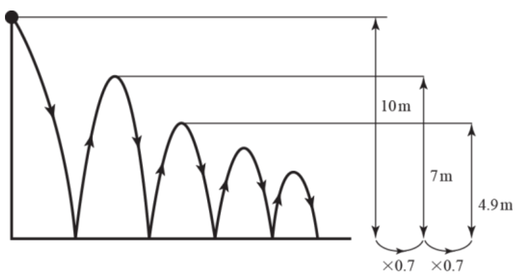
$$\text{No. grains} = \underbrace{1 + 2 + 4 + 8 + \ldots}_{64 \text{ terms}}$$

This is a geometric series with $a = 1$, $r = 2$ and $n = 64$.

As $|r| > 1$ use $S_n = \frac{a(r^n - 1)}{r - 1}$

$$\text{Number of grains} = \frac{1(2^{64} - 1)}{2 - 1} = 2^{64} - 1$$

14 a



After the 1st bounce it bounces to 7 cm
After the 2nd bounce it bounces to 4.9 cm (× 0.7)
After the 3rd bounce it bounces to 3.43 cm (× 0.7)
After the 4th bounce it bounces to 2.401 cm (× 0.7)

b Total distance travelled

$$= 10 + \underbrace{7 + 7}_{\text{1st bounce}} + \underbrace{4.9 + 4.9}_{\text{2nd bounce}} + \underbrace{\ldots}_{\text{3rd bounce}}$$

$$= 2 \times \underbrace{(10 + 7 + 4.9 + \ldots)}_{\substack{6 \text{ terms} \\ a=10,\, r=0.7,\, n=6}} - 10$$

$$= 2 \times \frac{10(1 - 0.7^6)}{1 - 0.7} - 10$$

$$= 48.8234 \text{ m}$$

15 a $a = 10$, $r = 1.1$

$$S_n = \frac{10(1.1^n - 1)}{1.1 - 1} = 1000$$
$$1.1^n - 1 = 10$$
$$1.1^n = 11$$
$$n = \frac{\log 11}{\log 1.1}$$
$= 25.16$
So 26 days

b On the 25th day:
$ar^{24} = 10 \times 1.1^{24} = 98.5$ miles

16 Jan. 1st, year 1 = €500
Dec. 31st, year 1 = 500 × 1.035
Jan. 1st, year 2 = 500 × 1.035 + 500
Dec. 31st, year 2
= (500 × 1.035 + 500) × 1.035
= 500 × 1.035^2 + 500 × 1.035
⋮
Dec. 31st, year n
= 500 × 1.035^n + . . . + 500 × 1.035^2
+ 500 × 1.035

$$= 500 \times \underbrace{\left(1.035^n + \ldots + 1.035^2 + 1.035\right)}$$

A geometric series with a = 1.035, r = 1.035 and n.

Use $S_n = \dfrac{a\left(r^n - 1\right)}{r - 1}$

Dec. 31st year $n = 500 \times \dfrac{1.035\left(1.035^n - 1\right)}{1.035 - 1}$

Set this equal to €20 000

$$20\,000 = 500 \times \frac{1.035\left(1.035^n - 1\right)}{1.035 - 1}$$

$$\left(1.035^n - 1\right) = \frac{20\,000 \times \left(1.035 - 1\right)}{500 \times 1.035}$$

$1.035^n - 1$ = 1.3526570. . .
1.035^n = 2.3526570. . .
log (1.035^n) = log 2.3526570. . .
n log (1.035) = log 2.3526570. . .

$$n = \frac{\log 2.3526570\ldots}{\log 1.035}$$

n = 24.9 years (3 s.f.)
It takes Liu Wei 25 years to save €20 000