Exercise 5F

1 a i $\sum_{r=1}^{5}(3r+1)=4+7+10+13+16$

ii $S_5=50$

b i $\sum_{r=1}^{6}3r^2=3+12+27+48+75+108$

ii $S_6=273$

c i $\sum_{r=1}^{5}\sin(90r°)=1+0+(-1)+0+1$

ii $S_5=1$

d i $\sum_{r=5}^{8}2\left(-\frac{1}{3}\right)^r=-\frac{2}{243}+\frac{2}{729}-\frac{2}{2187}+\frac{2}{6561}$

ii $S_4=-\frac{40}{6561}$

2 a i $2+4+6+8=\sum_{r=1}^{4}2r$

ii $S_4=20$

b i $2+6+18+54+162=\sum_{r=1}^{5}\left(2\times3^{r-1}\right)$

ii $S_5=242$

c i

$6+4.5+3+1.5+0-1.5=\sum_{r=1}^{6}\left(-\frac{3}{2}r+\frac{15}{2}\right)$

ii $S_6=13.5$

3 a i $7+13+19+\ldots+157=\sum_{r=1}^{n}(6r+1)$

$6n+1=157$
$n=26$

ii $\sum_{r=1}^{26}(6r+1)$

3 b i

$\frac{1}{3}+\frac{2}{15}+\frac{4}{75}+\ldots+\frac{64}{46\,875}=\sum_{r=1}^{n}\left(\frac{1}{3}\times\left(\frac{2}{5}\right)^{r-1}\right)$

$\frac{1}{3}\times\left(\frac{2}{5}\right)^{n-1}=\frac{64}{46\,875}$

$\left(\frac{2}{5}\right)^{n-1}=\frac{64}{15\,625}$

$n=\frac{\log(0.004096)}{\log(0.4)}+1=7$

ii $\sum_{r=1}^{7}\left(\frac{1}{3}\times\left(\frac{2}{5}\right)^{r-1}\right)$

c i $8-1-10-19-\ldots-127=\sum_{r=1}^{n}(17-9r)$

$17-9n=-127$
$n=16$

ii $\sum_{r=1}^{16}(17-9r)$

4 a $\sum_{r=1}^{20}(7-2r)=5+3+1+\ldots-33$

$a=5,\ l=-33,\ n=20$

$S_{20}=\frac{20}{2}(5-33)$
$=-280$

b $\sum_{r=1}^{10}3\times4^r=12+48+192+\ldots+3\,145\,728$

$a=12,\ r=4,\ n=10$

$S_{10}=\frac{12(4^{10}-1)}{4-1}$
$=4\,194\,300$

c $\sum_{r=1}^{100}(2r-8)=-6-4-2+\ldots+192$

$a=-6,\ l=192,\ n=100$

$S_{100}=\frac{100}{2}(-6+192)$
$=9300$

4 d $\sum_{r=1}^{\infty} 7\left(-\frac{1}{3}\right)^r = -\frac{7}{3}+\frac{7}{9}-\frac{7}{27}+\ldots$

$a = -\frac{7}{3},\ r = -\frac{1}{3}$

$S_\infty = \frac{-\frac{7}{3}}{1+\frac{1}{3}}$

$= -\frac{7}{4}$

5 a $\sum_{r=9}^{30}\left(5r-\frac{1}{2}\right) = 44\frac{1}{2}+49\frac{1}{2}+\ldots+149\frac{1}{2}$

$a = 44\frac{1}{2},\ l = 149\frac{1}{2},\ n = 22$

$S_{22} = \frac{22}{2}\left(44\frac{1}{2}+149\frac{1}{2}\right)$

$= 2134$

b $\sum_{r=100}^{200}(3r+4) = 304+307+310+\ldots+604$

$a = 304,\ l = 604,\ n = 101$

$S_{101} = \frac{101}{2}(304+604)$

$= 45\,854$

c

$\sum_{r=5}^{100} 3\times 0.5^r = 0.09375+0.046875+0.$

$0234375+\ldots$

$a = 0.09375,\ r = 0.5,\ n = 96$

$S_{96} = \frac{0.09375(1-0.5^{96})}{1-0.5}$

$= 0.1875$

d $\sum_{i=5}^{100} 1 = 1+1+1+\ldots+1$

$a = 1,\ l = 1,\ n = 96$

$S_{96} = \frac{96}{2}(1+1)$

$= 96$

6 $\sum_{r=1}^{n} 2r$

The sequence is 2, 4, 6, …
So $a = 2$ and $d = 2$.

$S_n = \frac{n}{2}(2a+(n-1)d)$

$= \frac{n}{2}(2(2)+(n-1)2)$

$= \frac{n}{2}(2+2n)$

$= n(1+n)$

$= n+n^2$ as required

7 $\sum_{r=1}^{n}(2r-1)$

The sequence is 1, 3, 5, …
So $a = 1$ and $d = 2$.

$S_n = \frac{n}{2}(2a+(n-1)d)$

$= \frac{n}{2}(2(1)+(n-1)2)$

$= \frac{n}{2}(2n)$

$= n^2$

From question **6**

$\sum_{r=1}^{n} 2r = n+n^2$

Therefore

$\sum_{r=1}^{n} 2r - \sum_{r=1}^{n}(2r-1) = (n+n^2)-n^2 = n$

as required.

8 a $\sum_{r=1}^{k} 4(-2)^r$

The sequence is −8, 16, −32, …
So $a = -8$ and $r = -2$.

$S_n = \frac{a(r^n-1)}{r-1}$

$S_k = \frac{-8((-2)^k-1)}{-2-1}$

$= \frac{8}{3}((-2)^k-1)$

8 b $\sum_{r=1}^{k}(100-2r)$

The sequence is 98, 96, 94, …
So $a = 98$ and $d = -2$.

$$S_n = \frac{n}{2}\left(2a+(n-1)d\right)$$
$$S_k = \frac{k}{2}\left(2(98)+(k-1)(-2)\right)$$
$$= \frac{k}{2}(198-2k)$$
$$= k(99-k) \quad (= 99k - k^2)$$

c $\sum_{r=10}^{k}(7-2r)$

The sequence is −13, −15, −17, …
So $a = -13$, $d = -2$, $n = k - 9$.

$$S_n = \frac{n}{2}\left(2a+(n-1)d\right)$$
$$S_k = \frac{(k-9)}{2}\left(2(-13)+(k-9-1)(-2)\right)$$
$$= (k-9)\left(\frac{-26}{2}-\frac{2(k-10)}{2}\right)$$
$$= (k-9)\left(-13-(k-10)\right)$$
$$= (k-9)(-3-k) \quad (= 6k - k^2 + 27)$$

9 $\sum_{r=10}^{\infty} 200\times\left(\frac{1}{4}\right)^r$

So $a = \frac{25}{131072}$ and $r = \frac{1}{4}$.

$$S_\infty = \frac{a}{1-r}$$
$$= \frac{25}{131072} \div \frac{3}{4}$$
$$= \frac{25}{98304}$$

10 a $\sum_{r=1}^{k}(8+3r) = 377$

So $a = 11$ and $d = 3$.

$$S_n = \frac{n}{2}\left(2a+(n-1)d\right)$$
$$S_k = \frac{k}{2}\left(2(11)+(k-1)(3)\right)$$
$$= \frac{k}{2}(19+3k)$$

So $\frac{k}{2}(19+3k) = 377$

$$k(19+3k) = 754$$
$$3k^2 + 19k - 754 = 0$$
$$(3k+58)(k-13) = 0 \text{ as required.}$$

b $k = 13$; reject other solution $k \neq \frac{-58}{3}$

11 a $\sum_{r=1}^{k} 2\times 3^r = 59046$

$a = 6$, $r = 3$

$$S_n = \frac{a\left(r^n-1\right)}{r-1}$$
$$S_k = \frac{6\left(3^k-1\right)}{3-1}$$
$$= 3\left(3^k-1\right)$$

So $3\left(3^k-1\right) = 59046$

$$3^k - 1 = 19682$$
$$3^k = 19683$$
$$k\log 3 = \log 19683$$
$$k = \frac{\log 19683}{\log 3} \text{ as required.}$$

11 b $k = \frac{\log 19683}{\log 3} = 9$

So

$$\sum_{r=k+1}^{13} 2\times 3^r = \sum_{r=10}^{13} 2\times 3^r = \sum_{r=1}^{13} 2\times 3^r - \sum_{r=1}^{9} 2\times 3^r$$

$a = 6$ and $r = 3$

$$\sum_{r=1}^{13} 2\times 3^r = \frac{6\left(3^{13}-1\right)}{3-1}$$
$$= 3\left(3^{13}-1\right)$$

$$\sum_{r=1}^{9} 2\times 3^r = \frac{6\left(3^{9}-1\right)}{3-1}$$
$$= 3\left(3^{9}-1\right)$$

$$\sum_{r=1}^{13} 2\times 3^r - \sum_{r=1}^{9} 2\times 3^r = 3\left(3^{13}-1\right) - 3\left(3^{9}-1\right)$$
$$= 3^{14} - 3^{10}$$
$$= 4723920$$

12 a A geometric sequence is

$1 + 3x + 9x^2 + \ldots$

$r = 3x$

If the series is convergent then $|r| < 1$

$|3x| < 1$

$-\frac{1}{3} < x < \frac{1}{3}$

b $\sum_{r=1}^{\infty} (3x)^{r-1} = 2$

$a = 1$ and $r = 3x$

$$S_\infty = \frac{a}{1-r}$$
$$= \frac{1}{1-3x}$$

Since $\sum_{r=1}^{\infty} (3x)^{r-1} = 2$

$$\frac{1}{1-3x} = 2$$

$$1 - 3x = \frac{1}{2}$$

$$3x = \frac{1}{2}$$

$$x = \frac{1}{6}$$

Challenge

Many possible solutions, such as

$$\sum_{r=1}^{10}\left(a+(r-1)d\right) = \sum_{r=11}^{14}\left(a+(r-1)d\right)$$

$$\sum_{r=11}^{14}\left(a+(r-1)d\right) = \sum_{r=1}^{14}\left(a+(r-1)d\right) - \sum_{r=1}^{10}\left(a+(r-1)d\right)$$

$$\sum_{r=1}^{10}\left(a+(r-1)d\right) = \sum_{r=1}^{14}\left(a+(r-1)d\right) - \sum_{r=1}^{10}\left(a+(r-1)d\right)$$

$$2\sum_{r=1}^{10}\left(a+(r-1)d\right) = \sum_{r=1}^{14}\left(a+(r-1)d\right)$$

$$S_n = \frac{n}{2}\left(2a+(n-1)d\right)$$

$$S_{10} = \frac{10}{2}\left(2a+(10-1)d\right)$$
$$= 5\left(2a+9d\right)$$

$$S_{14} = \frac{14}{2}\left(2a+(14-1)d\right)$$
$$= 7\left(2a+13d\right)$$

$$10\left(2a+9d\right) = 7\left(2a+13d\right)$$

$$6a = d$$