Exercise 4D

1 **a** $(3+x)^5$

x^3 term $= \binom{5}{3}3^2(x)^3 = 10 \times 9x^3 = 90x^3$

Coefficient = 90

b $(1+2x)^5$

x^3 term $= \binom{5}{3}1^2\,(2x)^3 = 10 \times 8x^3 = 80x^3$

Coefficient = 80

c $(1-x)^6$

x^3 term $= \binom{6}{3}(1)^3(-x)^3 = 20 \times (-x^3) = -20x^3$

Coefficient = −20

d $(3x+2)^5$

x^3 term $= \binom{5}{2}(3x)^3 2^2 = 10 \times 108x^3 = 1080x^3$

Coefficient = 1080

e $(1+x)^{10}$

x^3 term $= \binom{10}{3}1^7(x)^3 = 120 \times 1x^3 = 120x^3$

Coefficient = 120

f $(3-2x)^6$

x^3 term $= \binom{6}{3}3^3(-2x)^3 = 20 \times (-216x^3) = -4320x^3$

Coefficient = −4320

g $(1+x)^{20}$

x^3 term $= \binom{20}{3}1^{17}x^3 = 1140 \times x^3 = 1140x^3$

Coefficient = 1140

h $(4-3x)^7$

x^3 term $= \binom{7}{3}4^4(-3x)^3 = 35 \times (-6912x^3) = -241\,920x^3$

Coefficient = −241 920

1 i $\left(1-\frac{1}{2}x\right)^6$

x^3 term $= \binom{6}{3}1^3\left(-\frac{1}{2}x\right)^3 = 20 \times -\frac{1}{8}x^3 = -2.5x^3$ or $-\frac{5}{2}x^3$

Coefficient $= -2.5$ or $-\frac{5}{2}$

j $\left(3+\frac{1}{2}x\right)^7$

x^3 term $= \binom{7}{3}3^4\left(\frac{1}{2}x\right)^3 = 35 \times 81 \times \frac{1}{8}x^3 = 354.375x^3$ or $\frac{2835}{8}x^3$

Coefficient $= 354.375$ or $\frac{2835}{8}$

k $\left(2-\frac{1}{2}x\right)^8$

x^3 term $= \binom{8}{3}2^5\left(-\frac{1}{2}x\right)^3 = 56 \times 32 \times -\frac{1}{8}x^3 = -224x^3$

Coefficient $= -224$

l $\left(5+\frac{1}{4}x\right)^5$

x^3 term $= \binom{5}{3}5^2\left(\frac{1}{4}x\right)^3 = 10 \times 25 \times \frac{1}{64}x^3 = 3.90625x^3$ or $\frac{125}{32}$

Coefficient $= 3.90625$ or $\frac{125}{32}$

2 $(2 + ax)^6$

x^2 term $= \binom{6}{2}2^4(ax)^2 = 15 \times 16a^2x^2 = 240a^2x^2$

$240a^2 = 60$

$a^2 = \frac{1}{4}$

$a = \pm\frac{1}{2}$

3 $(3 + bx)^5$

x^3 term $= \binom{5}{3}3^2(bx)^3 = 10 \times 9b^3x^3 = 90b^3x^3$

$90b^3 = -720$

$b^3 = -8$

$b = -2$

4 $(3-ax)^4 = 3^4 + \binom{4}{1}3^3(-ax) + \binom{4}{2}3^2(-ax)^2 + \binom{4}{3}3^1(-ax)^3 + (-ax)^4$

$= 81 + 4 \times (-27ax) + 6 \times 9a^2x^2 + 4 \times (-3a^3x^3) + a^4x^4$

$= 81 - 108ax + 54a^2x^2 - 12a^3x^3 + a^4x^4$

$(2+x)(3-ax)^4 = (2+x)(\ 81 + 108ax + 54a^2x^2 - 12a^3x^3 + a^4x^4)$

x^3 term $= 2 \times (-12a^3x^3) + x \times 54a^2x^2$

$= -24a^3x^3 + 54a^2x^3$

$-24a^3 + 54a^2 = 30$

$-4a^3 + 9a^2 = 5$

$0 = 4a^3 - 9a^2 + 5$

$0 = (a-1)(4a^2 - 5a - 5)$

Either $a = 1$ or $4a^2 - 5a - 5 = 0 \Rightarrow a = \dfrac{5 \pm \sqrt{25+80}}{8} = \dfrac{5 \pm \sqrt{105}}{8}$

Possible values of a are 1, $\dfrac{5+\sqrt{105}}{8}$ and $\dfrac{5-\sqrt{105}}{8}$.

5 **a** $(1-2x)^p = 1^p + \binom{p}{1}1^{p-1}(-2x) + \binom{p}{2}1^{p-2}(-2x)^2 + \ldots$

$\binom{p}{1} = \dfrac{p!}{1!(p-1)!} = p$

$\binom{p}{2} = \dfrac{p!}{2!(p-2)!} = \dfrac{p(p-1)}{2}$

So $(1-2x)^p = 1 + p(-2x) + \dfrac{p(p-1)}{2}(-2x)^2 + \ldots$

$= 1 - 2px + 2p(p-1)x^2 + \ldots$

x^2 term $= 2p(p-1)x^2$

$2p(p-1) = 40$

$p^2 - p - 20 = 0$

$(p-5)(p+4) = 0$

$p > 0$, so $p = 5$

b Coefficient of $x = -2p = -10$

c x^3 term $= \binom{p}{3}1^{p-3}(-2x)^3$

$= \dfrac{p(p-1)(p-2)}{3!}(-2x)^3$

$= \dfrac{5 \times 4 \times 3}{3!}(-8x^3)$

$= -80x^3$

Coefficient of $x^3 = -80$

6 **a** $(5+px)^{30} = 5^{30} + \binom{30}{1}5^{29}(px) + \binom{30}{2}5^{28}(px)^2 + \ \ldots$

$= 5^{30} + 5^{29}(30px) + 5^{28}(435p^2x^2) + \ldots$

6 **b** $5^{28}(435p^2) = 29 \times 5^{29}(30p)$

$435p^2 = 29 \times 5(30p)$

$435p^2 = 4350p$

$p^2 = 10p$

$p^2 - 10p = 0$

$p(p - 10) = 0$

$p = 0$ or $p = 10$

p is a non-zero constant, so $p = 10$

7 **a** $(1+qx)^{10} = 1^{10} + \binom{10}{1}1^9(qx) + \binom{10}{2}1^8(qx)^2 + \binom{10}{3}1^7(qx)^3 + \ldots$

$= 1 + 10qx + 45q^2x^2 + 120q^3x^3 + \ldots$

b Coefficient of x^3 is $120q^3$

Coefficient of x is $10q$

So $120q^3 = 108 \times 10q$

$\Rightarrow 120q^3 - 1080q = 0$

$\Rightarrow 120q(q^2 - 9) = 0$

$\Rightarrow 120q(q + 3)(q - 3) = 0$

$q = 0$, $q = -3$ or $q = 3$

But as q is non-zero, $q = \pm 3$.

8 **a** $(1+px)^{11} = 1^{11} + \binom{11}{1}1^{10}(px) + \binom{11}{2}1^9(px)^2 + \ldots$

$= 1 + 11px + 55p^2x^2 + \ldots$

b $11p = 77$ and $55p^2 = q$

$p = 7$

$q = 55 \times 7^2 = 2695$

$p = 7$, $q = 2695$

9 **a** $(1+px)^{15} = 1^{15} + \binom{15}{1}1^{14}(px)^1 + \binom{15}{2}1^{13}(px)^2 + \ldots$

$= 1 + 15px + 105p^2x^2 + \ldots$

b $15p = -q$ and $105p^2 = 5q$

$21p^2 = q$

Substituting $15p = -q$ into $21p^2 = q$:

$21p^2 = -15p$

$21p^2 + 15p = 0$

$3p(7p + 5) = 0$

$p = 0$ or $-\frac{5}{7}$

p is a non-zero constant, so $p = -\frac{5}{7}$

$q = -15 \times -\frac{5}{7} = \frac{75}{7} = 10\frac{5}{7}$

$p = -\frac{5}{7}$, $q = 10\frac{5}{7}$

10 $(1+x)^{30}$

x^9 term $= \binom{30}{9} 1^{21} x^9 = 14\,307\,150x^9$

x^{10} term $= \binom{30}{10} 1^{20} x^{10} = 30\,045\,015x^{10}$

$p = 14\,307\,150$ and $q = 30\,045\,015$

$\frac{q}{p} = \frac{30\,045\,015}{14\,307\,150} = 2.1$ (to 2 s.f.)

Challenge

a $(3 - 2x^2)^9$

x^4 term $= \binom{9}{2} 3^7 \left(-2x^2\right)^2$

$= 36 \times 2187 \times 4x^4$

$= 314\,928x^4$

The coefficient of x^4 in the binomial expansion of $(3 - 2x^2)^9$ is 314 928.

b $\left(\frac{5}{x} + x^2\right)^8$

x^4 term $= \binom{8}{4} \left(\frac{5}{x}\right)^4 \left(x^2\right)^4$

$= 70 \times \left(\frac{625}{x^4}\right) \times x^8$

$= 43\,750x^4$

The coefficient of x^4 in the binomial expansion of $\left(\frac{5}{x} + x^2\right)^8$ is 43 750.