Exercise 4C

1 a $(1+x)^4 = 1^4 + \binom{4}{1}1^3x + \binom{4}{2}1^2x^2 + \binom{4}{3}1x^3 + x^4$

$= 1 + 4x + 6x^2 + 4x^3 + x^4$

b $(3+x)^4 = 3^4 + \binom{4}{1}3^3x + \binom{4}{2}3^2x^2 + \binom{4}{3}3x^3 + x^4$

$= 81 + 108x + 54x^2 + 12x^3 + x^4$

c $(4-x)^4 = 4^4 + \binom{4}{1}4^3(-x) + \binom{4}{2}4^2(-x)^2 + \binom{4}{3}4(-x)^3 + (-x)^4$

$= 256 - 256x + 96x^2 - 16x^3 + x^4$

d $(x+2)^6 = x^6 + \binom{6}{1}x^5 2 + \binom{6}{2}x^4 2^2 + \binom{6}{3}x^3 2^3 + \binom{6}{4}x^2 2^4 + \binom{6}{5}x2^5 + 2^6$

$= x^6 + 12x^5 + 60x^4 + 160x^3 + 240x^2 + 192x + 64$

e $(1+2x)^4 = 1^4 + \binom{4}{1}1^3(2x) + \binom{4}{2}1^2(2x)^2 + \binom{4}{3}1(2x)^3 + (2x)^4$

$= 1 + 8x + 24x^2 + 32x^3 + 16x^4$

f $\left(1-\frac{1}{2}x\right)^4 = 1^4 + \binom{4}{1}1^3(-\frac{1}{2}x) + \binom{4}{2}1^2(-\frac{1}{2}x)^2 + \binom{4}{3}1(-\frac{1}{2}x)^3 + (-\frac{1}{2}x)^4$

$= 1 - 2x + \frac{3}{2}x^2 - \frac{1}{2}x^3 + \frac{1}{16}x^4$

2 a $(1+x)^{10} = 1^{10} + \binom{10}{1}1^9x + \binom{10}{2}1^8x^2 + \binom{10}{3}1^7x^3 + \ldots$

$= 1 + 10 \times 1x + 45 \times 1x^2 + 120 \times 1x^3 + \ldots$

$= 1 + 10x + 45x^2 + 120x^3 + \ldots$

b $(1-2x)^5 = 1^5 + \binom{5}{1}1^4(-2x) + \binom{5}{2}1^3(-2x)^2 + \binom{5}{3}1^2(-2x)^3 + \ldots$

$= 1 \times 1 + 5 \times (-2x) + 10 \times 4x^2 + 10 \times (-8x^3) + \ldots$

$= 1 - 10x + 40x^2 - 80x^2 + \ldots$

c $(1+3x)^6 = 1^6 + \binom{6}{1}1^5(3x) + \binom{6}{2}1^4(3x)^2 + \binom{6}{3}1^3(3x)^3 + \ldots$

$= 1 \times 1 + 6 \times 3x + 15 \times 9x^2 + 20 \times 27x^3 + \ldots$

$= 1 + 18x + 135x^2 + 540x^3 + \ldots$

d $(2-x)^8 = 2^8 + \binom{8}{1}2^7(-x) + \binom{8}{2}2^6(-x)^2 + \binom{8}{3}2^5(-x)^3 + \ldots$

$= 1 \times 256 + 8 \times (-128x) + 28 \times 64x^2 + 56 \times (-32x^3) + \ldots$

$= 256 - 1024x + 1792x^2 - 1792x^3 + \ldots$

2 e $\left(2-\frac{1}{2}x\right)^{10} = 2^{10} + \binom{10}{1}2^9\left(-\frac{1}{2}x\right) + \binom{10}{2}2^8\left(-\frac{1}{2}x\right)^2 + \binom{10}{3}2^7\left(-\frac{1}{2}x\right)^3 + \ldots$
$= 1 \times 1024 + 10 \times (-256x) + 45 \times 64x^2 + 120 \times (-16x^3) + \ldots$
$= 1024 - 2560x + 2880x^2 - 1920x^3 + \ldots$

f $(3-x)^7 = 3^7 + \binom{7}{1}3^6(-x) + \binom{7}{2}3^5(-x)^2 + \binom{7}{3}3^4(-x)^3 + \ldots$
$= 1 \times 2187 + 7 \times (-729x) + 21 \times 243x^2 + 35 \times (-81x^3) + \ldots$
$= 2187 - 5103x + 5103x^2 - 2835x^3 + \ldots$

3 a $(2x+y)^6 = (2x)^6 + \binom{6}{1}(2x)^5 y + \binom{6}{2}(2x)^4 y^2 + \binom{6}{3}(2x)^3 y^3 + \ldots$
$= 64x^6 + 192x^5y + 240x^4y^2 + 160x^3y^3$

b $(2x+3y)^5 = (2x)^5 + \binom{5}{1}(2x)^4(3y) + \binom{5}{2}(2x)^3(3y)^2 + \binom{5}{3}(2x)^2(3y)^3 + \ldots$
$= 32x^5 + 240x^4y + 720x^3y^2 + 1080x^2y^3 + \ldots$

c $(p-q)^8 = p^8 + \binom{8}{1}p^7(-q) + \binom{8}{2}p^6(-q)^2 + \binom{8}{3}p^5(-q)^3 + \ldots$
$= p^8 - 8p^7q + 28p^6q^2 - 56p^5q^3 + \ldots$

d $(3x-y)^6 = (3x)^6 + \binom{6}{1}(3x)^5(-y) + \binom{6}{2}(3x)^4(-y)^2 + \binom{6}{3}(3x)^3(-y)^3 + \ldots$
$= 729x^6 - 1458x^5y + 1215x^4y^2 - 540x^3y^3 + \ldots$

e $(x+2y)^8 = x^8 + \binom{8}{1}x^7(2y) + \binom{8}{2}x^6(2y)^2 + \binom{8}{3}x^5(2y)^3 + \ldots$
$= x^8 + 16x^7y + 112x^6y^2 + 448x^5y^3 + \ldots$

f $(2x-3y)^9 = (2x)^9 + \binom{9}{1}(2x)^8(-3y) + \binom{9}{2}(2x)^7(-3y)^2 + \binom{9}{3}(2x)^6(-3y)^3 + \ldots$
$= 512x^9 - 6912x^8y + 41\,472x^7y^2 - 145\,152x^6y^3 + \ldots$

4 a $(1+x)^8 = 1^8 + \binom{8}{1}1^7x + \binom{8}{2}1^6x^2 + \binom{8}{3}1^5x^3 + \ldots$
$= 1 + 8x + 28x^2 + 56x^3 + \ldots$

b $(1-2x)^6 = 1^6 + \binom{6}{1}1^5(-2x) + 1^4\binom{6}{2}(-2x)^2 + \binom{6}{3}1^3(-2x)^3 + \ldots$
$= 1 - 12x + 60x^2 - 160x^3 + \ldots$

c $\left(1+\frac{x}{2}\right)^{10} = 1^{10} + \binom{10}{1}1^9\left(\frac{x}{2}\right) + \binom{10}{2}1^8\left(\frac{x}{2}\right)^2 + \binom{10}{3}1^7\left(\frac{x}{2}\right)^3 + \ldots$
$= 1 + 5x + \frac{45}{4}x^2 + 15x^3 + \ldots$

4 d $(1-3x)^5 = 1^5 + \binom{5}{1}1^4(-3x) + \binom{5}{2}1^3(-3x)^2 + \binom{5}{3}1^2(-3x)^3 + \ldots$

$= 1 - 15x + 90x^2 - 270x^3 + \ldots$

e $(2+x)^7 = 2^7 + \binom{7}{1}2^6x + \binom{7}{2}2^5x^2 + \binom{7}{3}2^4x^3 + \ldots$

$= 128 + 448x + 672x^2 + 560x^3 + \ldots$

f $(3-2x)^3 = 3^3 + \binom{3}{1}3^2(-2x) + \binom{3}{2}3(-2x)^2 + (-2x)^3$

$= 27 - 54x + 36x^2 - 8x^3$

g $(2-3x)^6 = 2^6 + \binom{6}{1}2^5(-3x) + \binom{6}{2}2^4(-3x)^2 + \binom{6}{3}2^3(-3x)^3 + \ldots$

$= 64 - 576x + 2160x^2 - 4320x^3 + \ldots$

h $(4+x)^4 = 4^4 + \binom{4}{1}4^3x + \binom{4}{2}4^2x^2 + \binom{4}{3}4x^3 + \ldots$

$= 256 + 256x + 96x^2 + 16x^3 + \ldots$

i $(2+5x)^7 = 2^7 + \binom{7}{1}2^6(5x) + \binom{7}{2}2^5(5x)^2 + \binom{7}{3}2^4(5x)^3 + \ldots$

$= 128 + 2240x + 16\,800x^2 + 70\,000x^3 + \ldots$

5 $(2-x)^6 = 2^6 + \binom{6}{1}2^5(-x) + \binom{6}{2}2^4(-x)^2 + \ldots$

$= 64 - 192x + 240x^2 + \ldots$

6 $(3-2x)^5 = 3^5 + \binom{5}{1}3^4(-2x) + \binom{5}{2}3^3(-2x)^2 + \ldots$

$= 243 - 810x + 1080x^2 + \ldots$

7 $\left(x+\frac{1}{x}\right)^5 = x^5 + \binom{5}{1}x^4\left(\frac{1}{x}\right) + \binom{5}{2}x^3\left(\frac{1}{x}\right)^2 + \binom{5}{3}x^2\left(\frac{1}{x}\right)^3 + \binom{5}{4}x\left(\frac{1}{x}\right)^4 + \left(\frac{1}{x}\right)^5$

$= x^5 + 5x^3 + 10x + \frac{10}{x} + \frac{5}{x^3} + \frac{1}{x^5}$

Challenge

a $(a+b)^4 = a^4 + \binom{4}{1}a^3b + \binom{4}{2}a^2b^2 + \binom{4}{3}ab^3 + b^4$
$= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$(a-b)^4 = a^4 + \binom{4}{1}a^3(-b) + \binom{4}{2}a^2(-b)^2 + \binom{4}{3}a(-b)^3 + (-b)^4$
$= a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$

$(a+b)^4 - (a-b)^4 = (a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4) - (a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4)$
$= 8a^3b + 8ab^3$
$= 8ab(a^2 + b^2)$

b $82\,896 = 17^4 - 5^4$
$a = 11$ and $b = 6$
$= 8 \times 11 \times 6 \times (11^2 + 6^2)$
$= 8 \times 11 \times 6 \times 157$
$= 2^3 \times 11 \times 2 \times 3 \times 157$
$= 2^4 \times 3 \times 11 \times 157$