Exercise 2D

1 Substitute $y = 0$ into $(x-1)^2 + (y-3)^2 = 45$

$$(x-1)^2 + (-3)^2 = 45$$
$$(x-1)^2 + 9 = 45$$
$$(x-1)^2 = 36$$
$$x - 1 = \pm\sqrt{36}$$
$$x - 1 = \pm 6$$

So $x - 1 = 6 \Rightarrow x = 7$
and $x - 1 = -6 \Rightarrow x = -5$
The circle meets the x-axis at (7, 0) and (–5, 0).

2 Substitute $x = 0$ into $(x-2)^2 + (y+3)^2 = 29$

$$(-2)^2 + (y+3)^2 = 29$$
$$4 + (y+3)^2 = 29$$
$$(y+3)^2 = 25$$
$$y + 3 = \pm\sqrt{25}$$
$$y + 3 = \pm 5$$

So $y + 3 = 5 \Rightarrow y = 2$
and $y + 3 = -5 \Rightarrow y = -8$
The circle meets the y-axis at (0, 2) and (0, –8).

3 Substitute $y = x + 4$ into $(x-3)^2 + (y-5)^2 = 34$

$$(x-3)^2 + ((x+4)-5)^2 = 34$$
$$(x-3)^2 + (x+4-5)^2 = 34$$
$$(x-3)^2 + (x-1)^2 = 34$$
$$x^2 - 6x + 9 + x^2 - 2x + 1 = 34$$
$$2x^2 - 8x + 10 = 34$$
$$2x^2 - 8x - 24 = 0$$
$$x^2 - 4x - 12 = 0$$
$$(x-6)(x+2) = 0$$

So $x = 6$ and $x = –2$
Substitute $x = 6$ into $y = x + 4$
$y = 6 + 4$
$y = 10$
Substitute $x = –2$ into $y = x + 4$
$y = –2 + 4$
$y = 2$
The coordinates of A and B are (6, 10) and (–2, 2).

4 Rearranging $x + y + 5 = 0$
$y + 5 = –x$
$y = –x –5$
and $x^2 + 6x + y^2 + 10y - 31 = 0$
$$(x+3)^2 + (y+5)^2 = 65$$

Substitute $y = –x –5$ into $(x+3)^2 + (y+5)^2 = 65$

$$(x+3)^2 + ((-x-5)+5)^2 = 65$$
$$(x+3)^2 + (-x-5+5)^2 = 65$$
$$(x+3)^2 + (-x)^2 = 65$$
$$x^2 + 6x + 9 + x^2 = 65$$
$$2x^2 + 6x + 9 = 65$$
$$2x^2 + 6x - 56 = 0$$
$$x^2 + 3x - 28 = 0$$
$$(x+7)(x-4) = 0$$

So $x = –7$ and $x = 4$
Substitute $x = –7$ into $y = –x –5$
$y = – (–7) –5$
$y = 7–5$
$y = 2$
Substitute $x = 4$ into $y = x – 5$
$y = –(4) –5$
$y = –4 –5$
$y = –9$
So the line meets the circle at (–7, 2) and (4, –9).

5 $x^2 - 4x + y^2 = 21$

Completing the square gives $(x-2)^2 + y^2 = 25$

Substitute $y = x - 10$ into $(x - 2)^2 + y^2 = 25$

$$(x-2)^2 + (x-10)^2 = 25$$
$$x^2 - 4x + 4 + x^2 - 20x + 100 = 25$$
$$2x^2 - 24x + 104 = 25$$
$$2x^2 - 24x + 79 = 0$$

Now $b^2 - 4ac = (-24)^2 - 4(2)(79) = 576 - 632 = -56$

As $b^2 - 4ac < 0$ then $2x^2 - 24x + 79 = 0$ has no real roots.

So the line does not meet the circle.

6 a Rearranging $x + y = 11$

$y = 11 - x$

Substitute $y = 11 - x$ into $x^2 + (y - 3)^3 = 32$

$$x^2 + ((11-x)-3)^2 = 32$$
$$x^2 + (11-x-3)^2 = 32$$
$$x^2 + (8-x)^2 = 32$$
$$x^2 + 64 - 16x + x^2 = 32$$
$$2x^2 - 16x + 64 = 32$$
$$2x^2 - 16x + 32 = 0$$
$$x^2 - 8x + 16 = 0$$
$$(x-4)(x-4) = 0$$

The line meets the circle at $x = 4$ (only).

So the line is a tangent.

b Substitute $x = 4$ into $y = 11 - x$

$y = 11 - (4)$

$y = 11 - 4$

$y = 7$

The point of intersection is (4, 7).

7 a Substitute $y = 2x - 2$ into $(x-2)^2 + (y-2)^2 = 20$

$$(x-2)^2 + ((2x-2)-2)^2 = 20$$
$$(x-2)^2 + (2x-4)^2 = 20$$
$$x^2 - 4x + 4 + 4x^2 - 16x + 16 = 20$$
$$5x^2 - 20x + 20 = 20$$
$$5x^2 - 20x = 0$$
$$5x(x-4) = 0$$

So $x = 0$ and $x = 4$
Substitute $x = 0$ into $y = 2x - 2$
$y = 2(0) - 2$
$y = 0 - 2$
$y = -2$
Substitute $x = 4$ into $y = 2x - 2$
$y = 2(4) - 2$
$y = 8 - 2$
$y = 6$
So the coordinates of A and B are $(0, -2)$ and $(4, 6)$.

b The length of AB is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(4-0)^2 + (6-(-2))^2}$$
$$= \sqrt{4^2 + (6+2)^2}$$
$$= \sqrt{4^2 + 8^2}$$
$$= \sqrt{16 + 64}$$
$$= \sqrt{80}$$
$$= \sqrt{4 \times 20}$$
$$= \sqrt{4} \times \sqrt{20}$$
$$= 2\sqrt{20}$$

The radius of the circle $(x-2)^2 + (y-2)^2 = 20$ is $\sqrt{20}$.
So the length of the chord AB is twice the length of the radius.
AB is a diameter of the circle.

Alternative method: substitute $x = 2$, $y = 2$ into $y = 2x - 2$
$2 = 2(2) - 2 = 4 - 2 = 2$
So the line $y = 2x - 2$ joining A and B passes through the centre $(2, 2)$ of the circle.
So AB is a diameter of the circle.

8 a Substitute $x = 3$, $y = 10$ into $x + y = a$
$(3) + (10) = a$
So $a = 13$

8 b Substitute $x = 3$, $y = 10$ into $(x - p)^2 + (y - 6)^2 = 20$

$$(3-p)^2+(10-6)^2=20$$
$$(3-p)^2+4^2=20$$
$$(3-p)^2+16=20$$
$$(3-p)^2=4$$
$$(3-p)=\sqrt{4}$$
$$3-p=\pm 2$$
So $3-p=2 \Rightarrow p=1$

and $3-p=-2 \Rightarrow p=5$

9 a Substitute $y = x - 5$ into $(x - 4)^2 + (y + 7)^2 = 50$

$$(x-4)^2+(x-5+7)^2=50$$
$$x^2-8x+16+x^2+4x+4=50$$
$$2x^2-4x-30=0$$
$$x^2-2x-15=0$$
$$(x+3)(x-5)=0$$
$x = -3$ or $x = 5$

when $x = -3$, $y = -3 - 5 = -8$

when $x = 5$, $y = 5 - 5 = 0$

$A(-3, -8)$ and $B(5, 0)$ or vice versa

b Midpoint $= \left(\frac{-3+5}{2}, \frac{-8+0}{2}\right) = (1, -4)$

The gradient of the line segment $AB = \frac{y_2-y_1}{x_2-x_1}$

$$=\frac{0-(-8)}{5-(-3)}$$
$$=1$$

So the gradient of the line perpendicular to AB is -1.

Using $y-y_1=m(x-x_1)$, $m=-1$ and $(x_1, y_1) = (1, -4)$

So the equation of the perpendicular line is $y - (-4) = -(x - 1)$

$$y = -x - 3$$

c Centre of the circle $= (4, -7)$

Substitute $x = 4$ into $y = -x - 3$

$$y = -4 - 3 = -7$$

Therefore, the perpendicular bisector of AB passes through the centre of the circle $(4, -7)$

d Base $OB = 5$ units, height of the triangle $= 8$ units

Area $OAB = \frac{1}{2} \times 5 \times 8 = 20$ units^2

10 a Substitute $y = kx$ into $x^2 - 10x + y^2 - 12y + 57 = 0$

$$x^2 - 10x + (kx)^2 - 12kx + 57 = 0$$
$$(1 + k^2)x^2 - (10 + 12k)x + 57 = 0$$

For two distinct points of intersection, $b^2 - 4ac > 0$

$$(-(10 + 12k))^2 - 4(1 + k^2)(57) > 0$$
$$144k^2 + 240k + 100 - 228k^2 - 228 > 0$$
$$-84k^2 + 240k - 128 > 0$$
$$21k^2 - 60k + 32 < 0$$

b Using the formula, $k = \dfrac{60 \pm \sqrt{(-60)^2 - 4(21)(32)}}{2(21)}$

$$k = \frac{60 \pm \sqrt{912}}{42}$$

$k = 0.71$ or $k = 2.15$,
$0.71 < k < 2.15$

11 $x^2 + 2x + y^2 = k$
Completing the square gives
$(x + 1)^2 - 1 + y^2 = k$

$$y^2 = k + 1 - (x + 1)^2$$

Using the equation of the line $y = 4x - 1$

$$y^2 = (4x - 1)^2$$

Solving the equations simultaneously gives

$$k + 1 - (x + 1)^2 = (4x - 1)^2$$
$$k + 1 - x^2 - 2x - 1 = 16x^2 - 8x + 1$$
$$17x^2 - 6x - k + 1 = 0$$

The line and the circle do not intersect so there are no solutions.
Using the discriminant: $b^2 - 4ac < 0$

$$36 - 4(17)(-k + 1) < 0$$
$$36 - 68 + 68k < 0$$
$$68k < 32$$
$$k < \frac{8}{17}$$

12 Substitute $y = 2x + 5$ into $x^2 + kx + y^2 = 4$

$$x^2 + kx + (2x+5)^2 = 4$$

$$x^2 + kx + 4x^2 + 20x + 25 = 4$$

$$5x^2 + (20+k)x + 21 = 0$$

For one point of intersection, $b^2 - 4ac = 0$

$$(20+k)^2 - 4(5)(21) = 0$$

$$k^2 + 40k + 400 - 420 = 0$$

$$k^2 + 40k - 20 = 0$$

Using the formula, $k = \dfrac{-40 \pm \sqrt{40^2 - 4(1)(-20)}}{2(1)}$

$$= \frac{-40 \pm \sqrt{1680}}{2}$$

$$= -20 \pm \sqrt{420}$$

$$= -20 \pm 2\sqrt{105}$$

$k = -20 + 2\sqrt{105}$ or $k = -20 - 2\sqrt{105}$