Chapter review 1

1 a $$\frac{3x^4-21x}{3x}=\frac{3x^4}{3x}-\frac{21x}{3x}$$
$$=x^3-7$$

b $$\frac{x^2-2x-24}{x^2-7x+6}$$
$$=\frac{(x-6)(x+4)}{(x-6)(x-1)}$$
$$=\frac{x+4}{x-1}$$

c $$\frac{2x^2+7x-4}{2x^2+9x+4}$$
$$=\frac{(2x-1)(x+4)}{(2x+1)(x+4)}$$
$$=\frac{2x-1}{2x+1}$$

2 $$\begin{array}{r} 3x^2+5 \\ x+4\overline{)3x^3+12x^2+5x+20} \\ \underline{3x^3+12x^2} \\ 0+5x+20 \\ \underline{5x+20} \\ 0 \end{array}$$

So $\dfrac{3x^3+12x^2+5x+20}{x+4}=3x^2+5$

3 $$\begin{array}{r} 2x^2-2x+5 \\ x+1\overline{)2x^3+0x^2+3x+5} \\ \underline{2x^3+2x^2} \\ -2x^2+3x \\ \underline{-2x^2-2x} \\ 5x+5 \\ \underline{5x+5} \\ 0 \end{array}$$

So $\dfrac{2x^3+3x+5}{x+1}=2x^2-2x+5$

4 a $f(x) = 2x^3 - 2x^2 - 17x + 15$
$f(3) = 2(3)^3 - 2(3)^2 - 17(3) + 15$
$= 54 - 18 - 51 + 15$
$= 0$
So $(x - 3)$ is a factor of
$2x^3 - 2x^2 - 17x + 15$.

b $$\begin{array}{r} 2x^2+4x-5 \\ x-3\overline{)2x^3-2x^2-17x+15} \\ \underline{2x^3-6x^2} \\ 4x^2-17x \\ \underline{4x^2-12x} \\ -5x+15 \\ \underline{-5x+15} \\ 0 \end{array}$$

$2x^3 - 2x^2 - 17x + 15$
$= (x - 3)(2x^2 + 4x - 5)$
So $A = 2$, $B = 4$, $C = -5$

5 $$f(x)=16x^5-20x^4+8$$
$$f\left(\frac{1}{2}\right)=16\left(\frac{1}{2}\right)^5-20\left(\frac{1}{2}\right)^4+8$$
$$=16\left(\frac{1}{32}\right)-20\left(\frac{1}{16}\right)+8$$
$$=\frac{29}{4}$$

6 a $f(x) = x^3 + 4x^2 - 3x - 18$
$f(2) = (2)^3 + 4(2)^2 - 3(2) - 18$
$= 8 + 16 - 6 - 18$
$= 0$
So $(x - 2)$ is a factor of $x^3 + 4x^2 - 3x - 18$.

6 b

$$\begin{array}{r} x^2+6x+9 \\ x-2\overline{)x^3+4x^2-3x-18} \\ \underline{x^3-2x^2} \qquad\qquad\quad \\ 6x^2-3x \qquad\quad \\ \underline{6x^2-12x} \qquad\quad \\ 9x-18 \\ \underline{9x-18} \\ 0 \end{array}$$

$x^3 + 4x^2 - 3x - 18 = (x - 2)(x^2 + 6x + 9)$
$= (x - 2)(x + 3)^2$

So $p = 1$, $q = 3$

7 $f(x) = 2x^3 + 3x^2 - 18x + 8$
$f(2) = 2(2)^3 + 3(2)^2 - 18(2) + 8$
$= 16 + 12 - 36 + 8$
$= 0$

So $(x - 2)$ is a factor of $2x^3 + 3x^2 - 18x + 8$.

$$\begin{array}{r} 2x^2+7x-4 \\ x-2\overline{)2x^3+3x^2-18x+8} \\ \underline{2x^3-4x^2} \qquad\qquad\quad \\ 7x^2-18x \qquad \\ \underline{7x^2-14x} \qquad \\ -4x+8 \\ \underline{-4x+8} \\ 0 \end{array}$$

$2x^3 + 3x^2 - 18x + 8 = (x - 2)(2x^2 + 7x - 4)$
$= (x - 2)(2x - 1)(x +4)$

8 $f(x) = x^3 - 3x^2 + kx - 10$
$f(2) = 0$
$(2)^3 - 3(2)^2 + k(2) - 10 = 0$
$8 - 12 + 2k - 10 = 0$
$2k = 14$
$k = 7$

9 a $f(x) = 2x^2 + px + q$
$f(-3) = 0$
$2(-3)^2 + p(-3) + q = 0$
$18 - 3p + q = 0$
$3p - q = 18$ **(1)**
$f(4) = 21$
$2(4)^2 + p(4) + q = 21$
$32 + 4p + q = 21$
$4p + q = -11$ **(2)**
(1) + **(2)**:
$7p = 7$
$p = 1$
Substituting in **(2)**:
$4(1) + q = -11$
$q = -15$
Checking in (1):
$3p - q = 3(1) - (-15) = 3 + 15 = 18$✓
So $p = 1$, $q = -15$

b $f(x) = 2x^2 + x - 15$
$= (2x - 5)(x + 3)$

10 a $h(x) = x^3 + 4x^2 + rx + s$
$h(-1) = 0$
$(-1)^3 + 4(-1)^2 + r(-1) + s = 0$
$-1 + 4 - r + s = 0$
$r - s = 3$ **(1)**
$h(2) = 30$
$(2)^3 + 4(2)^2 + r(2) + s = 30$
$8 + 16 + 2r + s = 30$
$2r + s = 6$ **(2)**

(1) + **(2)**:
$3r = 9$
$r = 3$
Substituting in **(1)**
$3 - s = 3$
$s = 0$
Checking in **(2)**:
$2r + s = 2(3) + (0) = 6$✓
So $r = 3$, $s = 0$

b $h(x) = x^3 + 4x^2 + 3x$

$$h\left(\frac{1}{3}\right) = \left(\frac{1}{3}\right)^3 + 4\left(\frac{1}{3}\right)^2 + 3\left(\frac{1}{3}\right)$$
$$= \frac{1}{27} + \frac{4}{9} + 1$$
$$= 1\frac{13}{27}$$

The remainder is $1\frac{13}{27}$.

11 a $g(x) = 2x^3 + 9x^2 - 6x - 5$

$g(1) = 2(1)^3 + 9(1)^2 - 6(1) - 5$

$= 2 + 9 - 6 - 5$

$= 0$

So $(x - 1)$ is a factor of $2x^3 + 9x^2 - 6x - 5$.

$$\begin{array}{r} 2x^2+11x+5 \\ x-1\overline{)2x^3+9x^2-6x-5} \\ \underline{2x^3-2x^2} \\ 11x^2-6x \\ \underline{11x^2-11x} \\ 5x-5 \\ \underline{5x-5} \\ 0 \end{array}$$

$g(x) = 2x^3 + 9x^2 - 6x - 5$

$= (x - 1)(2x^2 + 11x + 5)$

$= (x - 1)(2x + 1)(x + 5)$

11 b $g(x) = 0$

$(x - 1)(2x + 1)(x + 5) = 0$

So $x = 1$, $x = -\frac{1}{2}$ or $x = -5$

12 a $f(x) = x^3 + x^2 - 5x - 2$

$f(2) = (2)^3 + (2)^2 - 5(2) - 2$

$= 8 + 4 - 10 - 2$

$= 0$

So $(x - 2)$ is a factor of $x^3 + x^2 - 5x - 2$.

b

$$\begin{array}{r} x^2+3x+1 \\ x-2\overline{)x^3+x^2-5x-2} \\ \underline{x^3-2x^2} \\ 3x^2-5x \\ \underline{3x^2-6x} \\ x-2 \\ \underline{x-2} \\ 0 \end{array}$$

$f(x) = x^3 + x^2 - 5x - 2$

$= (x - 2)(x^2 + 3x + 1)$

$f(x) = 0$ when $x = 2$

or $x^2 + 3x + 1 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-3 \pm \sqrt{(3)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{-3 \pm \sqrt{5}}{2}$$

So the solutions are

$x = 2$, $x = \frac{-3+\sqrt{5}}{2}$ and $x = \frac{-3-\sqrt{5}}{2}$.

13

$$\begin{array}{r} 2x^2-7x+3 \\ x+1\overline{)2x^3-5x^2-4x+3} \\ \underline{2x^3+2x^2} \\ -7x^2-4x \\ \underline{-7x^2-7x} \\ 3x+3 \\ \underline{3x+3} \\ 0 \end{array}$$

$2x^3 - 5x^2 - 4x + 3 = (x + 1)(2x^2 - 7x + 3)$

$= (x + 1)(2x - 1)(x - 3)$

The roots are $x = -1$, $x = \frac{1}{2}$ and $x = 3$.

So the positive roots are $x = \frac{1}{2}$ and $x = 3$.

14 $f(x) = x^3 - 5x^2 + px + 6$

Since the remainder obtained when $f(x)$ is divided by $(x + 2)$ is equal to the remainder obtained when the same expression is divided by $(x - 3)$,

$f(-2) = f(3)$

$(-2)^3 - 5(-2)^2 + p(-2) + 6$

$= (3)^3 - 5(3)^2 + p(3) + 6$

$-8 - 20 - 2p = 27 - 45 + 3p$

$-28 - 2p = 3p - 18$

$5p = -10$

$p = -2$

15 a $f(x) = x^3 - 2x^2 - 19x + 20$

$f(-4) = (-4)^3 - 2(-4)^2 - 19(-4) + 20$

$= -64 - 32 + 76 + 20$

$= 0$

The remainder is 0.

15 b

$$
\begin{array}{r}
x^2-6x+5 \\
x+4\overline{\smash{)}x^3-2x^2-19x+20} \\
\underline{x^3+4x^2} \\
-6x^2-19x \\
\underline{-6x^2-24x} \\
5x+20 \\
\underline{5x+20} \\
0
\end{array}
$$

$f(x) = x^3 - 2x^2 - 19x + 20$
$= (x + 4)(x^2 - 6x + 5)$
$= (x + 4)(x - 5)(x - 1)$
$f(x) = 0$ when
$x = -4$, $x = 5$ or $x = 1$

16 a $f(x) = 6x^3 + 17x^2 - 5x - 6$

$$f\left(\frac{2}{3}\right) = 6\left(\frac{2}{3}\right)^3 + 17\left(\frac{2}{3}\right)^2 - 5\left(\frac{2}{3}\right) - 6$$

$$= 6\left(\frac{8}{27}\right) + 17\left(\frac{4}{9}\right) - 5\left(\frac{2}{3}\right) - 6$$

$$= \frac{16}{9} + \frac{68}{9} - \frac{10}{3} - 6$$

$= 0$

So $(3x - 2)$ is a factor of $f(x)$.

$$
\begin{array}{r}
2x^2+7x+3 \\
3x-2\overline{\smash{)}6x^3+17x^2-5x-6} \\
\underline{6x^3-4x^2} \\
21x^2-5x \\
\underline{21x^2-14x} \\
9x-6 \\
\underline{9x-6} \\
0
\end{array}
$$

$f(x) = 6x^3 + 17x^2 - 5x - 6$
$= (3x - 2)(2x^2 + 7x + 3)$
So $a = 2$, $b = 7$, $c = 3$

b $f(x) = (3x - 2)(2x^2 + 7x + 3)$
$= (3x - 2)(2x + 1)(x + 3)$

c $(3x - 2)(2x + 1)(x + 3) = 0$
The real roots are $x = \frac{2}{3}$, $x = -\frac{1}{2}$ and $x = -3$.

17

$$\text{LHS} = \frac{x-y}{\left(\sqrt{x}-\sqrt{y}\right)} \times \frac{\left(\sqrt{x}+\sqrt{y}\right)}{\left(\sqrt{x}+\sqrt{y}\right)}$$

$$= \frac{(x-y)\left(\sqrt{x}+\sqrt{y}\right)}{x-y}$$

$$= \sqrt{x}+\sqrt{y}$$

= RHS

So $\dfrac{x-y}{\sqrt{x}-\sqrt{y}} \equiv \sqrt{x}+\sqrt{y}$

18 Completing the square:
$n^2 - 8n + 20 = (n - 4)^2 + 4$
The minimum value is 4, so $n^2 - 8n + 20$ is always positive.

19 $A(1,1)$, $B(3,2)$, $C(4,0)$ and $D(2, -1)$

The gradient of line $AB = \dfrac{2-1}{3-1} = \dfrac{1}{2}$

The gradient of line $BC = \dfrac{0-2}{4-3} = -2$

The gradient of line $CD = \dfrac{-1-0}{2-4} = \dfrac{1}{2}$

The gradient of line $AD = \dfrac{-1-1}{2-1} = -2$

AB and BC, BC and CD, CD and AD and AB and AD are all perpendicular.

Distance $AB = \sqrt{(3-1)^2+(2-1)^2}$
$= \sqrt{5}$

Distance $BC = \sqrt{(4-3)^2+(0-2)^2}$
$= \sqrt{5}$

Distance $CD = \sqrt{(2-4)^2+(-1-0)^2}$
$= \sqrt{5}$

Distance $AD = \sqrt{(2-1)^2+(-1-1)^2}$
$= \sqrt{5}$

All four sides are equal and all four angles are right angles, therefore $ABCD$ is a square.

20 $1 + 3 = \text{even}$
$3 + 5 = \text{even}$
$5 + 7 = \text{even}$
$7 + 9 = \text{even}$
So the sum of two consecutive positive odd numbers is always even.

21 To show something is untrue you only need to find one counter example.
Example: when $n = 6$,
$n^2 - n + 3 = 6^2 - 6 + 3 = 33$
which is not a prime number.
So the statement is untrue.

22 $$\text{LHS} = \left(x - \frac{1}{x}\right)\left(x^{\frac{4}{3}} + x^{-\frac{2}{3}}\right)$$
$$= x^{\frac{7}{3}} + x^{\frac{1}{3}} - x^{\frac{1}{3}} - x^{-\frac{5}{3}}$$
$$= x^{\frac{7}{3}} - x^{-\frac{5}{3}}$$
$$= x^{\frac{1}{3}}\left(x^2 - \frac{1}{x^2}\right)$$
$= \text{RHS}$

So $\left(x - \frac{1}{x}\right)\left(x^{\frac{4}{3}} + x^{-\frac{2}{3}}\right) \equiv x^{\frac{1}{3}}\left(x^2 - \frac{1}{x^2}\right)$

23 Remember, in an identity you can start from the RHS or the LHS. Here it is easier to start from the RHS.
$\text{RHS} = (x + 4)(x - 5)(2x + 3)$
$= (x + 4)(2x^2 - 7x - 15)$
$= 2x^3 + x^2 - 43x - 60$
$= \text{LHS}$
So $2x^3 + x^2 - 43x - 60 \equiv (x + 4)(x - 5)(2x + 3)$

24 $x^2 - kx + k = 0$ has two equal roots,
so $b^2 - 4ac = 0$
$k^2 - 4k = 0$
$k(k - 4) = 0$
$k = 4$ or 0.
So $k = 4$ is a solution.

25 Using Pythagoras' theorem:
The distance between opposite edges
$$= 2\left(\left(\sqrt{3}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2\right)$$
$$= 2\left(3 - \frac{3}{4}\right)$$
$$= \frac{9}{2}$$
$\frac{9}{2}$ is rational.

26 a Let the first even number be $2n$.
The next even number is $2n + 2$.
$(2n + 2)^2 - (2n)^2 = 4n^2 + 8n + 4 - 4n^2$
$= 8n + 4$
$= 4(2n + 1)$
$4(2n + 1)$ is a multiple of 4 so is always divisible by 4.
So the difference of the squares of two consecutive even numbers is always divisible by 4.

b Let the first odd number be $2n - 1$.
The next odd number is $2n + 1$.
$(2n + 1)^2 - (2n - 1)^2$
$= (4n^2 + 4n + 1) - (4n^2 - 4n + 1)$
$= 8n$
$8n$ is a multiple of 8, which is always divisible by 4, so the statement is also true for odd numbers.

27 a The assumption is that x is positive.

b When $x = 0$, $1 + 0^2 = (1 + 0)^2$

Challenge

1 a Diameter of circle = 1,
so side of outside square = 1
Using Pythagoras' theorem:
Perimeter of the inside square =

$$4\left(\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}\right)=\frac{4}{\sqrt{2}}=2\sqrt{2}$$

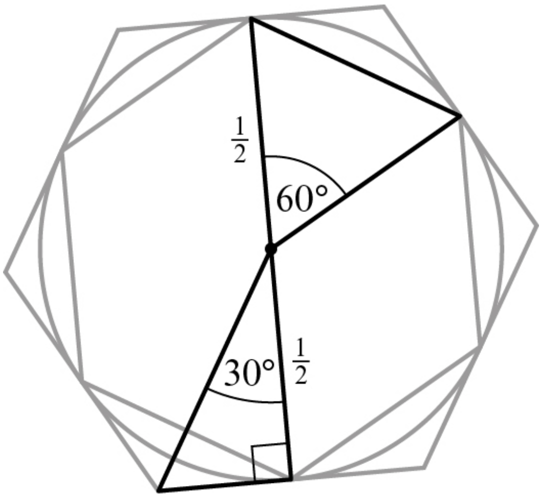
Perimeter of the outside square = 4 × 1 = 4
The circumference of the circle is between the perimeters of the two squares, so $2\sqrt{2} < \pi < 4$.

b Perimeter of inside hexagon = $6 \times \frac{1}{2} = 3$

because the triangles with 60° angles are equilateral.
Perimeter of outside hexagon

$= 6 \times \frac{\sqrt{3}}{3} = 2\sqrt{3}$

The circumference of the circle is between the perimeters of the two hexagons, so
$3 < \pi < 2\sqrt{3}$



2

$$\begin{array}{r l}
 & ax^2+(b+ap)x+(c+bp+ap^2) \\ \cline{2-2}
x-p \big) & ax^3+bx^2 \qquad +cx \qquad +d \\
 & \underline{ax^3-apx^2} \\
 & \qquad (b+ap)x^2+cx \\
 & \qquad \underline{(b+ap)x^2-(bp+ap^2)x} \\
 & \qquad\qquad (c+bp+ap^2)x+d \\
 & \qquad\qquad \underline{(c+bp+ap^2)x-(cp+bp^2+ap^3)} \\
 & \qquad\qquad\qquad d+cp+bp^2+ap^3
\end{array}$$

So $\frac{ax^3+bx^2+cx+d}{x-p} = ax^2 + (b + ap)x + (c + bp + ap^2)$ with remainder.
So, $d + cp + bp^2 + ap^3$
$f(p) = ap^3 + bp^2 + cp + d = 0$, which matches the remainder
$d + cp + bp^2 + ap^3 = 0$
Therefore $(x - p)$ is a factor of $f(x)$.