MEI

Mathematics in Education and Industry

MEI Structured Mathematics

Module Summary Sheets

C1, Introduction to Advanced Mathematics

(Version B—reference to new book)

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Mathematical processes and language

References:
Chapter 6
Pages 149-151

Exercise 6B
Q. 5, 7

Symbols
In making logical deductions we use the symbols

A $\Rightarrow$ B means statement A leads to statement B
A $\Leftarrow$ B means statement A follows from statement B
A $\Leftrightarrow$ B means statement A is equivalent to statement B

A $\Rightarrow$ B means A is ***sufficient*** for B
A $\Leftarrow$ B means A is ***necessary*** for B

E.g. Statement A:
The three angles of a triangle are equal
Statement B:
The three sides of a triangle are equal

Which of the following are correct?
A $\Rightarrow$ B, A $\Leftarrow$ B, A $\Leftrightarrow$ B.

A leads to B because if the angles are equal then the triangle is equilateral and so all three sides are equal.
But also if the sides are equal then the triangle is equilateral and so the angles are equal.
So A $\Leftrightarrow$ B is correct.

E.g. Statement A: $x^2 = x$
Statement B: $x = 1$

Which of the following are correct?
A $\Rightarrow$ B, A $\Leftarrow$ B, A $\Leftrightarrow$ B.

A does not lead to B as this is an incomplete statement.
However, if $x = 1$, then $x^2 = x$.
So A $\Leftarrow$ B is correct.

Note that if B had been $x = 0$ or 1 then A $\Leftrightarrow$ B would have been correct.

References:
Chapter 6
Pages 154-157

Exercise 6D
Q. 1, 5

Theorems are general statements. If they are true then they will be true in all circumstances. To prove a theorem therefore requires a proof that covers all possibilities. Showing the assertion to be true for a specific case is not good enough.

E.g. to prove that the sum of integers

$$1+2+3+....+n=\frac{1}{2}n(n+1)$$

Substitute n = 2, then $1+2=\frac{1}{2}n(n+1)=\frac{1}{2}\times 2\times 3$

This does **not** constitute a proof.

E.g. Sum of angles of triangle = 180^o
Suppose you draw a triangle and measure the angles and they come to 180^o. This fails to constitute a proof on two counts:
1. You cannot measure accurately so you can only assert that the sum is approximately 180^o.
2. It is only for one triangle anyway and not all triangles.

Disproving a theorem
An assertion can be disproved by a single example.

E.g. Is $n^2 + n + 41$ prime?
For $n = 0, 1, 2, 3$ etc, it is prime, but $n = 41$ gives a number which is not prime. This single "counter-example" shows that the theorem is not true.

References:
Chapter 6
Pages 152-153

Exercise 6C
Q. 1, 5

The converse of a theorem
This is a theorem stated the other way round.
If the theorem can be represented by A $\Rightarrow$ B then the converse is represented by A $\Leftarrow$ B .

E.g. Pythagoras' Theorem.
If a triangle is right-angled then $a^2 + b^2 = c^2$.
The converse is:
If $a^2 + b^2 = c^2$ then the triangle is right-angled.

The converse of a theorem is often, but not always true.

E.g. If a quadrilateral has four angles = 90^0 then opposite sides will be equal.
The converse is **not** true.
i.e. If a quadrilateral has opposite sides equal then it is not true that the angles are all equal to 90^0. (e.g. a rhombus.)

Mathematical Modelling

A situation in real life can sometimes be expressed exactly mathematically (e.g. the cost of a kilogram of apples is 47p).

Sometimes it cannot be done exactly, or it is not convenient to do so.

E.g. fitting a function to a curve from observed data.

E.g. making assumptions in order to be able to express it mathematically in a form which can be used.

In either situation we say that we are making a mathematical model.

Questions in the examinations (at all levels and in all strands) will refer to a mathematical model as a mathematical description of a real-life situation.

The Language of Algebra

References: Chapter 1 Pages 2-33

Word	Example	Description
Function	$f(x) = x^2 - 2x + 1$	Input a value, x, output a unique value $f(x)$ or y. A graph could be drawn. e.g. when $x = 3$, y or $f(x) = 4$.
Variable	$x, y, \ldots$	A value which can vary (take different values).
Constant	$1, c$	A fixed number.
Expression	$x^2 - 2x + 1$	An expression involves variables and constants (there is no = sign)
Equation	$x^2 - 2x + 1 = 0$	An equation can be solved for specific values of x.
Term	$-2x$	$-2x$ is a term of the expression $x^2 - 2x + 1$
Coefficient	-2	-2 is the coefficient of the x term in the expression $x^2 - 2x + 1$
Index	x^2	2 is the highest power (index) of x in the expression $x^2 - 2x + 1$

References: Chapter 1 Pages 2-5

Exercise 1A Q. 1(i),(v), 2(i), 3(i), 4(v), 5(v), 6(v), 7(v)

Simplifying Algebraic Expressions

Algebraic expressions can be simplified as follows:

Adding like terms:
E.g. $2x+3y-x+2y=x+5y$

Cancelling common factors in a fraction:
E.g. $\dfrac{6xy}{2y^2} = \dfrac{3x}{y}$

Multiplying out brackets:
E.g. $3(2x+5) = 6x+15$

By factorising:
E.g. $3x^2 - 6xy = 3x(x-2y)$

E.g. Simplify $2x+y-x-2y$
$= 2x-x+y-2y$
$= x-y$

E.g. Simplify $2(2x+3y)-3(4x-y)$
$= 4x+6y-12x+3y$
$= -8x+9y$

E.g. Simplify $\dfrac{3(4x+7y)-2(6x+3y)}{5xy}$
$= \dfrac{12x+21y-12x-6y}{5xy} = \dfrac{15y}{5xy} = \dfrac{3}{x}$

E.g. Factorise $81x^2y^2z - 36xyz^2$
H.C.F. of the two terms is $9xyz$
$\Rightarrow 9xyz(9xy-4z)$

References: Chapter 1 Pages 7-9

Exercise 1B Q. 1(iii), (x), 3, 5

Linear Equations
Linear equations involve only a single power of x.
A general order of approach:
- Clear fractions.
- Multiply out brackets.
- Gather terms & simplify.
- Divide.

E.g. Solve $2(x+3) = 1 - \dfrac{x}{3}$
$[\times 3]: \Rightarrow 6(x+3) = 3 - x$
$\Rightarrow 6x+18-3+x = 0$
$\Rightarrow 7x = -15$
$[\div 7]: \Rightarrow x = -\dfrac{15}{7}$

References: Chapter 1 Pages 11-12

Exercise 1C Q. 3, 6, 12

Transposing formulae
(Changing the subject of formulae)
Follows the same rules as solving linear equations.
- Clear fractions.
- Multiply out brackets.
- Gather terms & simplify.
- Factorise if necessary.
- Divide.

E.g. Transpose for x (make x the subject) in the formula $y = \dfrac{x+1}{3x-2}$

$y = \dfrac{x+1}{3x-2} \Rightarrow y(3x-2) = x+1$

$\Rightarrow 3xy - 2y = x+1 \Rightarrow 3xy - x = 1+2y$

$\Rightarrow x(3y-1) = 1+2y \Rightarrow x = \dfrac{1+2y}{3y-1}$

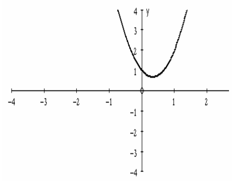
References:
Chapter 1
Pages 13-18

Quadratic functions

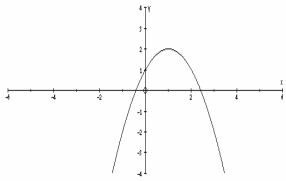
are of the form $f(x) = ax^2 + bx + c$
Their graphs are parabolas.

If $a > 0$ then $f(x)$ has a minimum point.
If $a < 0$ then $f(x)$ has a maximum point.

E.g. $f(x)=3x^2-2x+1$



E.g. $f(x)=-x^2+2x+1$



References:
Chapter 1
Pages 14-18

Exercise 1D
Q. 1(v), 2(ii), 3(i)
5(ii)

Quadratic factorisation

$(x-a)(x-b) = x^2-(a+b)x+ab$.
Therefore, to factorise a quadratic function you need to find the two numbers a and b such that their sum is the coefficient of x and their product is the constant term.
E.g. $x^2-8x+15$ has the two numbers 3 and 5
$\Rightarrow x^2-8x+15=(x-3)(x-5)$

E.g. Factorise $x^2+13x+12$
$1+12=13$ and $1\times 12=12$
$\Rightarrow\ x^2+13x+12=(x+1)(x+12)$

E.g. Factorise x^2-x-6
$-3+2=-1$ and $-3\times 2=-6$
$\Rightarrow\ x^2-x-6=(x-3)(x+2)$

References:
Chapter 1
Pages18-24

Exercise 1D
Q. 6(ii), 7(ii),
8(ii), 11

Quadratic equations

are of the form $ax^2+bx+c=0$

The values of x that satisfy this equation are called the roots.

The solution of the equation is the set of all the roots. Graphically this is where the curve $y = ax^2+bx+c$ cuts the x-axis. A quadratic equation can have 0, 1 or 2 roots.

Quadratic equations may be solved by factorising and putting each factor equal to zero, or by using the formula:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

If the "discriminant", b^2-4ac is negative then there is no square root and so there are no roots.
If $b^2-4ac = 0$ then the two roots are coincident.

E.g. Solve the equation $3x^2+5x-2=0$
$3x^2+5x-2=0$
$(3x-1)(x+2)=0$
$(3x-1)=0 \Rightarrow x=\frac{1}{3}$
$(x+2)=0 \Rightarrow x=-2$

E.g. Solve the equation $2x-3=\frac{4}{x}$
$2x^2-3x-4=0$
(equation does not factorise)
$x=\frac{3\pm\sqrt{9-4\times 2\times(-4)}}{4}$
$x=\frac{1}{4}(3\pm 6.403)\Rightarrow x=2.35$ or -0.85

References:
Chapter 1
Page 24

Exercise 1D
Q. 8 (iii), (iv)

Completing the square is the process of putting a quadratic expression in the form

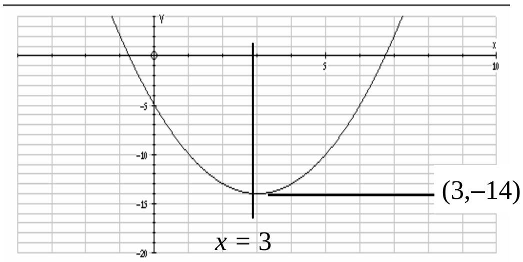
$$a(x-p)^2+q.$$

Hence we can say that the function will have a minimum (or maximum) point at (p, q).
An alternative method is to that shown is to compare coefficients.

$(x-p)^2+q\equiv x^2-2px+p^2+q$
Therefore: $x^2-6x-5\equiv(x-3)^2-9-5\equiv(x-3)^2-14$

E.g. Complete the square for x^2-6x-5.
Hence state the coordinates of the minimum point and the equation of the axis of symmetry.
$x^2-6x-5\equiv(x-3)^2-9-5\equiv(x-3)^2-14$

Has min. point at $(3,-14)$, Axis of symmetry at $x=3$



Summary C1 Topic 2: **Algebra—3; Simultaneous Equations and Inequalities**

References:
Chapter 1
Pages 28-30

Exercise 1E
Q. 1(ii), 3

Linear simultaneous equations

Equations in two (or more) variables that are true simultaneously are known as simultaneous equations.

Two linear equations in two variables may be solved by:

- Substitution
  (one variable is made the subject of one Equation and substituted in the other)
- Elimination
  (both equations are manipulated so that a coefficient of one variable is the same. That variable is then eliminated.)
- Graphical solution – see page 7

E.g. Find the point of intersection of the lines $2x - 3y = 7$ and $3x + 4y = 2$.

$2x - 3y = 7$ (i)
$3x + 4y = 2$ (ii)
(i)x4 $8x - 12y = 28$ (iii)
(ii)x3 $9x + 12y = 6$ (iv)
(iii) + (iv) $17x = 34 \Rightarrow x = 2$
Sub in (ii) : $6 + 4y = 2 \Rightarrow y = -1$

E.g. Find the point of intersection of the lines $x - y = 2$ and $4x + 5y = 17$

$x - y = 2 \Rightarrow y = x - 2$
Sub. into $4x + 5y = 17 \Rightarrow 4x + 5(x-2) = 17$
$\Rightarrow 9x - 10 = 17 \Rightarrow 9x = 27 \Rightarrow x = 3 \Rightarrow y = 1$

References:
Chapter 1
Page 30

Exercise 1E
Q. 5(ii), 6

Simultaneous equations when one is non-linear

Make one variable the subject of the linear equation and substitute in the other equation. In this unit the result will be a quadratic equation.

E.g. Solve simultaneously the following equations

$y = 2x - 1$ and $y = x^2 - 3x + 3$

Substitute for y: $2x - 1 = x^2 - 3x + 3$
$\Rightarrow x^2 - 5x + 4 = 0$
$\Rightarrow (x-1)(x-4) = 0$
$\Rightarrow x = 1, 4 \Rightarrow (1,1), (4,7)$

References:
Chapter 4
Pages 123-124

Exercise 4A
Q. 1(i), (ii)

Inequalities

can be solved using the same rules as for equations except:

- When multiplying by a negative number the direction of the inequality sign reverses.
- Care needs to be taken to maintain the inequality sign.

You should be familiar with inequality diagrams.

E.g. Solve the inequality $26 - 5x < 11$

$26 - 5x < 11$
$26 - 11 < 5x$ or $-5x < 11 - 26$
$15 < 5x$ or $-5x < -15$
$3 < x$ or $x > 3$

References:
Chapter 4
Pages 124-125

Exercise 4A
Q. 2(i), (v)

Quadratic inequalities

You are advised always to sketch a graph.

For the inequality, $ax^2 + bx + c > 0$, sketch the curve $y = ax^2 + bx + c$.

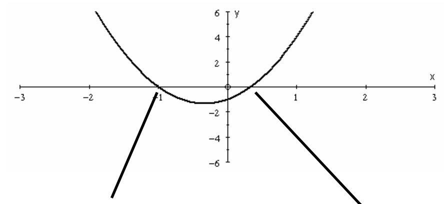
Find where the curve crosses the x-axis (say at a and b) and this gives one of the two results, $a < x < b$ or $x < a$ and $x > b$.

There are two ways of finding the solution.

(i) Sketch the graph accurately enough to determine where it crosses the x-axis.
(ii) Factorise the quadratic expression. This gives a product of two factors. For a "greater than" inequality both factors must be positive. For a "less than" inequality one must be positive and the other negative.

E. g. Solve $-3x^2 - 2x + 1 \le 0$

$[\times -1]$: $3x^2 + 2x - 1 \ge 0$
(This factorises) $\Rightarrow (3x-1)(x+1) \ge 0$
$\Rightarrow x \le -1, \quad x \ge \frac{1}{3}$ (See graph)



$x = -1$ $x = {}^1/_3$

Competence statements a2, a3, a4, a8, a9

References: Chapter 5 Pages 127-129

Surds and irrational numbers

The square root of a positive integer is either an integer or an irrational number. E.g. $\sqrt{4}=2$ E.g. $\sqrt{5}$

An irrational number is one that, when written as a decimal, neither terminates nor recurs, and so cannot be written as a fraction.

A number which includes such a value is called a surd. E.g. $2+\sqrt{3}$

Note that this is an exact value, while any decimal approximation (e.g. 2 + 1.732 = 3.732) is only an approximation. Any question that requires an exact answer is almost certainly going to involve surds. Take care when you see the requirement for an exact answer in a question.

Exercise 5A Q. 1(iii), 2(iii), (v), 3 (iii), (iv)

E.g. $2\sqrt{2}+3\sqrt{2}=5\sqrt{2}$

$\sqrt{18}+\sqrt{8}=3\sqrt{2}+2\sqrt{2}=5\sqrt{2}$

E.g. Find the roots of the equation $3x^2+4x-5=0$ exactly.

$$x=\frac{-4\pm\sqrt{16+60}}{6}=\frac{-4\pm\sqrt{76}}{6}=\frac{-4\pm2\sqrt{19}}{6}$$

$$=\frac{1}{3}\left(-2-\sqrt{19}\right) \text{ and } \frac{1}{3}\left(-2+\sqrt{19}\right)$$

N.B. Use of calculators here to find a value of the square root will result in an approximate answer. Note the requirement for an exact answer. ***Remember that calculators are not allowed in the examination so you will not be asked for an approximate answer.***

References: Chapter 5 Page 128

Rationalising the denominator

When a surd appears in the denominator, the faction may be "rationalised" by multiplying top and bottom by the surd so that the denominator becomes a rational number.

E.g. $\frac{2}{\sqrt{3}}=\frac{\sqrt{3}}{\sqrt{3}}\times\frac{2}{\sqrt{3}}=\frac{2\sqrt{3}}{3}$

$(2+\sqrt{3})(2-\sqrt{3})=4-3=1$ (Difference of squares)

$\frac{1}{(2+\sqrt{3})}$ has an irrational denominator which can be "rationalised" by multiplying by $(2-\sqrt{3})$.

e.g. $\frac{1}{(2+\sqrt{3})}=\frac{1}{(2+\sqrt{3})}\times\frac{(2-\sqrt{3})}{(2-\sqrt{3})}=(2-\sqrt{3})$

Exercise 5A Q. 4(i), (iii)

E.g. Simplify $\frac{2}{3+\sqrt{5}}$

$$\frac{2}{3+\sqrt{5}}=\frac{2}{3+\sqrt{5}}\times\frac{3-\sqrt{5}}{3-\sqrt{5}}=\frac{2(3-\sqrt{5})}{9-5}=\frac{1}{2}\left(3-\sqrt{5}\right)$$

E.g. Simplify $\frac{3+\sqrt{5}}{3-\sqrt{5}}$

$$\frac{3+\sqrt{5}}{3-\sqrt{5}}\times\frac{3+\sqrt{5}}{3+\sqrt{5}}=\frac{\left(3+\sqrt{5}\right)^2}{9-5}$$

$$=\frac{9+6\sqrt{5}+5}{4}=\frac{1}{2}\left(7+3\sqrt{5}\right)$$

References: Chapter 5 Pages 130-135

Indices

The Laws of indices

1. $a^m\times a^n=a^{m+n}$
2. $a^m\div a^n=a^{m-n}$
3. $(a^m)^n=a^{mn}$
4. $m=n$ in $2\Rightarrow a^0=1$
5. $m=0$ in 2 and using 4 $\Rightarrow 1\div a^n=a^{-n}$
6. $n=m=\frac{1}{2}$ in $1\Rightarrow\sqrt{a}=a^{\frac{1}{2}}$
7. more generally: $\sqrt[m]{a}=a^{\frac{1}{m}}$

In the number a^m, a is called the base and m is called the power or index.

Exercise 5B Q.4(iv), 6(i), 7(iv), 8(ii), 9(iii)

E.g. $2^2\times2^3=2^5$

$2^2\div2^3=2^{-1}=\frac{1}{2}$

$2^5=32$, $2^{-5}=\frac{1}{32}$, $\left(2^2\right)^3=2^6$; $64^{\frac{1}{3}}=4$

$9^{\frac{3}{2}}=\left(9^{\frac{1}{2}}\right)^3=3^3=27$; also $9^{\frac{3}{2}}=\left(9^3\right)^{\frac{1}{2}}=729^{\frac{1}{2}}=27$

N.B. Mixed Bases, for instance $2^3\times3^2$, cannot be evaluated using the laws of indices because the bases are not the same.

$2^3\times3^2=8\times9=72$

Note, however, that $2^3\times3^3=6^3=216$

References:
Chapter 2
Pages 34-40

Exercise 2A
Q. 1(i), (iv), 7

Properties of lines

For the line defined by the points (x_1, y_1) and (x_2, y_2)

Gradient = $\dfrac{y_2 - y_1}{x_2 - x_1}$

If a line has gradient m

Parallel lines also have gradient m

Perpendicular lines have gradient $-\dfrac{1}{m}$

Length = $\sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$

Mid point = $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$

References:
Chapter 2
Pages 42-51

Exercise 2B
Q.2 (v), (x)

Exercise 2C
Q.2(v), 3(v),
4(v), 6, 10

Equations of lines

The equation of a line passing through (x_1, y_1) with gradient m is

$$y - y_1 = m(x - x_1)$$

Gradient-intercept form of a straight line, with gradient m, intercept on y axis c, is

$$y = mx + c$$

The form $px + qy + r = 0$ is usually used to ensure that fractions are not included.

Rearranging will give gradient $= -\dfrac{p}{q}$ and intercept on the axes $\left(-\dfrac{r}{p}, 0\right)$ and $\left(0, -\dfrac{r}{q}\right)$.

Exercise 2B
Q. 1(ix), (xv)

Sketching straight line graphs

The sketch should show whether the slope is positive or negative and where the line cuts the axis.

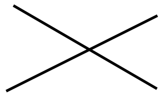
- Don't worry about putting scales on the axes
- Line cuts the y axis where $x = 0$.
- Line cuts the x axis where $y = 0$.

References:
Chapter 2
Pages 55-56

Exercise 2D
Q. 2, 5

Intersection of lines

Two non-parallel lines meet at a point.



The coordinates of this point satisfy both equations simultaneously.

Solve as simultaneous equations.

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Version B: page 7
Competence statements g1, g2, g3, g4, g5, g6, g7, g8

E.g. Find the gradient, length & mid point of the line between the points $(-2,3)$ & $(4,-1)$.

gradient $= \dfrac{-1-3}{4-(-2)} = \dfrac{-4}{6} = -\dfrac{2}{3}$

length $= \sqrt{(-1-3)^2 + (4-(-2))^2} = \sqrt{16+36} = \sqrt{52}$

mid point $= \left(\dfrac{-2+4}{2}, \dfrac{3+(-1)}{2}\right) = (1,1)$

E.g. Find the equation of a line with gradient -4, passing through the point $(2,-3)$.

$y + 3 = -4(x - 2) \Rightarrow 4x + y = 5$

E.g. Find the equation of a line passing through $(-1,2)$ and perpendicular to the line $2x + 5y = 1$.

$2x + 5y = 1$:

$y = -\dfrac{2}{5}x + \dfrac{1}{5}$ (putting into gradient-intercept form)

has gradient $-\dfrac{2}{5}$

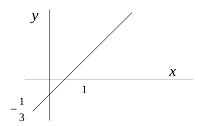
So perpendicular gradient is $\dfrac{5}{2}$ & required eqn is:

$y - 2 = \dfrac{5}{2}(x + 1) \Rightarrow 5x - 2y = -9$

E.g. Sketch the line $2x - 3y = 1$

y axis $(x = 0)$: $-3y = 1 \Rightarrow y = -\dfrac{1}{3}$

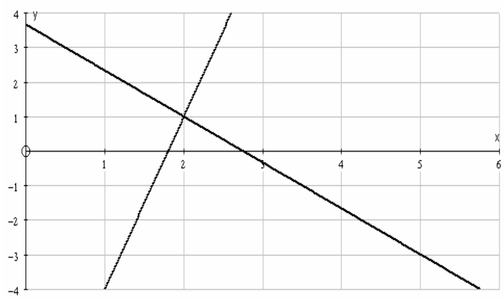
x axis $(y = 0)$: $2x = 1 \Rightarrow x = \dfrac{1}{2}$



Alternatively:
$y = \dfrac{2}{3}x - \dfrac{1}{3}$
cuts y axis at $-\dfrac{1}{3}$
Gradient (+ve) $= \dfrac{2}{3}$

E.g. Find the point of intersection of the lines
$4x + 3y = 11$, $5x - y = 9$

Plot the lines.



They intersect at (2, 1).

For the general shape of curves, see Topic 5

References:
Chapter 2
Pages 61-66

Exercise 2E
Q. 1(ii), 2(ii),8

The Circle

The circle with equation $(x-a)^2 + (y-b)^2 = r^2$ has centre (a, b) and radius r.

The circle with equation $x^2 + y^2 - 2hx - 2ky + c = 0$ simplifies to

$$(x-h)^2 + (y-k)^2 = h^2 + k^2 - c,$$

giving the radius, r, as $r^2 = h^2 + k^2 - c$

A line, in general, cuts a circle in two points. Substitution for y from the equation of the line into the equation of the circle will give a quadratic equation in x. The equation may have 0, 1 or 2 roots.
A chord (or diameter) will give two points; a tangent will have coincident roots . If the line does not cut the circle then this quadratic equation will have no roots.

References:
Chapter 2
Page 63

The Circle Theorems

- The angle at the centre is twice the angle at the circumference
- Angles in the same segment are equal
- The angle subtended by a diameter is a right angle
- The angle between a tangent and a chord at a point is equal to the angle in the alternate segment. (*not included in the specification.*)

References:
Chapter 2
Pages 68-70

Exercise 2F
Q.4, 6

Intersection of a line and a curve

A line may cut a curve in distinct points, or it may touch it.
A line that touches a curve is called a *Tangent.*
It is also possible that the line does not cut the curve.

The process of finding the points is usually as follows:

- Write the equation of the line in the form $y = ….$
- Substitute this expression for x into the equation of the curve.
- Solve the resulting equation in x.
- Substitute the values of x that satisfy this equation into the equation for the line to give the y values of the points of intersection.

If the line does not meet the curve then there will be no roots to the equation.
If the line touches the curve then there will be two coincident roots to the equation.

References:
Chapter 2
Pages 71-72

Intersection of curves

The intersection(s) of two curves can be found at this level only if the above procedure can be adopted - i.e. if one of the variables can be made the subject of one equation and then substituted into the other to give an equation (in either x or y) which can then be solved.
Two curves can intersect in any number of points; the resulting polynomial equation in x will be of the order of the number of intersections—I.e. if two curves cut in two points (e.g. two circles) then the equation to be solved will be quadratic.

E.g. Find the equation of the circle with centre (1,2) and radius 3.

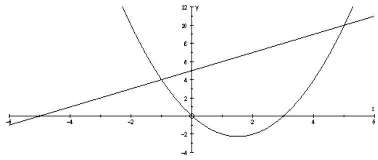
$(x-1)^2 + (y-2)^2 = 3^2$
gives $x^2 + y^2 - 2x - 4y - 5 = 0$

E.g. Find the radius and centre of the circle
$x^2 + y^2 - 6x + 4y + 9 = 0$

The equation simplifies to
$(x-3)^2 + (y+2)^2 = 9 + 4 - 9 = 4.$

Thus the centre is (3,–2) and the radius is 2.

E.g. Find where the line $y = x + 5$ cuts the curve $y = x^2 - 3x$.

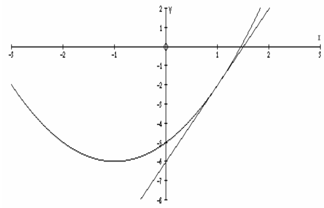


Sub. for y:
$x + 5 = x^2 - 3x$

Gives $x^2 - 4x - 5 = 0$
$(x-5)(x+1) = 0 \quad \Rightarrow x = 5$ or -1

Sub. in line: $y = 10$ or 4
The line cuts the curve at (–1, 4) and (5, 10).

E.g. Find where the line $y = 4x - 6$ cuts the curve $y = x^2 + 2x - 5$.



Sub. for y:
$4x - 6 = x^2 + 2x - 5$
Gives $x^2 - 2x + 1 = 0$
$(x-1)^2 = 0$
$\Rightarrow x = 1$ twice.
Sub. in line: $y = -2$.

E.g. Find where the curves $2y = x^2 + 2$ and $x^2 + y^2 = 13$ intersect.

1st equation gives $x^2 = 2y - 2$
Substitute into 2nd equation: $y^2 + 2y - 2 = 13$
$\Rightarrow y^2 + 2y - 15 = 0$
$\Rightarrow (y-3)(y+5) = 0 \Rightarrow y = 3, -5$
$y = -5$ has no solution for x.
$y = 3$ gives $x^2 = 4 \Rightarrow x = \pm 2$
$\Rightarrow$ intersections at $(-2,3)$ and $(2,3)$

Summary C1 Topic 4: ***Polynomials—1; Manipulating Polynomials***

References: Chapter 3 Pages 77-78

Polynomials are expressions of the form

$$ax^n + bx^{n-1} + ...px + q$$

n is a positive integer; $a,b,...p,q$ are real numbers.

E.g. $3x^5 + 2x^4 - 2x^2 - 1$ is a polynomial of degree 5.

References: Chapter 3 Pages 78-81

Exercise 3A Q. 3, 7, 16

Adding and subtracting polynomials
Remove brackets, collect like terms

Multiplying polynomials
Careful setting out can facilitate the gathering together of like terms.

Dividing polynomials
Set out as for an arithmetic division.

E.g. $\left(2x^3 - x^2 + 4x - 2\right) - \left(x^3 + 3x^2 - x + 4\right)$
$= 2x^3 - x^2 + 4x - 2 - x^3 + 3x^2 + x - 4 = x^3 + 2x^2 + 5x - 6$

E.g. $\left(x^2 - 3x + 2\right)\left(2x^2 + x - 5\right)$
$= 2x^4 + x^3 - 5x^2$
$\quad - 6x^3 - 3x^2 + 15x$
$\quad + 4x^2 + 2x - 10$
$= 2x^4 - 5x^3 - 4x^2 + 17x - 10$

E.g. Divide $x^3 + x^2 - 6x + 4$ by $x - 1$

$$\begin{array}{r} x^2 + 2x - 4 \\ (x-1)\overline{)x^3 + x^2 - 6x + 4} \\ \underline{x^3 - x^2} \\ 2x^2 - 6x \\ \underline{2x^2 - 2x} \\ -4x + 4 \\ \underline{-4x + 4} \end{array}$$

References: Chapter 3 Pages 82-86

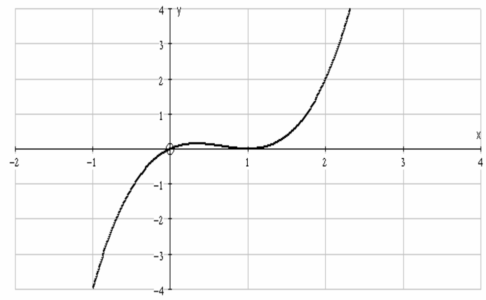
Exercise 3B Q. 3, 4, 6

Sketching Polynomial Functions

- Know the basic shapes.
- Find where the curve cuts the y-axis $(x = 0)$
- Find where the curve cuts the x-axis $(y = 0)$
  (solve the resulting equation by factorising where possible)
- Coordinates of turning points can be found by differentiation.

Inequalities
Can be solved by sketching a graph.

E.g. Sketch the curve $y = x(x - 1)^2$.
The curve is a cubic, so has two turning points.
As x gets large so does y.
When $x = 0$, $y = 0$
When $y = 0$, $x = 0$ and 1 (twice)



References: Chapter 3 Pages 89-91

The Factor Theorem

$x - a$ is a factor of $f(x) \Leftrightarrow f(a) = 0$
If a polynomial can be factorised then the factors can be found by trial of various numbers, a.

E.g. Suppose $f(x) = (x - a)\,p(x)$ where $p(x)$ is a polynomial of order one less than the order of $f(x)$.
Then if $x = a$ the value of $f(x)$ is 0 whatever the value of $p(x)$.

E.g. Solve, by factorising, $x^3 + x^2 - 4x - 4 = 0$
$f(x) = x^3 + x^2 - 4x - 4$
$f(1) = -6$ (So $x - 1$ is not a factor)
$f(-1) = 0 \Rightarrow x + 1$ is a factor
$f(2) = 0 \Rightarrow x - 2$ is a factor
The third factor is $x + 2$, by considering the last term.
So $f(x) = (x+1)(x-2)(x+2)$
$\Rightarrow x = -1, -2, 2$

References: Chapter 3 Pages 92-93

Exercise 3C Q. 2, 5

The Remainder Theorem
If $f(x)$ is divided by $(x - a)$ then the remainder is $f(a)$.
Note that when the remainder is 0 $(x - a)$ is a factor, so this is an extension of the Factor Theorem.

E.g. Find the remainder when $x^3 + x^2 - 4x - 4$ is divided by $(x - 1)$, $(x + 1)$
$f(x) = x^3 + x^2 - 4x - 4$
$f(1) = -6 \Rightarrow$ The remainder is -6
$f(-1) = 0 \Rightarrow$ The remainder is 0 (so $(x + 1)$ is a factor.

E.g. $ax^3 - x^2 + 4$ is divisible by $2x + 1$. Find a.
$f\left(-\frac{1}{2}\right) = -\frac{1}{8}a - \frac{1}{4} + 4 = 0$
$\Rightarrow a = 30$

Summary C1 Topic 4: ***Polynomials—2; Binomial Expansion***

References: Chapter 3 Pages 108-110

Exercise 3F Q. 1(iii),(v)

Binomial expansions

When a linear expression of the form ($a + x$) is multiplied by itself a number of times then a polynomial results. This polynomial can be written down noting the following:

For $(a + x)^n$

Each term is of the same power, starting with a^n with decreasing powers of a and increasing powers of x.

Each term has associated with it a "coefficient".

For $(a + x)^n$ each term is positive for positive a.

For $(a - x)^n$ the terms are alternating in sign for positive a.

For small positive n the coefficients can be found from Pascal's Triangle (See Students Handbook for a larger version).

$$\begin{array}{ccccccccc} &&&&1&&&& \\ &&&1&&1&&& \\ &&1&&2&&1&& \\ &1&&3&&3&&1& \\ 1&&4&&6&&4&&1 \end{array}$$

E.g. Simplify $(1 + x)^3 + (1 - x)^3$.
Hence find $1.01^3 + 0.99^3$

$(1 + x)^3 + (1 - x)^3 = 1 + 3x + 3x^2 + x^3$
$\quad +1 - 3x + 3x^2 - x^3$
$\quad = 2 + 6x^2$

Put $x = 0.01$
Gives $1.01^3 + 0.99^3 = 2 + 6(0.0001)$
$= 2.0006$

E.g. Expand $(2 + x)^4$.

$(2 + x)^4 = 2^4 + 4.2^3.x + 6.2^2.x^2 + 4.2\,x^3 + x^4$
$= 16 + 32x + 24x^2 + 8x^2 + x^4$.

E.g. next two lines are:

1 5 10 10 5 1
1 6 15 20 15 6 1

References: Chapter 3 Pages 110-113

Exercise 3F Q. 2 (iii),(iv)

The formula for a binomial coefficient

The formula for the coefficient of the rth term of $(a+x)^n$ is given by

$$\binom{n}{r} = {}^nC_r = \frac{n!}{(n-r)!r!} \text{ where } n! = n(n-1)(n-2).......3.2.1$$

$$\Rightarrow \binom{n}{r} = \frac{n(n-1)(n-2).....(n-r+1)}{1.2.3.4......r}$$

$$\text{E.g.} \binom{7}{3} = \frac{7.6.5}{1.2.3} = 35$$

The expansions are therefore as follows:

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + ...x^n$$

$$= 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + ...$$

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + ...b^n$$

E.g. $\binom{7}{5} = \frac{7.6.5.4.3}{1.2.3.4.5} = \frac{7.6}{1.2} = 21 = \binom{7}{2}$

$\binom{8}{3} = \frac{8.7.6}{1.2.3} = 56$

$\binom{10}{5} = \frac{10.9.8.7.6}{1.2.3.4.5} = 252$

E.g. Find the coefficient of x^3 in $(1 - 2x)^5$.

$(1 - 2x)^5 = 1 - 5.2.x + 10.2^2.x^2 - 10.2^3\,x^3 + ...$

Coefficient is –80.

E.g. Find the term independent of x in the expansion of $\left(2x - \frac{3}{x}\right)^4$.

$$\left(2x - \frac{3}{x}\right)^4 = (2x)^4 - 4.(2x)^3\left(\frac{3}{x}\right) + 6(2x)^2\left(\frac{3}{x}\right)^2 + ...$$

Term independent of x is $6(2x)^2\left(\frac{3}{x}\right)^2$

$$= 6.2^2.3^2 = 216$$

References: Chapter 3 Page 114

Relationships between binomial coefficients

$${}^nC_r = {}^nC_{n-r}$$

$${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$$

$${}^nC_r + {}^nC_{r+1} = \frac{n!}{r!(n-r)!} + \frac{(n+1)!}{(r+1)!(n-r-1)!}$$

$$= \frac{n!}{r!(n-r-1)!}\left(\frac{1}{n-r} + \frac{n+1}{r+1}\right) = \frac{n!}{(r+1)!(n-r)!}(r+1+n-r)$$

$$= \frac{(n+1)!}{(r+1)!(n-r)!} = {}^{n+1}C_{r+1}$$

Substitute $x = 1$ into $(1+x)^n$ to give

$${}^nC_0 + {}^nC_1 + ... + {}^nC_n = 2^n$$

E.g. ${}^3C_0 = 1,\ {}^3C_1 = 3,\ {}^3C_2 = 3,\ {}^3C_3 = 1$

${}^3C_0 + {}^3C_1 + {}^3C_2 + {}^3C_3 = 1 + 3 + 3 + 1 = 8 = 2^3$

Note also that the sum of the lines of Pascal's Triangle are the powers of 2.

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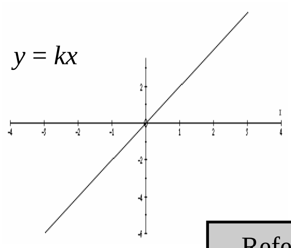
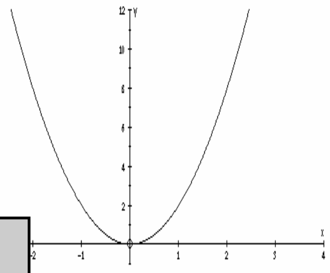
Competence statements f7, f8, f9

Plotting and sketching

A curve can be *plotted* point by point from its equation. The general shape may be missed, however, if the range of values of x is not large enough or not small enough.
A curve can be *sketched*, and so the general shape discovered The general shape of "standard" curves should be known.

$y = kx$

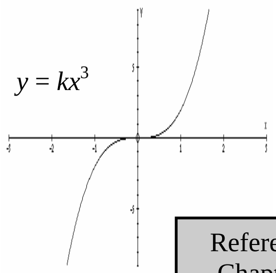


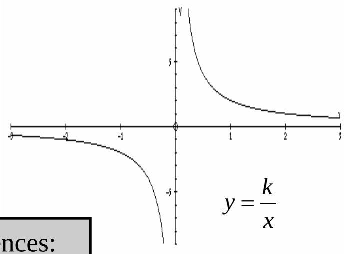


$y = kx^2$

References:
Chapter 2
Pages 60-62

$y = kx^3$





$y = \frac{k}{x}$

References:
Chapter 3
Pages 82-87

References:
Chapter 1
Pages 1-19

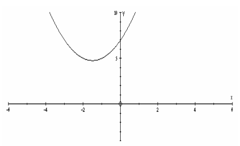
References:
Chapter 3
Pages 97-99

Exercise 3D
Q.1 (v), (vi)

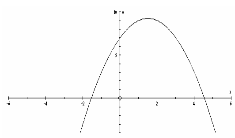
Finding the minimum value of a quadratic by completing the square.

(See page 4 for this topic).

The quadratic function $y = ax^2 + bx + c$ is a parabola of the following forms



$a > 0$

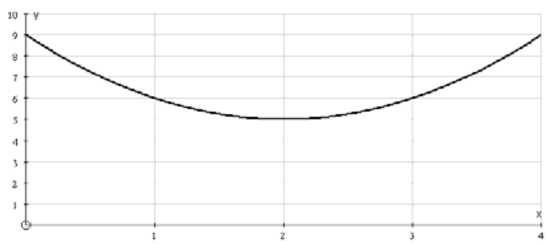


$a < 0$

E.g. Find the minimum value of $x^2 - 4x + 9$.

$(x - 2)^2 = x^2 - 4x + 4$
$\Rightarrow x^2 - 4x + 9 = x^2 - 4x + 4 + 5 = (x - 2)^2 + 5$

So minimum value is –5 when $x = 2$.



References:
Chapter 3
Pages 101-105

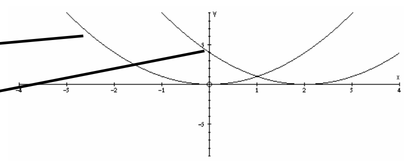
Exercise 3E
Q. 1(i), (vi)

Using transformations to sketch the curves of functions

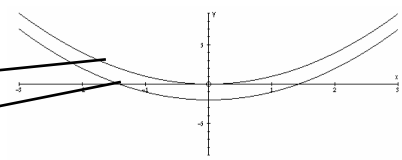
$y = \mathrm{f}(x - a)$ is the curve $y = \mathrm{f}(x)$ translated a units in the +ve x direction.

$y = \mathrm{f}(x) + a$ is the curve $y = \mathrm{f}(x)$ translated a units in the +ve y direction.

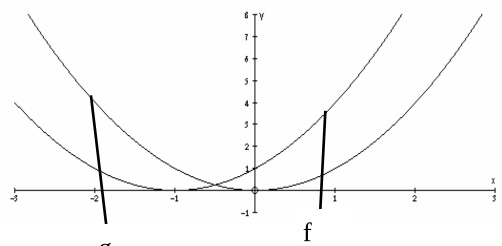
$y = x^2$
$y = (x - 2)^2$



$y = x^2$
$y = x^2 - 2$



E.g. $\mathrm{f}: x \to x^2$ $\quad$ $\mathrm{g}: x \to (x + 1)^2$
i.e. $\mathrm{g}(x) = \mathrm{f}(x + 1)$



g $\quad$ f

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