OCR Core 1 Module Revision Sheet

The C1 exam is 1 hour 30 minutes long. You are **not** allowed **any** calculator[1].

Before you go into the exam make sure you are fully aware of the contents of the formula booklet you receive. Also be sure not to panic; it is not uncommon to get stuck on a question (I've been there!). Just continue with what you can do and return at the end to the question(s) you have found hard. If you have time check all your work, especially the first question you attempted... always an area prone to error.

J.M.S.

Preliminaries

- Changing the subject of an equation. For example in $y = \sqrt{x+6}$, y is the subject of the equation. To change the subject to x we merely need to re–arrange to $x = y^2 - 6$. A harder example is make to make g the subject of $T = 2\pi\sqrt{\frac{l}{g}}$.

$$2\pi\sqrt{\frac{l}{g}} = T \qquad \Rightarrow \qquad \frac{l}{g} = \left(\frac{T}{2\pi}\right)^2 \qquad \Rightarrow \qquad g = l\left(\frac{2\pi}{T}\right)^2 = \frac{4\pi^2 l}{T^2}.$$

- To solve simultaneous equations, isolate x or y from one of the equations and substitute into the other. For example, solve

$$x + 2y = 1,$$
$$x^2 - 2y^2 = 31.$$

  From the first we find $x = 1 - 2y$ and putting into the second we find $(1-2y)^2 - 2y^2 = 31$, which simplifies to $y^2 - 2y - 15 = 0$. This gives $y = -3$ or $y = 5$. To calculate the x values we put the y solutions into either original equation. The solutions are $(-9, 5)$ and $(7, -3)$. [Give your solutions as coordinates to show which x and y values go together.]

- If you ever need to find where two lines or curves cross, then merely view it as a pair of simultaneous equations to be solved.

- You must also know how to handle algebraic fractions and how to write two algebraic fractional expressions as one fraction. The general rules are

$$\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd} \qquad \text{and} \qquad \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

  Therefore to write $x - \frac{x}{x+1}$ as a single fraction we do the following

$$x - \frac{x}{x+1} = \frac{x}{1} - \frac{x}{x+1} = \frac{x(x+1) - x}{x+1} = \frac{x^2}{x+1}.$$

- *Any* line or curve crosses the x-axis when $y = 0$. Similarly, any line or curve crosses the y-axis when $x = 0$. So to find where $y = x^2 + x - 12$ crosses the x-axis we solve $0 = x^2 + x - 12$ and find $(3, 0)$ and $(-4, 0)$. To find where it crosses the y-axis we put in $x = 0$ to discover $(0, -12)$.

- Always, always, always draw a sketch in any problem that is even vaguely geometric; the sooner you do, the sooner you'll get full marks.

[1] In my own school a few candidates were disqualified recently because they had it with them in the exam despite the fact they were not using it. Don't bring it in at all! Also, don't bring in any paper with formulae on that they give with certain pencil cases.

Coordinates, Points and Lines

- Mid point of (x_1, y_1), (x_2, y_2) is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$. Average the x-coordinates and average the y-coordinates.

- Distance from (x_1, y_1) to (x_2, y_2) is (by Pythagoras) $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$. Be careful about negatives! Remember $(2-(-3))^2 = (2+3)^2$.

- Gradient is defined to be
$$\frac{\text{difference in } y}{\text{difference in } x} = \frac{y_2-y_1}{x_2-x_1}.$$
If you need the gradient between two points you should visualise them first to see if you should be getting a positive or negative answer. This should also give you an idea of whether to expect a big (steep) of small (shallow) gradient.

- Two lines with gradients m_1 and m_2 are at right angles (perpendicular) if $m_1 \times m_2 = -1$. So if a line has gradient -3 then the line perpendicular to it has gradient $\frac{1}{3}$.

- Lines can be written in many forms, the most common being $y = mx + c$ and $ax + by = c$. Any form can be converted to any other. For example write $3x - 2y = 4$ in the form $y = mx + c$.
$$\begin{aligned} 3x - 2y &= 4 \\ 2y &= 3x - 4 \\ y &= \tfrac{3}{2}x - 2. \end{aligned}$$

- Given one point (x_1, y_1) and a gradient m the line is given by $y - y_1 = m(x - x_1)$.

Surds

- Know and understand the laws
$$\sqrt{a \times b} = \sqrt{a} \times \sqrt{b} \qquad \text{and} \qquad \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}.$$
In particular know how to deal with $\frac{\sqrt{44}}{2}$; it is *not* $\sqrt{22}$! It is $\frac{\sqrt{44}}{2} = \frac{\sqrt{44}}{\sqrt{4}} = \sqrt{\frac{44}{4}} = \sqrt{11}$. This comes up in solving quadratics by the formula; check that when you solve $x^2 + 4x - 2 = 0$ by the formula you obtain $x = -2 \pm \sqrt{6}$.

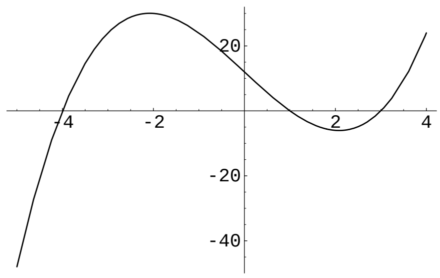
- You also need to be able to rationalise the denominator of certain types of surd expressions. For example to rationalise $\frac{9}{\sqrt{3}}$ is easy; just multiply by $\frac{\sqrt{3}}{\sqrt{3}}$ to obtain $\frac{9\sqrt{3}}{3} = 3\sqrt{3}$. In harder examples you must multiply the top and bottom of the fraction by the denominator with the sign 'flipped'. For example
$$\frac{2+2\sqrt{3}}{5-2\sqrt{3}} = \frac{2+2\sqrt{3}}{5-2\sqrt{3}} \times \boxed{\frac{5+2\sqrt{3}}{5+2\sqrt{3}}} = \frac{10+4\sqrt{3}+10\sqrt{3}+12}{25+10\sqrt{3}-10\sqrt{3}-12} = \frac{22+14\sqrt{3}}{13}.$$

Some Important Graphs

- Know the shape of the graph $y = x^n$ for $n = \{1, 2, 3, 4 \ldots\}$.
- If the power is even, then the graph will be $\bigcup$-shaped. They all pass through the points $(-1, 1)$, $(0, 0)$ and $(1, 1)$. The bigger the power, the faster it goes to infinity. Slightly more subtle is the point that in the range $-1 < x < 1$ then the higher the power, the *smaller* y-value (because $0.2 \times 0.2 \times 0.2 < 0.2 \times 0.2$). They are all 'even functions' with the y-axis as a line of symmetry.
- If the power is odd then they will (with the exception of $y = x^1 = x$, which is a straight line) be shaped like $y = x^3$. They all pass through $(-1, -1)$, $(0, 0)$ and $(1, 1)$. Similar arguments as for even powers exist here. They are all 'odd functions' with the origin being a point of rotational symmetry.
- The family of curves $y = ax^2 + bx + c$ are parabolas. If a is positive then you get a "happy" $\bigcup$-type curve. If a is negative then you get a "sad" $\bigcap$-type curve. They have a line of symmetry and a vertex (turning point) that you can discover by completing the square (see later).
- If you have a curve that is factorised then you can sketch it easily. For example

$$y = (x - 1)(x + 4)(x - 3)$$

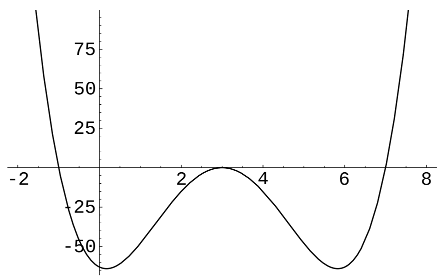
is a cubic curve that crosses the x-axis at $(1, 0)$, $(-4, 0)$ and $(3, 0)$. It crosses the y-axis when $x = 0$, which gives $(0, 12)$. If x is huge, y is huge and positive and if x is massively negative, then so is y. So



- If a factor is repeated, then it merely touches the x-axis at that point. So

$$y = (x - 3)^2(x + 1)(x - 7)$$

is a quartic curve that crosses the x-axis at $(-1, 0)$ and $(7, 0)$, but only touches at $(3, 0)$.



Quadratics

- Factorising quadratics. To check whether a given quadratic factorises calculate the discriminant $b^2 - 4ac$; if it is a perfect square $(4, 49, 81$ etc.) then it factorises.
- When the x^2 coefficient (the number in front of the x^2) is one this is easy. Just spot two numbers which multiply to the constant and add to the x coefficient. For example with $x^2 + 8x + 15$ we need to find two numbers which multiply to 15 and sum to 8; clearly 3 and 5. So $x^2 + 8x + 15 = (x + 3)(x + 5)$.
- If the x^2 coefficient is not one then more work is required. You need to multiply the x^2 coefficient by the constant term and then find 2 numbers which multiply to this and sum to the x coefficient. For example with $6x^2 + x - 12$ we calculate $6 \times -12 = -72$ so the two numbers are clearly 9 and -8. So

$$\begin{aligned} 6x^2 + x - 12 &= 6x^2 + 9x - 8x - 12 & &= 6x^2 - 8x + 9x - 12 \\ &= 3x(2x + 3) - 4(2x + 3) & &= 2x(3x - 4) + 3(3x - 4) \\ &= (3x - 4)(2x + 3) & &= (2x + 3)(3x - 4). \end{aligned}$$

  Notice that it does not matter which way round we write the $9x$ and $-8x$.
- For quadratics that cannot be factorised we need to use the formula. For $ax^2 + bx + c = 0$ the solution is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

- The $b^2 - 4ac$ part is called the *discriminant*. If it is positive then there are two *distinct* roots. If it is zero then there exists only one root and it is *repeated*. If it is negative then there are no roots. For example: find the values of k such that $x^2 + (k + 3)x + 4k = 0$ has only one root. We need the discriminant to be zero, so

$$\begin{aligned} b^2 - 4ac &= 0 \\ (k + 3)^2 - 16k &= 0 \\ k^2 - 10k + 9 &= 0 \\ k &= 9 \text{ or } k = 1. \end{aligned}$$

- Completing the square. All about halving the x coefficient into the bracket and then correcting the constant term. For example $x^2 - 6x + 10 = (x - 3)^2 - 9 + 10 = (x - 3)^2 + 1$. If the x^2 coefficient isn't one then need to factorise it out. For example

$$\begin{aligned} -2x^2 + 4x - 8 &= -2[x^2 - 2x] - 8 \\ &= -2[(x - 1)^2 - 1] - 8 \\ &= -2(x - 1)^2 - 6. \end{aligned}$$

  From this we can find the maximum or minimum of the quadratic. For $y = -2(x - 1)^2 - 6$ it is when $x = 1$ (to make the bracket 0) and therefore $y = -6$. In this case $(1, -6)$ is a maximum due to negative x^2 coefficient.

  We can also find the vertical line of symmetry by completing the square. For example

$$\begin{aligned} 3x^2 + 5x + 1 &= 3[x^2 + \tfrac{5}{3}x] + 1 \\ &= 3[(x + \tfrac{5}{6})^2 - \tfrac{25}{36}] + 1 \\ &= 3(x + \tfrac{5}{6})^2 - \tfrac{25}{12} + \tfrac{12}{12} \\ &= 3(x + \tfrac{5}{6})^2 - \tfrac{13}{12}. \end{aligned}$$

From this we see that the vertex is at $(-\frac{5}{6}, -\frac{13}{12})$ and consequently the line of symmetry is $x = -\frac{5}{6}$.

- You must be on the lookout for *quadratics in disguise*. You spot these when there are two powers on the variable and one is *twice* the other (or can be manipulated into such an equation[2]). Most students like to solve these by means of a substitution (although some students don't need to do this). For example to solve $x^4 + 2x^2 = 8$ work as follows:

$$\begin{aligned} x^4 + 2x^2 - 8 &= 0 \qquad \text{getting everything to one side} \\ u^2 + 2u - 8 &= 0 \qquad \text{substituting } u = x^2 \\ (u+4)(u-2) &= 0 \\ u = -4 \text{ or } u = 2 \qquad &\Rightarrow \qquad x^2 = -4 \text{ or } x^2 = 2 \end{aligned}$$

  But $x^2 = -4$ has no solutions, so $x = \pm\sqrt{2}$.

- For those who don't like substituting, just factorise and solve:

$$\begin{aligned} 2x^{\frac{2}{3}} &= 5x^{\frac{1}{3}} + 3 \\ 2x^{\frac{2}{3}} - 5x^{\frac{1}{3}} - 3 &= 0 \\ (2x^{\frac{1}{3}} + 1)(x^{\frac{1}{3}} - 3) &= 0 \end{aligned}$$

  So $x^{\frac{1}{3}} = -\frac{1}{2}$ or $x^{\frac{1}{3}} = 3$. Therefore cubing we find $x = -\frac{1}{8}$ or $x = 27$.

- Don't be one of the cretins who sees something like $x^4 + 4x^2 = 9$ and then *thinks* that they are square rooting to obtain $x^2 + 2x = 3$. Remember $\sqrt{x^4 + 4x^2} \neq x^2 + 2x$.

  Likewise $x + \sqrt{x} + 3 = 0$ does not square to $x^2 + x + 9 = 0$.

Differentiation

- We now turn to calculus[3]. Calculus $\equiv$ Differentiation + Integration. You will discover integration in C2.

- Differentiation allows us to calculate the 'gradient function' $\frac{dy}{dx}$. This tells us how the gradient on the original function y changes with x. $\frac{dy}{dx}$ *is* the gradient of a curve. So if you need to find where on a curve the gradient is 7, then you solve $\frac{dy}{dx} = 7$.

- Two alternative notations for derivatives are $\frac{dy}{dx} \equiv f'(x) \equiv y'$.

- The "rules" are;

$$\begin{aligned} y &= \text{constant} &&\Rightarrow \frac{dy}{dx} = 0, \\ y &= ax &&\Rightarrow \frac{dy}{dx} = a, \\ y &= ax^n &&\Rightarrow \frac{dy}{dx} = anx^{n-1}. \end{aligned}$$

  Notice that the first two are merely subsets of the third; the third is the daddy; the big cheese; the head honcho...

[2]For example $3x^3 = 5 + \frac{2}{x^3}$ can be manipulated into $3x^6 - 5x^3 - 2 = 0$ where one power is twice the other.
[3]Isaac Newton. Arguably the greatest physicist ever. [Gottfried Leibniz also came up with it a bit later.]

- For example:

$$y = 4x^4 - 3x^2 + 2x - 5 \quad \Rightarrow \quad \frac{dy}{dx} = 16x^3 - 6x + 2,$$
$$y = 4x^{\frac{5}{4}} + 3x^{\frac{4}{5}} \quad \Rightarrow \quad \frac{dy}{dx} = 5x^{\frac{1}{4}} + \frac{12}{5}x^{-\frac{1}{5}}.$$

- You must expand brackets or carry out divisions *before* you differentiate[4]. For example:

$$y = x^2(x-3)^2 \quad \Rightarrow \quad y = x^4 - 6x^3 + 9x^2 \quad \Rightarrow \quad \frac{dy}{dx} = 4x^3 - 18x^2 + 18x,$$
$$y = \frac{x^7 + x}{x^6} \quad \Rightarrow \quad y = x + x^{-5} \quad \Rightarrow \quad \frac{dy}{dx} = 1 - 5x^{-6} = 1 - \frac{5}{x^6}.$$

- We can use differentiation to find the equation of tangents and normals to curves at specified points. For example find the equation of the normal to the curve $y = x^3 + 2x^2 - 5x - 1$ when $x = 1$.

  Firstly we need the y-coordinate: $x = 1 \quad \Rightarrow \quad y = -3$.

  Secondly $\frac{dy}{dx} = 3x^2 + 4x - 5$. Into this we put $x = 1$, so $\frac{dy}{dx} = 2$. Therefore the *normal* has gradient $-\frac{1}{2}$. So

$$y - y_1 = m(x - x_1)$$
$$y + 3 = -\tfrac{1}{2}(x - 1)$$
$$x + 2y + 5 = 0.$$

- If asked to show that $y + 5x + 17 = 0$ is tangent to the curve $y = x^2 + 3x - 1$, there are two methods to do this:

  1. Find where $y = x^2 + 3x - 1$ and $y + 5x + 17 = 0$ cross. Solving simultaneously we gain the quadratic $-5x - 17 = x^2 + 3x - 1$ which simplifies and factorises to $(x + 4)(x + 4) = 0$. This gives a *repeated* root, so the line intersects the curve once and we can therefore conclude that the line *must* be a tangent. [I prefer this method.]
  2. The line $y + 5x + 17 = 0$ has gradient -5. Therefore we need to find where on $y = x^2 + 3x - 1$ the gradient is -5. Therefore we differentiate $y = x^2 + 3x - 1$ to get $\frac{dy}{dx} = 2x + 3$ and put $\frac{dy}{dx} = -5$. This gives $x = -4$. On the curve, when $x = -4$, $y = 3$, so to find the equation of the tangent

$$y - y_1 = m(x - x_1)$$
$$y - 3 = -5(x + 4)$$
$$x + 5y + 17 = 0, \text{ as required.}$$

- Another example: Given that the curve $y = ax^3 + 4x^2 + bx + 1$ passes through the point $(-1, 5)$ and (at that point) the tangent is parallel to the line $y + 4x + 1 = 0$. Find a and b.

  There's quite a bit going on here, so take it a bit at a time. Since the curve passes through $(-1, 5)$, then $x = -1$ and $y = 5$ must be a solution to the curve's equation, so $5 = -a + 4 - b + 1$ which simplifies to $a + b = 0$. The line given has gradient -4, so we need to set $\frac{dy}{dx} = -4$ when $x = -1$. So $\frac{dy}{dx} = 3ax^2 + 8x + b$ which gives $-4 = 3a - 8 + b$. These solve to $a = 2$, $b = -2$.

[4] $y = \frac{x^3+x}{x^2}$ does *not* differentiate to $\frac{dy}{dx} = \frac{3x^2+1}{2x}$ and $y = (x^3 + 2)(2x^2 + 5x)$ does *not* differentiate to $\frac{dy}{dx} = 3x^2(4x + 5)$!

Inequalities

- Treat linear inequalities like equations except when multiplying or dividing by a negative number when you reverse the sign. For example

$$\begin{aligned} 2x+4 &< 3x+2 \\ -x &< -2 \\ x &> 2. \end{aligned}$$

- To solve quadratic inequalities:

  1. Get all terms over one side so that quadratic > 0 or quadratic < 0 in such a way that the x^2 term is always positive. This will ensure a 'happy' curve.
  2. Solve quadratic $= 0$ to find where it crosses x-axis.
  3. Sketch the graph and read off solution. If it is quadratic > 0 then it is the region(s) above the x-axis, and if quadratic < 0 then it is region below the x-axis.
  4. If one region then express as one triple inequality (e.g. $-2 < x < 5$) and if two regions then two *separate* inequalities (e.g. $x > 5$ or $x < -2$).

- For example solve the inequality $-7x \geqslant 4 - 2x^2$. Firstly get the $2x^2$ on the other side to make it positive to get $2x^2 - 7x - 4 \geqslant 0$. Then solve the equality $2x^2 - 7x - 4 = 0 = (2x+1)(x-4)$, so $x = -\frac{1}{2}$ or $x = 4$. So we have a happy quadratic that crosses the x-axis at $-\frac{1}{2}$ and 4. The inequality is asking for where the curve is bigger than (or equal to) zero, and this is to the right of $x = 4$ and the left of $x = -\frac{1}{2}$. Therefore the solution is $x \leqslant -\frac{1}{2}$ or $x \geqslant 4$.

- Don't fall into the trap of seeing $x^2 < 16$ and saying $x < \pm 4$! Be disciplined and get zero on one side; $x^2 - 16 < 0$ so $(x-4)(x+4) < 0$ so we have happy curve that crosses at 4 and -4. Where is the curve less than zero? Between -4 and 4 so solution is $-4 < x < 4$.

Index Notation

- $(ab)^m = a^m \times b^m$. For example $6^5 = 2^5 \times 3^5$.

- When multiplying a number raised to different powers the powers *add*. Therefore $a^m \times a^n = a^{m+n}$. You can think of this as follows $2^2 \times 2^4 = (2 \times 2) \times (2 \times 2 \times 2 \times 2) = 2^6$.

- Know that $a^{-m} = \frac{1}{a^m}$. Remember this by the standard result that $2^{-1} = \frac{1}{2}$. "When moving something from the bottom line of a fraction to the top (or vice versa), the sign changes."

- From the above two results we can obtain the result $\frac{a^m}{a^n} = a^{m-n}$. This is derived thus; $\frac{a^m}{a^n} = a^m \times a^{-n} = a^{m-n}$ as required.

- We can also derive the important result $a^0 = 1$ for any $a \neq 0$. Derived by considering something like this; $a^0 = a^{1-1} = \frac{a^1}{a^1} = \frac{a}{a} = 1$.

- Know that $(a^m)^n = a^{mn}$. Think about it like this; $(a^3)^4 = a^3 \times a^3 \times a^3 \times a^3 = a^{12}$.

- The n^{th} root of a number can be expressed as a power thus; $\sqrt[n]{a} = a^{\frac{1}{n}}$.

- A few examples:

  1. Write 8 as a power of 4; well $8 = 2^3 = \left(4^{\frac{1}{2}}\right)^3 = 4^{\frac{3}{2}}$.

2. Simplify $\sqrt[4]{16^3} = \left((2^4)^3\right)^{\frac{1}{4}} = (2^{12})^{\frac{1}{4}} = 2^3 = 8.$
3. Simplify $\dfrac{12x^8y^{\frac{3}{2}}}{6x^6y^{\frac{5}{2}}} = 2x^{8-6}y^{\frac{3}{2}-\frac{5}{2}} = 2x^2y^{-1}.$
4. Simplify $\sqrt{x^6y^4} \times \sqrt[3]{x^3y^{-6}} = (x^6y^4)^{\frac{1}{2}} \times (x^3y^{-6})^{\frac{1}{3}} = x^3y^2x^1y^{-2} = x^4.$

- OCR is particularly 'hot' on linking differentiation, indices and surds. For example, find the equation of the normal to $y = 6x^{\frac{5}{2}} - 4x^{\frac{3}{2}}$ when $x = 2$ in the form $ax + by = c$.

  When $x = 2$, $y = 6 \times 2^{\frac{5}{2}} - 4 \times 2^{\frac{3}{2}} = 24\sqrt{2} - 8\sqrt{2} = 16\sqrt{2}$.

  Differentiating we find $\frac{dy}{dx} = 15x^{\frac{3}{2}} - 6x^{\frac{1}{2}}$.

  So, when $x = 2$, $\frac{dy}{dx} = 15 \times 2\sqrt{2} - 6\sqrt{2} = 24\sqrt{2}$. Therefore the gradient of the normal is $-\frac{1}{24\sqrt{2}} = -\frac{\sqrt{2}}{48}$. Therefore the equation of the normal is

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 16\sqrt{2} &= -\tfrac{\sqrt{2}}{48}(x - 2) \\ \sqrt{2}x + 48y &= 770\sqrt{2}. \end{aligned}$$

Graphs of nth Power Functions

- Be able to sketch $y = \frac{1}{x^n}$ for $n = \{1, 2, 3 \ldots\}$.

  If n is even then graphs look like $y = \frac{1}{x^2}$ with the y-axis being a line of symmetry.

  If n is odd then the graph looks like $y = \frac{1}{x}$.

- Shock, horror! The differentiation rule for $y = ax^n$ still works with fractional and negative powers:

$$\frac{dy}{dx} = anx^{n-1}.$$

Polynomials

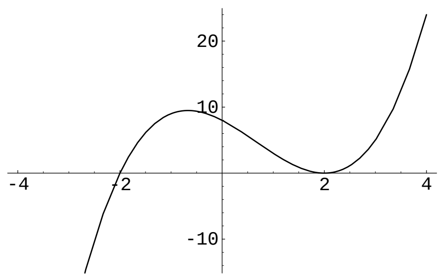
- Must be able to visualise a polynomial curve quickly. This is determined by two things;

  1. Whether the largest power of x is odd or even. x^{1000} has very different shape from x^{1001}.
  2. Whether the coefficient (number in front of) the largest power of x is positive or negative.

- A polynomial of order n has *at most* $n - 1$ stationary points. For example a cubic curve has up to two stationary points, but it might have none or one.

- If $y = x^3 + 2x^2 + 3x + 4$ then the curve crosses the y-axis at 4 (just the constant at the end). To find the x-axis intercept(s) need to solve $0 = x^3 + 2x^2 + 3x + 4$.

- To sketch a polynomial it is best to factorise it. For example given $y = x^3 - 2x^2 - 4x + 8$ we can write $y = (x + 2)(x - 2)^2$. So to sketch we know that it is a cubic with positive x^3 coefficient. y-axis intercept is at 8. It crosses the x-axis at $x = -2$, but only touches it at $x = 2$ due to the repeated root.



Transforming Graphs

- Given $y = f(x)$ then:

Replacement	Graph Shape
None	Normal Graph
x by $x - a$	Graph translated by a vector $\binom{a}{0}$
x by $-x$	Graph reflected in the y-axis
x by $\frac{x}{2}$	Graph stretched by a factor of 2 parallel to the x-axis $\leftarrow\!\rightarrow$
x by $2x$	Graph stretched by a factor of $\frac{1}{2}$ parallel to the x-axis $\rightarrow\!\leftarrow$
y by $y - b$	Graph translated by a vector $\binom{0}{b}$
y by $-y$	Graph reflected in the x-axis
y by $\frac{y}{2}$	Graph stretched by a factor of 2 parallel to the y-axis $\updownarrow$
y by $2y$	Graph stretched by a factor of $\frac{1}{2}$ parallel to the y-axis $\downarrow\uparrow$

- For example: Find the equation of $y^2 + 2x^2 = 2x + 1$ after the translation $\binom{1}{-1}$. So we replace x by $x - 1$ and y by $y + 1$. Therefore

$$(y + 1)^2 + 2(x - 1)^2 = 2(x - 1) + 1 \quad \Rightarrow \quad y^2 + 2y + 2x^2 = 6x - 4.$$

- For example: Explain the transformation that maps

$$y = \frac{2}{\sqrt{1 + x}} \text{ onto } y = \frac{1}{\sqrt{3 + x}}.$$

Rewriting the second equation as $2y = \frac{2}{\sqrt{1+(x+2)}}$ we can see y has been replaced by $2y$ and x has been replaced by $x + 2$. Therefore the curve has been translated by $\binom{-2}{0}$ and also stretched by a factor of $\frac{1}{2}$ parallel to the y-axis; i.e. every y-value has halved.

Investigating Shapes of Graphs

- Stationary points are where the gradient of curve is zero. They are either maxima, minima or points of inflection. To find the turning points of a curve we must find $\frac{dy}{dx}$ and then set $\frac{dy}{dx} = 0$ and solve for x.
- To determine the nature of a turning point we can consider the sign of the gradient either side of the turning point. Present this in a table. In the example of $y = x^2 + 2x + 3$ we find $\frac{dy}{dx} = 2x + 2$ so we solve $0 = 2x + 2$ to give the turning point when $x = -1$:

x	$x < -1$	-1	$x > -1$
dy/dx	negative	0	positive
		minimum	

- We can also use the second derivative to determine the nature of a turning point. This is found by differentiating the function twice;

$$y = 2x^3 + 3x^2 - 2x + 4 \quad \Rightarrow \quad \frac{dy}{dx} = 6x^2 + 6x - 2 \quad \Rightarrow \quad \frac{d^2y}{dx^2} = 12x + 6.$$

  You then evaluate the second derivative with the x value at the turning point and look at its sign. If it is positive it is a minimum, if it is negative it is a maximum. If it is zero then it is *probably* a point of inflection, but you need to do the above analysis either side of the turning point.

- For example, determine the nature of the stationary points on $y = 4x^3 - 21x^2 + 18x + 3$. So

$$y = 4x^3 - 21x^2 + 18x + 3 \quad \Rightarrow \quad \frac{dy}{dx} = 12x^2 - 42x + 18 = 0 \quad \Rightarrow \quad x = 3 \text{ or } x = \tfrac{1}{2}.$$

  Therefore the stationary points are $(3, -24)$ and $(\frac{1}{2}, \frac{29}{4})$. We therefore need the second derivative and evaluate it at 3 and $\frac{1}{2}$.

$$\begin{aligned} \frac{d^2y}{dx^2} &= 24x - 42 \\ \left.\frac{d^2y}{dx^2}\right|_{x=3} &= 24 \times 3 - 42 = 30 > 0 \text{ therefore } (3, -24) \text{ is a minimum.} \\ \left.\frac{d^2y}{dx^2}\right|_{x=\frac{1}{2}} &= 24 \times \tfrac{1}{2} - 42 = -30 < 0 \text{ therefore } (\tfrac{1}{2}, \tfrac{29}{4}) \text{ is a maximum.} \end{aligned}$$

- A function will be *increasing* when $\frac{dy}{dx}$ is positive and *decreasing* when $\frac{dy}{dx}$ is negative[5]. This is obvious if you consider a sketch.

  For example, find the set of values of x for which $y = -2x^3 + 3x^2 + 12x + 1$ is decreasing. First differentiate and recognise we want $\frac{dy}{dx}$ to be negative.

$$\begin{aligned} y &= -2x^3 + 3x^2 + 12x + 1 \\ \frac{dy}{dx} &= -6x^2 + 6x + 12 \\ 0 &> -6x^2 + 6x + 12 \qquad \text{(note that we place } \frac{dy}{dx} < 0\text{)} \\ x^2 - x - 2 &> 0 \\ (x - 2)(x + 1) &> 0. \end{aligned}$$

  This quadratic inequality solves to $x < -1$ or $x > 2$ which are the values of x for which the curve is decreasing.

Applications of Differentiation

- Differentiation can be used to work out "rates of change". In GCSE Physics you learnt that acceleration is the rate of change of velocity; you will then have learnt that $a = \frac{v-u}{t}$. However, at a higher level rates of change are calculated by differentiating with respect to time. So we now view acceleration as $a = \frac{dv}{dt}$.

[5] Technically a curve is 'increasing' when $\frac{dy}{dx} \geqslant 0$ and 'strictly increasing' when $\frac{dy}{dx} > 0$; likewise for decreasing, but don't get too het up about it; just get the sign the right way round.

- As we have already seen, differentiation allows us to calculate the stationary point of a curve $y = f(x)$. We do this by calculating $\frac{dy}{dx}$ and setting it equal to zero. We can use this to help is in practical problems where we might want to maximise a quantity (e.g. profit) or to minimise a quantity (e.g. cost).

- Worked example: An open topped cuboidal box is to be made from a rectangular piece of metal 10cm by 16cm. Squares are to be cut from each corner and then the four flaps are to be folded up. Find the maximum volume attainable for the box and prove that it is a maximum.

  1. Let x be the side length of the squares cut away, where $0 < x < 5$.
  2. The volume of the box would therefore be
  $$V = x(16 - 2x)(10 - 2x) = 160x - 52x^2 + 4x^3.$$
  We imagine a graph of V against x and hope that there is a stationary point in the range $0 < x < 5$.
  3. Differentiate V with respect to x and set equal to zero to find the stationary point;
  $$\frac{dV}{dx} = 160 - 104x + 12x^2 = 0 \quad \Rightarrow \quad x = 2 \text{ or } x = \tfrac{20}{3}.$$
  4. Notice that x can't be $\frac{20}{3}$ because it is outside the range $0 < x < 5$. So we only consider $x = 2$.
  5. If $x = 2$ then $V = 2 \times 12 \times 6 = 144$.
  6. To demonstrate that $x = 2$ is a maximum we need the second derivative and evaluate it at $x = 2$.
  $$\frac{d^2V}{dx^2} = -104 + 24x = -104 + 24 \times 2 = -56 < 0 \text{ therefore a maximum.}$$

Circles

- Circles with centre $(0, 0)$ and radius r are expressed by $x^2 + y^2 = r^2$.
- Circles with centre (a, b) and radius r are expressed by $(x - a)^2 + (y - b)^2 = r^2$.
- By 'completing the square' you can convert circles of the form $x^2 + y^2 + \alpha x + \beta y + \gamma = 0$ into the form $(x - a)^2 + (y - b)^2 = r^2$. For example
$$\begin{aligned} x^2 + y^2 + 6x - 4y + 9 &= 0 \\ x^2 + 6x + y^2 - 4y + 9 &= 0 \\ (x + 3)^2 - 9 + (y - 2)^2 - 4 + 9 &= 0 \\ (x + 3)^2 + (y - 2)^2 &= 4. \end{aligned}$$
- When finding the intersection of a line and a circle it is easiest to substitute in the value of y from the line into the circle and solve the resulting quadratic. For example; find where the line $y = 2x - 1$ intersects to circle $(x - 3)^2 + (y - 2)^2 = 25$.
$$\begin{aligned} (x - 3)^2 + (y - 2)^2 &= 25 \\ (x - 3)^2 + (2x - 3)^2 &= 25 \\ x^2 - 6x + 9 + 4x^2 - 12x + 9 - 25 &= 0 \\ 5x^2 - 18x - 7 &= 0. \end{aligned}$$

Solve the quadratic (in this case by the formula) and then find the y values by substituting both x values into $y = 2x - 1$ (the original line). There will usually be 2 points of intersection (where the discriminant of the resulting quadratic will be positive) except if the line doesn't intersect the circle at all (discriminant negative) or if the line is a tangent to the circle (discriminant equals zero).

- The gradient of the tangent to a circle is perpendicular to the radius of the circle at that point. For example: The point $B(1, 7)$ lies on the circle $(x-3)^2 + (y-4)^2 = 13$. Find the equation of the tangent to the circle at B. The centre of the circle is $(3, 4)$, so the gradient of the radius at B is $-\frac{3}{2}$. Therefore the gradient of the tangent is $\frac{2}{3}$ and will pass through B, so the tangent will be:

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 7 &= \tfrac{2}{3}(x - 1) \\ 0 &= 2x - 3y + 19. \end{aligned}$$

- You must always remember the GCSE theorem that if a triangle is constructed within a circle with one side being a diameter of the circle, then it is a right angled triangle. To demonstrate this one is often required to show that the gradients of certain line segments are perpendicular (i.e. $m_1 \times m_2 = -1$).

  For example; $A(2, 1)$, $B(4, 13)$ and $C(-3, 8)$. The line segment AB is the diameter of a circle and C is a point on its circumference. Find the area of triangle ABC. We know angle $A\hat{C}B$ must be a right angle, so

$$\begin{aligned} \text{Area } ABC &= \tfrac{1}{2} \times \text{base} \times \text{height} \\ &= \tfrac{1}{2} \times (\text{length } AC) \times (\text{length } CB) \\ &= \tfrac{1}{2} \times \sqrt{5^2 + 7^2} \times \sqrt{7^2 + 5^2} \\ &= 37 \text{ units}^2. \end{aligned}$$

- Also, given two points that lie on a circle's circumference, the centre of the circle lies on the perpendicular bisector of the two points.