AQA

ASSESSMENT and QUALIFICATIONS ALLIANCE

Not Formulae and Statistical Tables for GCE Mathematics and GCE Statistics

The Not-Formula Book

Core 1

Everything you need to remember that the formula book won't tell you

The Not-Formula Book for C1

Everything you need to know for Core 1 that *won't* be in the formula book

Examination Board: AQA

Brief

This document is intended as an aid for revision. Although it includes some examples and explanation, it is primarily not for learning content, but for becoming familiar with the requirements of the course as regards formulae and results. It cannot replace the use of a text book, and nothing produces competence and familiarity with mathematical techniques like practice. This document was produced as an addition to classroom teaching and textbook questions, to provide a summary of key points and, in particular, any formulae or results you are expected to know and use in this module.

Contents

Chapter 1 – Advancing from GCSE maths: algebra review

An equation of the form $ax + b = 0$, where a and b are constants, is said to be a **linear equation** with **variable** x.

The method of solving linear equations is to collect all the terms involving x on one side of the equation and everything else on the other side.

Sometimes linear equations involve fractions. To convert this to a linear equation with integer coefficients, multiply through by the lowest common multiple of the denominators.

Eg:

$$\frac{x}{4} + \frac{2}{3} = 2x$$

$(\times 12)$ $\qquad$ $(\times 12)$

$$3x + 8 = 24x$$

$(-3x)$ $\qquad$ $(-3x)$

$$8 = 21x$$

$(\div 21)$ $\qquad$ $(\div 21)$

$$\mathbf{\frac{8}{33}} = \boldsymbol{x}$$

One method for solving **simultaneous linear equations** is the **elimination method**. Coefficients of one of the variables are made equal and then the equations are effectively added or subtracted from one another to eliminate this variable.

Eg:

(1) $\quad 5x - 2y = 6$

(2) $\quad 2x + 4y = 9$

(1)×2 $\quad 10x - 4y = 12$

(2)+(1’) $\quad 12x = 21 \quad \Rightarrow \quad x = \frac{21}{12} = \frac{7}{4} = 1.75$

Sub into (2) $\quad 2\left(\frac{7}{4}\right) + 4y = 9 \quad \Rightarrow \quad 4y = 9 - \frac{7}{2} = \frac{11}{2} \quad \Rightarrow \quad y = \frac{11}{8} = 1.375$

More generally, a wider range of **simultaneous equations** can be solved more easily using **substitution**.

Eg:

(1)	$y - 1 = 2x$
(2)	$y^2 - 3x^2 = 6$
Rearrange (1)	$y = 2x - 1$
Sub into (2)	$(2x - 1)^2 - 3x^2 = 6$
Rearrange (2')	$x^2 - 4x - 5 = 0$
Solve (2')	$(x + 1)(x - 5) = 0 \quad \Rightarrow \quad x = -1 \quad or \quad x = 5$
Sub into (1')	$y = 2(-1) - 1 = -3 \quad or \quad y = 2(5) - 1 = 9$
Write out solutions	$x = -1 \;\; y = -3 \quad or \quad x = 5 \;\; y = 9$

Note: Take care to pair up the correct values of x and y – they represent the coordinates of the crossing points for the two equations.

When solving an **inequality**, if you **multiply or divide by a negative**, you must **reverse the sign**.

Note: The reason for this is clear when you consider $3 < 5$. By subtracting 8 from each side we get $-5 < -3$, which is of course perfectly correct. But notice that this is equivalent to $-3 > -5$ which not only has different signs to the original statement but also has the inequality sign reversed.

A **function** is any rule that produces an **output** value for a given **input**. The function f, when applied to the input value x, is written as $f(x)$.

Eg:
The function which subtracts 2 and then squares the result would be written as: $f(x) = (x - 2)^2$

Note: Numbers may be substituted into functions simply by replacing x:

$$f(5) = (5 - 2)^2 = 9$$

Chapter 2 – Surds

The counting numbers, or **natural numbers** are a subset of all whole numbers, or **integers**, which are themselves a subset of **rational numbers** (numbers which can be written as a fraction). The rationals are a subset of **real numbers**, which also include numbers called **irrational numbers** which cannot be written as fractions (and therefore also not as either terminating or recurring decimals).

Examples of irrational numbers include π, e, $\sqrt{2}$, $\sqrt[3]{5}$. Irrational numbers involving roots, such as $\sqrt{2}$ or $6-\sqrt[4]{3}$, are known as **surds.**

Multiplication or **division** within a root can be brought outside the root. In general:

$$\sqrt{ab}=\sqrt{a}\times\sqrt{b} \quad and \quad \sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}$$

Eg:
Simplify $\sqrt{108}+2\sqrt{27}$

$$\sqrt{108}=\sqrt{36}\times\sqrt{3}=6\sqrt{3} \quad and \quad \sqrt{27}=\sqrt{9}\times\sqrt{3}=3\sqrt{3}$$

$$\sqrt{108}+2\sqrt{27}=6\sqrt{3}+3\sqrt{3}=9\sqrt{3}$$

Note: The equivalent for addition and subtraction is **not** valid. Eg $\sqrt{9+16}\neq\sqrt{9}+\sqrt{16}$

If a fraction is written with a **surd in the denominator**, it is often useful to be able to rewrite it so that the denominator is a rational number. This is known as **rationalising the denominator**, and is accomplished by multiplying top and bottom of the fraction by a number which will have the desired effect.

Eg:

$$\frac{4}{\sqrt{3}}=\frac{4\sqrt{3}}{3}$$

$$\frac{\sqrt{3}}{1+\sqrt{2}}=\frac{\sqrt{3}\left(1-\sqrt{2}\right)}{-1}=\sqrt{3}\left(\sqrt{2}-1\right)$$

$$\frac{5}{4-3\sqrt{5}}=\frac{5\left(4+3\sqrt{5}\right)}{16-45}=\frac{-5\left(4+3\sqrt{5}\right)}{29}$$

Chapter 3 – Coordinate geometry of straight lines

The **distance between two points** can be calculated by constructing a right-angled triangle between the coordinates and applying Pythagoras' Theorem:

$$\textit{Distance between } (x_1, y_1) \textit{ and } (x_2, y_2): \quad \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The **midpoint** of the line between the points (x_1, y_1) and (x_2, y_2) is given by:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Note: This is simply the average of the x coordinates and the average of the y coordinates.

The **gradient** of the line between the points (x_1, y_1) and (x_2, y_2) is given by:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Note: This is often described as 'y step over x step', or 'rise over run'.

Lines with gradients m_1 and m_2 are **parallel** if $m_1 = m_2$. They are **perpendicular** if $m_1 m_2 = -1$.

Note: The concepts behind the above results are more important (and more easily memorable) than the formulae used to describe them mathematically. The result below is the only one where memorising the formula gives an advantage to quickly solving problems.

Given the **gradient**, m, and a **single point**, (x_1, y_1), the **equation** of a line can be generated using:

$$y - y_1 = m(x - x_1)$$

Given **two points**, the **equation** of a line can be generated using:

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

Note: The above result is simply a combination of the definition of gradient and the gradient & point formula above. As such it is not necessary to memorise this form if you are already confident with finding the gradient between two points and can recall $y - y_1 = m(x - x_1)$.

The **crossing point** of two lines directly corresponds with the **values of x and y** which satisfy both equations. This can be found either by reading off the graph, or – more precisely – by solving the equations simultaneously.

Chapter 4 – Quadratics and their graphs

A **quadratic function** can be written in the form $y = ax^2 + bx + c$, and the shape of the graph produced is known as a **parabola**. It is symmetrical, and resembles either a U shape (for $a > 0$) or an inverted U (for $a < 0$). If it crosses the x-axis, the equation $ax^2 + bx + c = 0$ has two distinct solutions. If it only touches at one point this will be at its maximum or minimum and the equation will have one (repeated) solution. If the graph doesn't touch the x-axis at all, the equation will have no real solutions.

A quadratic can sometimes be **factorised**; that is, written in the form $(x - a)(x - b) = 0$, and in this case the solutions are $x = a$ and $x = b$, giving crossing points of $(a, 0)$ and $(b, 0)$.

Note: These values of a and b could be positive or negative, or even fractional. For more difficult examples, an alternative method of solving might be used (completing the square or using the quadratic formula).

Note: Some simple cases can be solved more easily. For instance, if $c = 0$, one factor will always be x, giving one of the solutions as $x = 0$, and if $b = 0$, the equation can be directly rearranged to make x the subject, giving the solutions as $x = \pm\sqrt{-\frac{c}{a}}$.

Any quadratic can be rearranged into the **completed square form**:

$$A(x + B)^2 + C$$

Note: This form allows you not only to solve a quadratic equation through a few quick steps of rearrangement, but also provides a useful method for identifying a maximum or minimum. Note that the stationary point will occur when $x = -B$, and will take the value $y = C$; that is, $(-B, C)$.

The transformation represented by a change from $y = f(x)$ to $y = f(x - a) + b$ is a **translation** of $\begin{bmatrix} a \\ b \end{bmatrix}$. This means the graph has effectively moved a to the right and b up.

The solutions of any quadratic (if they exist) can be found using the **quadratic formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Note: This formula is a direct result of solving the general equation $ax^2 + bx + c = 0$ by completing the square.

The **discriminant** of a quadratic enables us to determine whether the equation will have 0, 1 or 2 real roots. For $b^2 - 4ac < 0$, since we can't square root a negative, there are no real roots.
For $b^2 - 4ac = 0$, there is one (repeated) root, and for $b^2 - 4ac > 0$ there are two roots.

Chapter 5 – Polynomials

A **polynomial** of degree n is an expression of the form:

$$ax^n + bx^{n-1} + cx^{n-2} + \cdots + px^2 + qx + r$$

Where a, b, c etc are constants, $a \neq 0$, and n is a positive integer (the degree is the highest power).

Note: Two polynomials can be added to make a new polynomial. To combine, simply collect like terms.

Note: The number in front of a particular power of x is known as its coefficient.

Eg: Find the coefficient of x^3 in $(2x^4 + 8x^3 - x) - (5x^3 - 3x^2 + 4)$

$$(2x^4 + 8x^3 - x) - (5x^3 - 3x^2 + 4) = 2x^4 + 8x^3 - x - 5x^3 + 3x^2 + 4 = 2x^4 + 3x^3 + 3x^2 - x + 4$$

The coefficient of x^3 is 3.

Note: Recall that when multiplying brackets it is necessary to multiply every term in the first bracket by every term in the second.

Eg: Multiply out: $(x^2 - 3x + 2)(2x + 5)$

$(x^2 - 3x + 2)(2x + 5) = 2x^3 + 5x^2 - 6x^2 - 15x + 4x + 10 = 2x^3 - x^2 - 11x + 10$

Chapter 6 – Factors, remainders and cubic graphs

The **factor theorem**:

$$(x-a)\ is\ a\ factor\ of\ polynomial\ P(x) \quad \Leftrightarrow \quad P(a)=0$$

Note: This means that if we know a factor of a polynomial, we can find the corresponding root (or solution), and vice versa.

To **factorise** a **cubic**, first find a root by trial and error, then use the factor theorem to generate a linear factor. Finally, use **inspection** to find the remaining (quadratic) factor, and factorise this, where possible.

Eg: Factorise fully $2x^3+3x^2-16x+15$

$$P(x)=2x^3-x^2-16x+15$$

$P(0)=15$ 0 not a root.
$P(1)=0$ 1 is a root.

$$P(1)=0 \quad \Rightarrow \quad (x-1)\ \ is\ a\ factor$$

$$2x^3-x^2-16x+15=(x-1)(\ldots)$$

By inspection, we need $2x^2$ in the second bracket to give the $2x^3$ term in the cubic.
We also need -15 in the second bracket to give the 15 term in the cubic.

$$P(x)=(x-1)(2x^2+\cdots-15)$$

To get the x^2 term we need to examine the x terms from each, as well as the combination of the x^2 and constant terms. Since we already have $-1\times 2x^2$ giving $-2x^2$, we need x^2 to give the result $-x^2$. This must be achieved by x multiplied by the second bracket's x term which therefore must be simply x.

$$P(x)=(x-1)(2x^2+x-15)$$

Finally, factorise the quadratic if possible:

$$P(x)=(x-1)(2x-5)(x+3)$$

To sketch the graph of a cubic:
Step 1: Determine the overall direction by looking at the sign of the coefficient of x^3.
Step 2: Find the y-intercept by looking at the constant term (evaluate y at $x=0$).
Step 3: Find any points where the graph crosses the x-axis by solving $y=0$. Repeated roots represent points where the graph touches the axis but does not cross it.
Step 4: Plot the points and sketch the graph. Check some additional points if necessary.

When a polynomial is divided by a linear expression, the following is true: $$P(x) = (x-a)Q(x) + R$$ Where: $P(x)$ is a polynomial of degree n $(x-a)$ is the divisor of degree 1 (linear) $Q(x)$ is the quotient of degree $n-1$ R is the remainder of degree 0 (constant)

Note: By considering the case when $x = a$, we produce the remainder theorem.

The **remainder theorem**: When a polynomial $P(x)$ is divided by $(x-a)$, the remainder is $R = P(a)$ and vice versa.

Note: The factor theorem is a corollary of the remainder theorem, for the case where $R = 0$.

Eg: When the polynomial $x^3 + ax^2 - x + 12$ is divided by $(x-3)$ it has a remainder of 9. Find a.

$$P(3) = 9 \quad \Rightarrow \quad 3^3 + a(3^2) - 3 + 12 = 9 \quad \Rightarrow \quad 36 + 9a = 9 \quad \Rightarrow \quad a = -3$$

Note: Often it will be necessary to combine knowledge of the factor theorem and the remainder theorem and use simultaneous equations to solve a problem.

Eg: The polynomial $3x^4 - 5x^2 + px - q$ has $(2x+3)$ as a factor, but gives a remainder 6 when divided by $(x-4)$. Find p and q.

By the factor theorem:

$$3\left(-\frac{3}{2}\right)^4 - 5\left(-\frac{3}{2}\right)^2 + p\left(-\frac{3}{2}\right) - q = 0$$

$$\Rightarrow \quad \frac{243}{16} - \frac{45}{4} - \frac{3p}{2} - q = 0 \quad \Rightarrow \quad 63 - 24p - 16q = 0$$

By the remainder theorem:

$$3(4)^4 - 5(4)^2 + p(4) - q = 6 \quad \Rightarrow \quad 682 + 4p - q = 0$$

Solving simultaneously:

(1) $\quad 63 - 24p - 16q = 0$

(2) $\quad 682 + 4p - q = 0$

(2)$\times 6$ $\quad 4092 + 24p - 6q = 0$

(1)+(2') $\quad 4155 - 22q = 0 \quad \Rightarrow \quad q = \frac{4155}{22}$

Sub into (1) $\quad 63 - 24p - 16\left(\frac{4155}{22}\right) = 0$

$$\Rightarrow \quad p = -\frac{10849}{88}$$

Chapter 7 – Simultaneous equations and quadratic inequalities

Simultaneous equations can sometimes be solved by **elimination** of one or other of the variables.

Simultaneous equations can be used to find the intersection of two lines, a line and a curve or even two curves. Where elimination is unsuitable, **substitution** is another valid method.

Note: See chapter 1 for detailed examples of these methods.

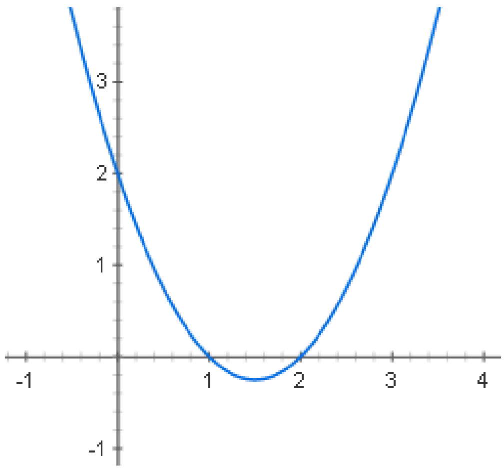
When dealing with the **intersection of a line and a parabola**, the resulting equation will be a quadratic. Two solutions correspond to two crossing points, no solutions to no crossing points, and exactly one solution (a repeated root) means the line is a tangent to the curve at that point.

To solve a **quadratic inequality**, use the equivalent equation to determine 'critical values', then consider the graph to determine the range of solutions for the inequality.

Eg: Solve the inequality $x^2 + 2x - 2 > 5x - 4$.

$$\textit{This is equivalent to:} \quad x^2 - 3x + 2 > 0$$

$$\textit{Critical values at:}$$
$$x^2 - 3x + 2 = 0 \quad \Rightarrow \quad (x-1)(x-2) = 0 \quad \Rightarrow \quad x = 1 \ \textit{and} \ x = 2$$



$$\textit{The graph is above the axis before } x = 1 \textit{ and after } x = 2$$

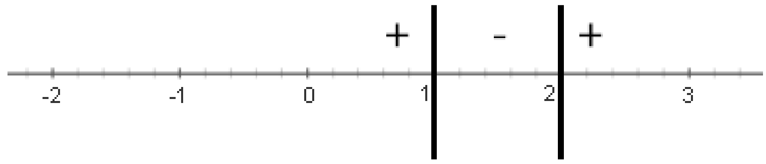
$$\textit{Therefore } x^2 - 3x + 2 > 0 \quad \textit{for} \quad x < 1 \quad \textit{or} \quad x > 2$$

A quadratic inequality can also be solved using a **sign diagram** which indicates whether the graph is positive or negative in each region.

Eg: Find all values of x which satisfy the inequality $(x-1)(x-2)>0$.

$$Identify\ critical\ values:\quad x=1\quad or\quad x=2$$

Sign diagram:



$$x<1\quad or\quad x>2$$

Writing a quadratic in **completed square form** can help to identify the signs for each region. It may be useful to note the following two results (visualising them as a graph can help):

$$x^2>a^2\quad\Rightarrow\quad x<-a\quad or\quad x>a$$

$$x^2<a^2\quad\Rightarrow\quad -a<x<a$$

Eg: Solve the inequality $x^2+4x-2\leq 0$

$$x^2+4x-2\leq 0\quad\Rightarrow\quad (x+2)^2-6\leq 0\quad\Rightarrow\quad (x+2)^2\leq 6$$

$$\Rightarrow\quad -\sqrt{6}\leq x+2\leq\sqrt{6}$$

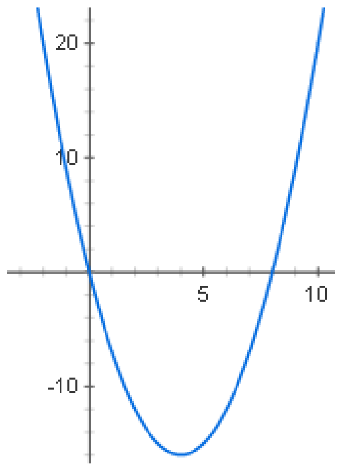
Note: Frequently knowledge of the discriminant is combined with solving quadratic inequalities.

Eg: For what values of k does the quadratic $x^2-kx+2k=0$ have no solutions?

$$x^2-kx+2k=0\ has\ no\ solutions$$
$$\Rightarrow\quad b^2-4ac<0$$
$$\Rightarrow\quad k^2-8k<0$$

$$Critical\ values\ at:\quad k(k-8)=0$$
$$\Rightarrow\quad k=0\quad or\quad k=8$$

$$0<k<8$$



Chapter 8 – Coordinate geometry of circles

The **equation of a circle** with centre $(0,0)$ and radius r is given by:

$$x^2 + y^2 = r^2$$

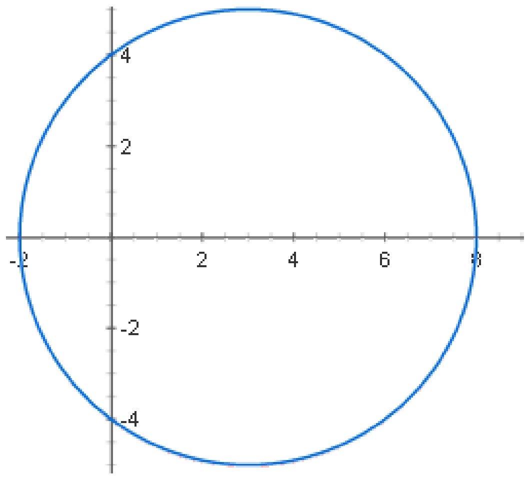
Note: This can be understood by considering any point on the circle and constructing a right-angled triangle, with the line from the origin to the point as the hypotenuse. By Pythagoras' Theorem, the square of the radius must be equal to the sum of the squares of the x and y coordinates.

The **general equation of a circle** with centre (a, b) and radius r is given by:

$$(x - a)^2 + (y - b)^2 = r^2$$

Note: This is simply a translation on the original circle – centre $(0,0)$ – by the translation vector $\begin{bmatrix} a \\ b \end{bmatrix}$.

Eg: The circle $(x - 3)^2 + y^2 = 25$:



Note: The most common mistakes to watch out for when interpreting a circle equation are getting the signs wrong for the centre coordinates or forgetting to square root the number on the right to get the radius.

Recall that a **translation** by the vector $\begin{bmatrix} a \\ b \end{bmatrix}$ represents a movement of a in the positive x direction and a movement of b in the positive y direction.

Note: This transformation can be applied repeatedly. It is often easiest to consider where the centre of a circle is, and where the centre of the new circle will be after the translation.

Eg: Translate the circle $(x - 4)^2 + (y + 1)^2 = 15$ by vector $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

$$Centre\ (4, -1)\ and\ radius\ \sqrt{15}.$$

$$New\ circle: centre\ (6, -2)\ and\ the\ same\ radius.$$

$$\Rightarrow \quad (x - 6)^2 + (y + 2)^2 = 15$$

To **sketch** a **circle**:
Step 1: Find the radius and centre from the equation.
Step 2: Mark the centre.
Step 3: Use the radius to mark on the four points directly above, below and to either side.
Step 4: Draw the circle through these four points, indicating where the circle crosses the axes, if applicable.

Note: It may be necessary, when finding the centre and radius of a circle, to rearrange the equation into the desired form. This is accomplished by completing the square separately for x and y.

Eg: Find the centre and radius of the circle whose equation is $x^2 + y^2 - 4x + 6y - 3 = 0$

$$x^2 + y^2 - 4x + 6y - 3 = 0 \quad \Longrightarrow \quad \{x^2 - 4x\} + \{y^2 + 6y\} - 3 = 0$$

$$\Longrightarrow \quad \{(x-2)^2 - 4\} + \{(y+3)^2 - 9\} - 3 = 0$$

$$\Longrightarrow \quad (x-2)^2 + (y+3)^2 = 16 \quad \Longrightarrow \quad Centre: (2, -3) \quad Radius: 4$$

To **find the equation of a circle** given the **end points of the diameter**, calculate the midpoint and the distance between the points (see chapter 3). This will give you the centre and the diameter. Halve the diameter to get the radius, then put into the form $(x-a)^2 + (y-b)^2 = r^2$.

The **perpendicular bisector** of any **chord** passes through the centre.

Note: This means the centre of a circle can be found by finding the intersection of any two such perpendicular bisectors. All that is required is two chords (which means a minimum of 3 points).

Eg: Find the coordinates of the centre of the circle through the points $(0,1), (3,-4)$ and $(2,2)$.

Using $m = \frac{y-step}{x-step}$ and $y - y_1 = m(x - x_1)$, then midpoints and $m_1 = -\frac{1}{m_2}$ for perpendicular lines:

Points $(0,1)$ and $(3,-4)$: $\quad Chord\ 1\ line\ equation: \quad y - 1 = \left(-\frac{5}{3}\right)(x - 0) \quad \Longrightarrow \quad y = -\frac{5}{3}x + 1$

$$Perpendicular\ bisector\ of\ chord\ 1: \quad y - (-1.5) = \left(\frac{3}{5}\right)(x - 1.5) \quad \Longrightarrow \quad y = \frac{3}{5}x - \frac{12}{5}$$

Points $(0,1)$ and $(2,2)$: $\quad Chord\ 2\ line\ equation: \quad y - 1 = \left(\frac{1}{2}\right)(x - 0) \quad \Longrightarrow \quad y = \frac{1}{2}x + 1$

$$Perpendicular\ bisector\ of\ chord\ 2: \quad y - 1.5 = (-2)(x - 1) \quad \Longrightarrow \quad y = -2x + 3.5$$

$$Bisectors\ cross\ at\ \frac{3}{5}x - \frac{12}{5} = -2x + 3.5 \quad \Longrightarrow \quad x = \frac{59}{26} \quad Therefore\ y = -2\left(\frac{59}{26}\right) + 3.5 = -\frac{27}{26}$$

$Centre: \left(\frac{59}{26}, -\frac{27}{26}\right)$ Note: By finding the distance from the centre to one of the points, we could also find the radius, and thus the equation of the circle, if required.

Like any other curve, and as discussed in chapter 7, finding the **crossing points** of a **line** and a **circle** can be found by solving simultaneously (using the substitution method).

Note: Just like with a line crossing a parabola, the discriminant can be used to determine if the line passes through the circle (cutting it at two points; positive discriminant), lies tangent to the circle (one, repeated, root; zero discriminant) or misses the circle completely (no solutions; negative discriminant).

Eg: For which values of k does the line $y = kx + 5$ lie tangent to the circle $(x-6)^2 + (y-7)^2 = 4$?

Substituting for y:

$$(x-6)^2 + (kx + 5 - 7)^2 = 4$$

Multiplying out and rearranging:

$$x^2 - 12x + 36 + k^2x^2 - 4kx + 4 = 4 \quad \Rightarrow \quad (1+k^2)x^2 + \big(-4(3+k)\big)x + 36 = 0$$

Using the discriminant condition:

$$\mathit{One\ solution\ when}\ b^2 - 4ac = 0 \quad \Rightarrow \quad 16(3+k)^2 - 4(36)(1+k^2) = 0$$

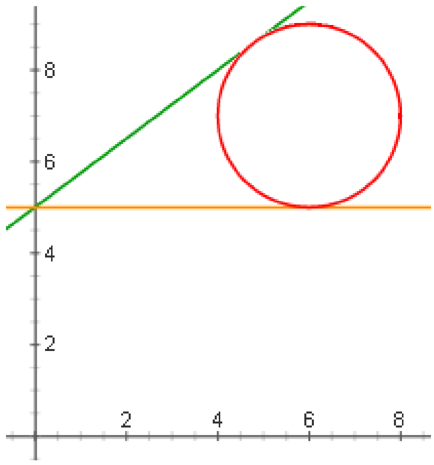
Simplifying:

$$(9 + 6k + k^2) - 9(1+k^2) = 0 \quad \Rightarrow \quad 9 + 6k + k^2 - 9 - 9k^2 = 0$$

Rearranging and solving:

$$-8k^2 + 6k = 0 \quad \Rightarrow \quad k(4k-3) = 0 \quad \Rightarrow \quad k = 0 \quad or \quad k = \frac{3}{4}$$

(optional) Verify solution graphically:



A **tangent** to the circle at a particular point is a line which touches the circle only at that point. It is always **perpendicular** to the radius.

A **normal** to the circle at a particular point is a line passing through a circle which is **perpendicular** to the tangent at that point.

Note: The normal line at any point on a circle will pass through the centre, since it is perpendicular to the tangent which is perpendicular to the radius.

To find the **equation of a tangent or normal** at a particular point:
Step 1: Find the centre of the circle.
Step 2: Calculate the gradient of the line segment from the centre to your point.
Step 3i: For a normal, use this gradient and your point in the formula $y - y_1 = m(x - x_1)$.
Step 3ii: For a tangent, first find the tangent gradient by using $m_1 = -\frac{1}{m_2}$, then use the formula.

Eg: Find the equation of the tangent to the circle $(x+1)^2 + (y-3)^2 = 35$ at the point (2,7).

$$Centre: \quad (-1,3)$$

$$Gradient\ of\ line\ segment: \quad \frac{7-3}{2-(-1)} = \frac{4}{3}$$

$$Gradient\ of\ tangent: \quad -\frac{1}{\frac{4}{3}} = -\frac{3}{4}$$

$$Equation\ of\ tangent: \quad y - 7 = -\frac{3}{4}(x-2) \quad \Longrightarrow \quad y = -\frac{3}{4}x + \frac{17}{2}$$

Chapter 9 – Introduction to differentiation: gradient of curves

Differentiation is a method for finding the gradient of a curve at any given point. It can be thought of as the *rate of change* of y with respect to x.

For a **small change** in x and a corresponding small change in y, the **gradient of a chord** can be written as:

$$\frac{\delta y}{\delta x}$$

Note: In this case, 'chord' refers to a straight line segment drawn between two nearby points on a curve.

We define the **gradient** of a **curve** at a particular point as the gradient of the **tangent** to the curve at that point.

Note: As the end-points get closer together, the gradient of the chord approaches the gradient of the tangent at each point (ie, the gradient of the curve). This is the basis for differentiation – the limit to which the gradients of the ever-decreasing chords tends is the gradient of the curve at that point.

The limit of $\frac{\delta y}{\delta x}$ as $\delta y, \delta x \to 0$ is written as $\frac{dy}{dx}$ and is known as the **differential** of y with respect to x.

Note: The proof of this idea is given in the textbook, but while it is worth understanding it is not necessary to memorise the proof.

$$y = x^n \quad \Rightarrow \quad \frac{dy}{dx} = nx^{n-1}$$

Note: This can be thought of as "bring the power down in front, then reduce the power by one".
Note: If there is a constant multiplied by the x^n term, this is not affected by differentiating.

Eg:

$$y = 5x^4 \quad \Rightarrow \quad \frac{dy}{dx} = 20x^3$$

$$y = f(x) \pm g(x) \quad \Rightarrow \quad \frac{dy}{dx} = f'(x) \pm g'(x)$$

Note: $f'(x)$ is the notation sometimes used to denote the derivative of $f(x)$. It means the same as, but is less cumbersome than, $\frac{d(f(x))}{dx}$.

Eg: The differential of $2x^4 - 8x^2 + 4$ is $8x^3 - 16x$ (note that any constants differentiate to 0).

To find the gradient of a curve at a particular point, calculate $\frac{dy}{dx}$ and substitute in the x coordinate.

Eg: Find the gradient of the curve $y = (2x+3)(x^2-5)$ at the point $(2, -7)$.

Multiply out:

$$y = (2x+3)(x^2-5) = 2x^3 + 3x^2 - 10x - 15$$

Differentiate:

$$\frac{dy}{dx} = 6x^2 + 6x - 10$$

Substitute in $x = 2$:

$$At\ x = 2 \quad \frac{dy}{dx} = 6(2^2) + 6(2) - 10 = 26 \quad \Rightarrow \quad Gradient = 26$$

To find a **point on a curve** with a **given gradient**, differentiate then set your expression equal to the gradient and solve for x. Finally, substitute into the original equation for a corresponding value for y.

Eg: Find any points on the curve $y = x^3 - 3x$ where the gradient is 45.

Differentiate:

$$\frac{dy}{dx} = 3x^2 - 3$$

Rewrite using $\frac{dy}{dx} = 45$:

$$45 = 3x^2 - 3 \quad \Rightarrow \quad x^2 - 1 = 15 \quad \Rightarrow \quad x^2 = 16 \quad \Rightarrow \quad x = \pm 4$$

Substitute back into the original equation:

$$For\ x = 4: \quad y = 4^3 - 3(4) = 52 \quad \Rightarrow \quad (4,52)$$

$$For\ x = -4: \quad y = (-4)^3 - 3(-4) = -52 \quad \Rightarrow \quad (-4, -52)$$

Chapter 10 – Applications of differentiation: tangents, normals and rates of change

Given the gradient of a curve at a particular point, we can find the **equation of the tangent** using $y - y_1 = m(x - x_1)$ where (x_1, y_1) is the point on the curve and m is the gradient.

Note: We can calculate the gradient for a particular point on a curve using differentiation. (see chapter 9). Also note that, when dealing with circles, although this method does work, it requires knowledge beyond the scope of this module to apply, and it is still more straightforward to use the method described in chapter 8 using the centre and a normal line.

Eg: Find the equation of the tangent to the curve $y = 5x^3 - 6$ at the point $(1, -1)$.

Differentiate:

$$\frac{dy}{dx} = 15x^2$$

Substitute in $x = 1$:

$$m = 15(1^2) = 15$$

Use $y - y_1 = m(x - x_1)$:

$$y - (-1) = 15(x - 1) \quad \Rightarrow \quad y = 15x - 16$$

Recall that $\frac{dy}{dx}$ means the **rate of change** of y with respect to x.

$$\frac{dy}{dx} > 0 \quad \Rightarrow \quad y \text{ increases as } x \text{ increases}$$

$$\frac{dy}{dx} < 0 \quad \Rightarrow \quad y \text{ decreases as } x \text{ increases}$$

Note: Often the rate of change will be with respect to time.

Eg:
The height h metres of a skydiver at time t seconds is modelled by the equation:

$$h = 2000 + 5t - 5t^2$$

Calculate the skydiver's rate of descent when he has fallen 1000 metres.

Substitute in $h = 1000$ to find the time at which he is at a height of 1000:

$$1000 = 2000 + 5t - 5t^2 \quad \Longrightarrow \quad t^2 - t - 200 = 0$$

Solve (using the quadratic formula) to find t (note: only positive values are valid for this problem):

$$t = \frac{-(-1) \pm \sqrt{1 - 4(1)(-200)}}{2(1)} = \frac{1 \pm \sqrt{801}}{2} = 14.65 \;\; or \; -13.65 \quad \Longrightarrow \quad t = 14.65 \; seconds$$

Find the rate of descent (in other words, speed) by differentiating:

$$\frac{dh}{dt} = 5 - 10t$$

Substitute $t = 14.65$ to find the rate of descent at this time:

$$\frac{dh}{dt} = 5 - 10(14.65) = 5 - 146.5 = -141.5 \; metres \; per \; second$$

Note: This is negative as it is the rate of change of height, so the rate of *descent* is $141.5 m/s$.

If $f'(x) > 0$ for all values in a given interval, the function is said to be **increasing** for this interval.

If $f'(x) < 0$ for all values in a given interval, the function is said to be **decreasing** for this interval.

Note: You may be asked to show that a given function is either increasing or decreasing for a given range. You would do this by finding the gradient by differentiating and showing that the expression you have found fits the criteria above.

Chapter 11 – Maximum and minimum points and optimisation

We can find a point on a curve with a particular gradient (see chapter 9). If we look specifically for points where the gradient is zero, these must correspond to points on the curve which are either maxima, minima or turning points. These points are known as stationary points.

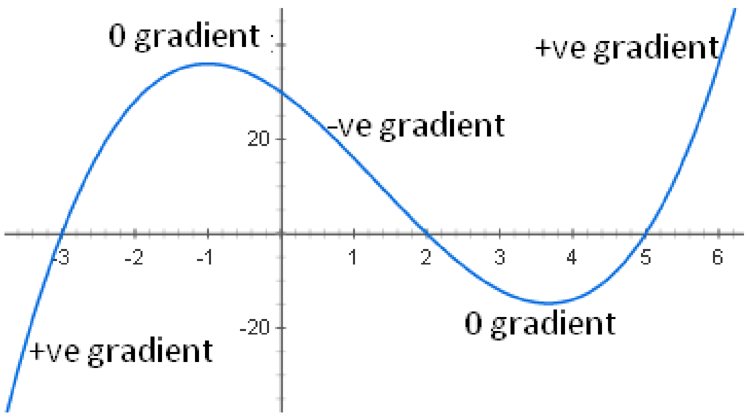
Stationary points occur when:

$$\frac{dy}{dx} = 0$$

For a **minimum** on a curve, such as the lowest point of a parabola, the gradient must be negative just before the minimum and positive just after.

For a **maximum**, the gradient is positive just before and negative just after.

Note: This is one way we can determine if a given stationary point is a maximum or a minimum.



If we **differentiate** a function a **second time** we get the **rate of change of the gradient**. This can be written as:

$$f''(x) \quad or \quad \frac{d^2y}{dx^2}$$

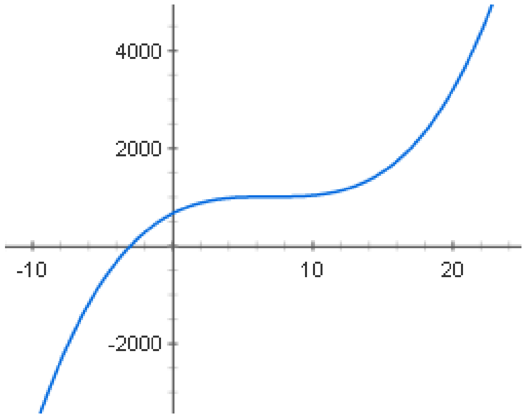
Note: These are usually said as "f-double-dashed of x" and "d-two-y by d-x-squared".

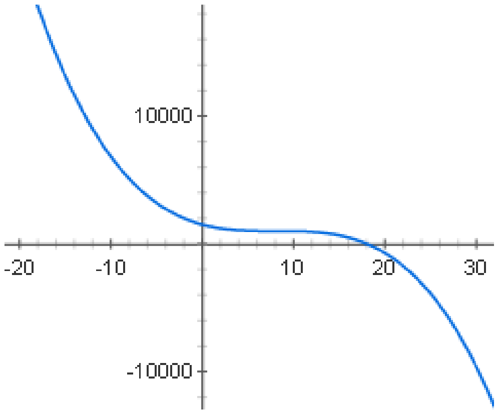
Another way of testing the **nature** of a **stationary point** is to calculate the **second derivative.**

$$\frac{dy}{dx} = 0 \quad and \quad \frac{d^2y}{dx^2} > 0 \quad \Longrightarrow \quad Minimum \qquad \frac{dy}{dx} = 0 \quad and \quad \frac{d^2y}{dx^2} < 0 \quad \Longrightarrow \quad Maximum$$

Note: Since this is the rate at which the gradient changes, if the stationary point is a minimum the gradient goes from negative to positive; it is increasing which means its rate of change is positive.

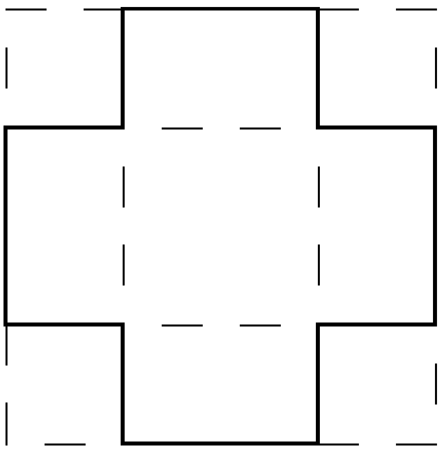
If the gradient is 0, the curve may be at a local maximum or minimum, or it might be at a **point of inflection**. This means the gradient goes to zero, but then continues with the same type of gradient (positive or negative) it had before.





A **point of inflection** can be either **positive** or **negative**. A positive point of inflection has a positive gradient either side, and a negative point of inflection has a negative gradient.

One common application of stationary points is **optimisation problems**. If a problem can be formulated algebraically and we are aiming to maximise or minimise something (eg maximum volume, or minimum surface area for a given shape), we can use differentiation to find the required value of x.



Eg: A box is to be made by cutting corners from a square sheet of cardboard and folding up the edges as shown. The original square measures $12cm$ by $12cm$. Find the maximum volume for the resulting box and the corresponding height.

Let x be the height of the cuboid; then:

$$V = (12 - 2x)x = 12x - 2x^2$$

Differentiate to find stationary points:

$$\frac{dV}{dx} = 12 - 4x = 0 \quad \Rightarrow \quad x = 3$$

Find the resulting volume:

$$C = \left(12 - 2(3)\right)(3) = 18cm^3$$

Chapter 12 – Integration

Integration is the **reverse of differentiation**. If you know the **derivative** of a function, the process of finding the function itself is known as **integration**.

If $f'(x)$ is the differential of $f(x)$ then:

$$\int f'(x)\,dx = f(x) + C$$

Note: This would be described as "The integral of $f'(x)$ with respect to x".
Note: The $+\,C$ on the end is due to the fact that any constant would differentiate to 0, therefore, for instance, $x^3 + 2$ differentiates to the same thing as $x^3 - 6$. By including this arbitrary 'constant of integration' we are describing all possible functions (the 'family of functions') which differentiate to the function we are integrating.

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

Note: This can be thought of as "increase the power by one, then divide by the new power".

Since integration in this way always yields an **arbitrary constant**, and therefore cannot give a single unique value, the integral is known as an **indefinite integral**.

Note: The specific solution for a particular case can be found provided **initial conditions** are provided – typically a value for y at $x = 0$, but could be any specific point on the curve.

Eg: A function has derivative $\frac{dy}{dx} = 3x^2 - 4$ and goes through the point (4,1). Find y in terms of x.

$$y = \int 3x^2 - 4\,dx = x^3 - 4x + C$$

$$Substituting\ in\ y = 1\ at\ x = 4: \quad 1 = 4^3 - 4(4) + C \quad \Rightarrow \quad C = -47$$

$$Rewriting\ with\ known\ value\ for\ C: \quad y = x^3 - 4x - 47$$

A result concerning **combinations of functions** has been given for differentiation (see chapter 9). An equivalent result is valid for **integration**:

$$\int Af(x) + Bg(x)dx = A\int f(x)\,dx + B\int g(x)\,dx$$

Note: It may often be necessary to multiply out brackets in order to turn a function into an easily integrable form.

In the same way that **differentiation** can be thought of as finding the **gradient** of a curve at any point, **integration** is a way of finding the **area** under a curve between any two points.

Finding the **area under a curve** using integration requires the use of **limits**. The resulting integral is known as a **definite integral**, since the effect of any arbitrary constant is cancelled out and is therefore superfluous.

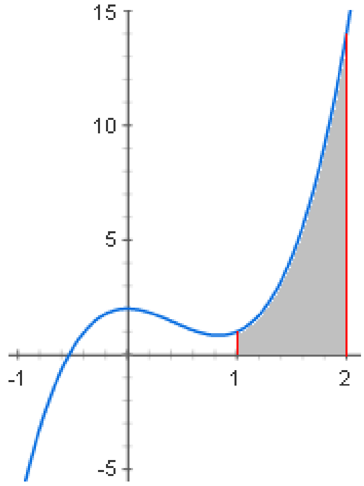
Note: It is important to be aware of the shape of the graph, since integration will yield a negative result for any section of the area which lies below the x axis.

Eg: Find the area bounded by the curve $y = 4x^5 - 5x^2 + 2$, the x axis and the lines $x = 1$ and $x = 2$.

$$\int_1^2 4x^3 - 5x^2 + 2\ d = \left[x^4 - \frac{5}{3}x^3 + 2x\right]_1^2$$

$$= \left(16 - \frac{40}{3} + 4\right) - \left(1 - \frac{5}{3} + 2\right)$$

$$= \frac{16}{3} \quad \Longrightarrow \quad Area = \frac{16}{3}$$



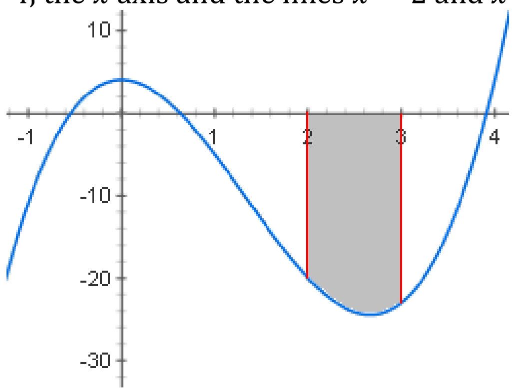
In this example, there is no problem because the entire region lies above the axis. The positive value given by integration is equal to the area.

Eg: Find the area bounded by the curve $y = 3x^3 - 12x^2 + 4$, the x axis and the lines $x = 2$ and $x = 3$.

$$\int_2^3 3x^3 - 12x^2 + 4\ dx = \left[\frac{3}{4}x^4 - 4x^3 + 4x\right]_2^3$$

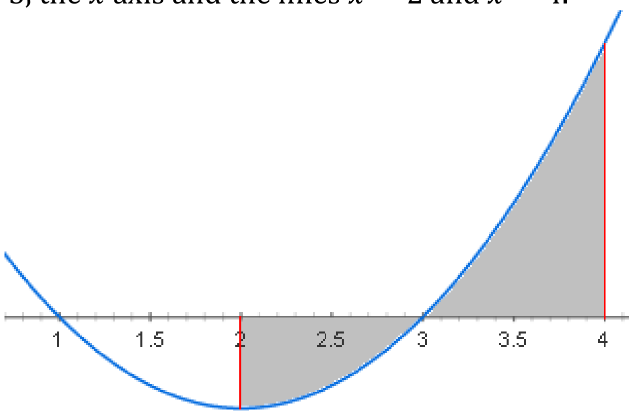
$$= \left(\frac{243}{4} - 108 + 12\right) - (12 - 32 + 8)$$

$$= -\frac{209}{4} \quad \Longrightarrow \quad Area = \frac{209}{4}$$



Here, since the whole region is below the axis, the integral is negative, but the same size as the area. All that is necessary is to change the sign from negative to positive.

Eg: Find the area bounded by the curve $y = x^2 - 4x + 3$, the x axis and the lines $x = 2$ and $x = 4$.



$$\int_2^3 x^2 - 4x + 3\,dx = \left[\frac{1}{3}x^3 - 2x^2 + 3x\right]_2^3$$

$$= (9 - 18 + 9) - \left(\frac{8}{3} - 8 + 6\right)$$

$$= -\frac{2}{3} \quad \Longrightarrow \quad Area = \frac{2}{3}$$

$$\int_3^4 x^2 - 4x + 3\,dx = \left[\frac{1}{3}x^3 - 2x^2 + 3x\right]_3^4$$

$$= \left(\frac{64}{3} - 32 + 12\right) - (9 - 18 + 9)$$

$$= \frac{4}{3} \quad \Longrightarrow \quad Area = \frac{4}{3}$$

$$\Longrightarrow \quad Total\ Area = \frac{2}{3} + \frac{4}{3} = 2$$

Here, because part of the integral would be negative, it would cancel out with some of the positive part, so we need to separate and integrate each region independently. In this case, the x axis crossing points are obvious because the function is a quadratic that easily factorises. In other cases, more work may be necessary.

The **area under a line** can be found using integration, just like any function of x, but it is generally quicker to consider it as a **trapezium**.

Eg: The area under the line $y = 3x - 1$ between $x = 4$ and $x = 6$ can be calculated by:

Using integration:

$$\int_4^6 3x - 1\,dx = \left[\frac{3}{2}x^2 - x\right]_4^6$$

$$= (54 - 6) - (24 - 4)$$

$$= 28 \quad \Longrightarrow \quad Area = 28$$

Using area of a trapezium:

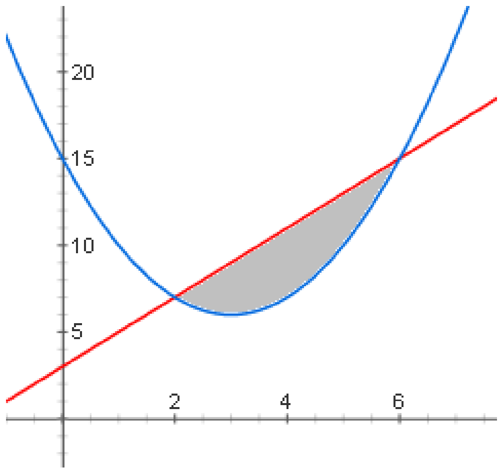
$$A = \frac{a + b}{2}h$$

$$a = 3(4) - 1 = 11 \qquad b = 3(6) - 1 = 17 \qquad h = 6 - 4 = 2$$

$$Area = \frac{11 + 17}{2}(2) = 28$$

To find the **area** of a region bounded by a **curve** and a **line**, it is necessary to calculate the points of intersection, and then calculate the difference between the area under the curve and the area under the line.

Eg: Find the area bounded by the curve $y = x^2 - 6x + 15$ and the line $y = 2x + 3$.



Find points of intersection:

$$x^2 - 6x + 15 = 2x + 3$$

$$x^2 - 8x + 12 = 0 \quad \Longrightarrow \quad (x-2)(x-6) = 0$$

$$\Longrightarrow \quad x = 2 \quad and \quad x = 6$$

Find area under line (note: this is a trapezium):

$$\frac{a+b}{2}h = \frac{(2(2)+3)+(2(6)+3)}{2}(6-2) = 44$$

Find area under curve:

$$\int_2^6 x^2 - 6x + 15\,dx = \left[\frac{1}{3}x^3 - 3x^2 + 15x\right]_2^6$$

$$= (72 - 108 + 90) - \left(\frac{8}{3} - 12 + 30\right) = \frac{116}{3}$$

Calculate the difference to find the area in between:

$$Area = 44 - \frac{116}{3} = \frac{16}{3}$$