Applications of Differentiation (ch 11 and ch 12)

1. $f(x) = 5x^3 - 15x^2$

(i) $f'(x) = 15x^2 - 30x$

(ii) $f''(x) = 30x - 30$

(iii) $15x^2 - 30x = 0$

$15x(x-2) = 0$

$x = 0$ and $x = 2$

(iv) $f''(0) = -30 < 0 \therefore x = 0$ is a max

$f''(2) = 60 - 30 = 30 > 0 \therefore x = 2$ is a min.

2.(i) $A = \frac{\cancel{x}(12-x)}{2} = \frac{12x - x^2}{2} = 6x - \frac{1}{2}x^2$

(ii) $\frac{dA}{dx} = 6 - x$

(iii) $6 - x = 0$

$x = 6$

$\frac{d^2A}{dx^2} = -1 < 0 \quad \therefore \; x = 6$ is a max.

Greatest area $= 6 \times 6 - \frac{1}{2} \times 6^2$

$= 36 - 18$

$= 18 cm^2$

3.(i) $y = x^3 - 6x^2 + 9x + 2$

$\frac{dy}{dx} = 3x^2 - 12x + 9$

(ii) $3x^2 - 12x + 9 = 0$

$x^2 - 4x + 3 = 0$ $\qquad \frac{d^2y}{dx^2} = 6x - 12$

$(x-3)(x-1) = 0$

$x = 3$ or $x = 1$

when $x = 3$, $\frac{d^2y}{dx^2} = 18 - 12 = 6 > 0$ $\quad \therefore$ minimum at $(3, 2)$

when $x = 1$, $\frac{d^2y}{dx^2} = 6 - 12 = -6 < 0$ $\quad \therefore$ maximum at $(1, 6)$

(iii) At $(-1, -14)$, $\frac{dy}{dx} = 3 + 12 + 9 = 24$.

$3x^2 - 12x + 9 = 24$

$3x^2 - 12x - 15 = 0$

$x^2 - 4x - 5 = 0$

$(x-5)(x+1) = 0$

$x = 5$ (or $x = -1$ $\rightarrow$ already given $(-1, -14)$)

$y = 125 - 150 + 45 + 2$

$= 22$

$(a, b) = (5, 22)$.

Find distance between $(-1, -14)$ and $(5, 22)$:

$\sqrt{(-1-5)^2 + (-14-22)^2} = \sqrt{36 + 36^2} = \sqrt{36(1+36)} = 6\sqrt{37}$

4. $f(x) = x^3 - 4x^2 - 3x + 1$

(i) $f'(x) = 3x^2 - 8x - 3$

(ii) $f''(x) = 6x - 8$

(iii) $3x^2 - 8x - 3 = 0$

$(3x+1)(x-3) = 0$

$x = -\frac{1}{3}$ or $x = 3$

(iv) At $x = -\frac{1}{3}$, $f''(-\frac{1}{3}) = -2 - 8 = -10 < 0$ $\quad \therefore \; x = -\frac{1}{3}$ is a maximum

At $x = 3$, $f''(3) = 18 - 8 = 10 > 0$ $\quad \therefore \; x = 3$ is ~~is~~ a minimum

5. (i) $y = 6x^3 - 12x^2 + 8x - 1$

$\frac{dy}{dx} = 18x^2 - 24x + 8$

$18x^2 - 24x + 8 = 0$

$9x^2 - 12x + 4 = 0$

$x = \frac{12 \pm \sqrt{144 - 4 \times 9 \times 4}}{18}$

$= \frac{12 \pm \sqrt{0}}{18}$

$= \frac{2}{3}$

$y = 6\left(\frac{2}{3}\right)^3 - 12\left(\frac{2}{3}\right)^2 + 8\left(\frac{2}{3}\right) - 1$

$= \frac{48}{27} - \frac{48}{9} + \frac{16}{3} - 1$

$= \frac{48}{27} - \cancel{\frac{144}{27}} + \cancel{\frac{144}{27}} - \frac{27}{27} = \frac{21}{27} = \frac{7}{9}$

$\therefore$ There is only one point with a gradient of 0. $P\left(\frac{2}{3}, \frac{7}{9}\right)$

(ii) $y = \frac{7}{9}$

6. (i) $y = 2x^2 - x^3$ $\quad A(2, 0)$

$\frac{dy}{dx} = 4x - 3x^2$

At A, $\frac{dy}{dx} = 4 \times 2 - 3 \times 4 = 8 - 12 = -4$

$y - 0 = -4(x - 2)$

$y = -4x + 8$

(ii) $-4x + 8 = 2x^2 - x^3$

$x^3 - 2x^2 - 4x + 8 = 0$

$(-2)^3 - 2(-2)^2 - 4(-2) + 8 = 0$ $\quad \therefore$ The lines cross when $x = -2$.

$y = -4(-2) + 8 = 16$

$y = 2(-2)^2 - (-2)^3 = 8 + 8 = 16$ $\quad \therefore$ B is the point $(-2, 16)$.

7. (i) $y = 2x^3 - 3x^2 - 12x - 7$

$\frac{dy}{dx} = 6x^2 - 6x - 12$

$6x^2 - 6x - 12 = 0$

$x^2 - x - 2 = 0$

$(x - 2)(x + 1) = 0$

$x = 2$ or $x = -1$

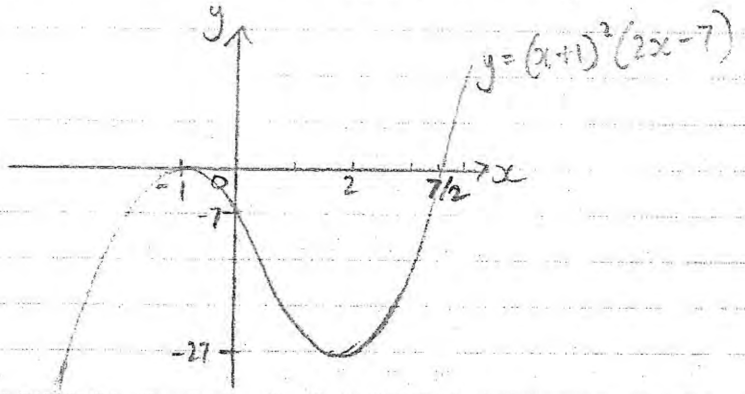
$y = -27$ $\quad y = 0$ $\quad \therefore$ stationary points at $(2, -27)$ and $(-1, 0)$

(ii) $\frac{d^2y}{dx^2} = 12x - 6$

When $x = 2$, $\frac{d^2y}{dx^2} = 24 - 6 = 18 > 0$ $\quad \therefore (2, -27)$ is a minimum

When $x = -1$, $\frac{d^2y}{dx^2} = -12 - 6 = -18 < 0$ $\quad \therefore (-1, 0)$ is a maximum.

(iii) $2x^3-3x^2-12x-7 \equiv (x+1)^2(2x-7)$



(iv) $2x^3-3x^2-12x-7=k$ has exactly one real solution when $k<-27$ or $k>0$

8 (i) $y=\frac{1}{3}x^3-9x$

$$\frac{dy}{dx}=x^2-9$$

(ii) $x^2-9=0$

$x=3$ or $x=-3$

$y=-18$ $\quad y=18$

$\therefore$ stationary points at $(3,-18)$ and $(-3,18)$

(iii) $\frac{d^2y}{dx^2}=2x$

when $x=3$, $\frac{d^2y}{dx^2}=6>0$ $\therefore$ $(3,-18)$ is a minimum

when $x=-3$, $\frac{d^2y}{dx^2}=-6<0$ $\therefore$ $(-3,18)$ is a maximum

(iv) $24x+3y+2=0$

$3y=-24x-2$

$y=-8x-\frac{2}{3}$ $\quad\therefore$ gradient of tangent $=-8$ at (p,q)

$x^2-9=-8$

$x^2=1$

$x=1$ or $\cancel{x=-1}$

$y=-8\frac{2}{3}$ $\quad \cancel{y=8\frac{2}{3}}$

(from $y=\frac{1}{3}x^3-9x$)

(p,q) could be $(1,-8\frac{2}{3})$ or $\cancel{(-1,8\frac{2}{3})}$

$y=-8x-\frac{2}{3}$ (tangent)

$=-8-\frac{2}{3}$

$=-8\frac{2}{3}$

$\therefore$ (p,q) is the point $(1,-8\frac{2}{3})$.