Exercise 3B

1 a Rearrange $x + y = 11$ to give:
$y = 11 - x$
Substitute into $xy = 30$:
$x(11 - x) = 30$
$11x - x^2 = 30$
$0 = x^2 - 11x + 30$
$0 = (x-5)(x-6)$
$x = 5$ or $x = 6$
Substitute into $y = 11 - x$:
When $x = 5$, $y = 11 - 5 = 6$
When $x = 6$, $y = 11 - 6 = 5$
Solutions are $x = 5$, $y = 6$ or $x = 6$, $y = 5$

b Rearrange $2x + y = 1$ to give:
$y = 1 - 2x$
Substitute into $x^2 + y^2 = 1$:
$x^2 + (1-2x)^2 = 1$
$x^2 + 1 - 4x + 4x^2 = 1$
$5x^2 - 4x = 0$
$x(5x - 4) = 0$
$x = 0$ or $x = \frac{4}{5}$
Substitute into $y = 1 - 2x$:
When $x = 0$, $y = 1$
When $x = \frac{4}{5}$, $y = 1 - \frac{8}{5} = -\frac{3}{5}$
Solutions are
$x = 0$, $y = 1$ or $x = \frac{4}{5}$, $y = -\frac{3}{5}$

c $y = 3x$
Substitute into $2y^2 - xy = 15$:
$2(3x)^2 - x(3x) = 15$
$18x^2 - 3x^2 = 15$
$15x^2 = 15$
$x^2 = 1$
$x = -1$ or $x = 1$
Substitute into $y = 3x$:
When $x = -1$, $y = -3$
When $x = 1$, $y = 3$
Solutions are
$x = -1$, $y = -3$ or $x = 1$, $y = 3$

d Rearrange $3a + b = 8$ to give:
$b = 8 - 3a$
Substitute into $3a^2 + b^2 = 28$:
$3a^2 + (8 - 3a)^2 = 28$
$3a^2 + 64 - 48a + 9a^2 = 28$
$12a^2 - 48a + 36 = 0$
Divide by 12:
$a^2 - 4a + 3 = 0$
$(a-1)(a-3) = 0$
$a = 1$ or $a = 3$
Substitute into $b = 8 - 3a$:
When $a = 1$, $b = 8 - 3 = 5$
When $a = 3$, $b = 8 - 9 = -1$
Solutions are
$a = 1$, $b = 5$ or $a = 3$, $b = -1$.

e Rearrange $2u + v = 7$ to give:
$v = 7 - 2u$
Substitute into $uv = 6$:
$u(7 - 2u) = 6$
$7u - 2u^2 = 6$
$0 = 2u^2 - 7u + 6$
$0 = (2u - 3)(u - 2)$
$u = \frac{3}{2}$ or $u = 2$
Substitute into $v = 7 - 2u$:
When $u = \frac{3}{2}$, $v = 7 - 3 = 4$
When $u = 2$, $v = 7 - 4 = 3$
Solutions are
$u = \frac{3}{2}$, $v = 4$ or $u = 2$, $v = 3$

f Rearrange $3x + 2y = 7$ to give:
$2y = 7 - 3x$
$y = \frac{1}{2}(7 - 3x)$
Substitute into $x^2 + y = 8$:
$x^2 + \frac{1}{2}(7 - 3x) = 8$
Multiply by 2:
$2x^2 + (7 - 3x) = 16$
$2x^2 - 3x - 9 = 0$
$(2x + 3)(x - 3) = 0$
$x = -\frac{3}{2}$ or $x = 3$

1 f Substitute into $y=\frac{1}{2}(7-3x)$:

When $x=-\frac{3}{2}$, $y=\frac{1}{2}\left(7+\frac{9}{2}\right)=\frac{23}{4}$

When $x=3$, $y=\frac{1}{2}(7-9)=-1$

Solutions are

$x=-\frac{3}{2}$, $y=\frac{23}{4}$ or $x=3$, $y=-1$

2 a Rearrange $2x + 2y = 7$ to give:

$2x=7-2y$

$x=\frac{1}{2}(7-2y)$

Substitute into $x^2-4y^2=8$:

$\left(\frac{1}{2}(7-2y)\right)^2-4y^2=8$

$\frac{1}{4}(7-2y)^2-4y^2=8$

Multiply by 4:

$(7-2y)^2-16y^2=32$

$49-28y+4y^2-16y^2=32$

$0=12y^2+28y-17$

$0=(6y+17)(2y-1)$

$y=-\frac{17}{6}$ or $y=\frac{1}{2}$

Substitute into $x=\frac{1}{2}(7-2y)$:

When $y=-\frac{17}{6}$, $x=\frac{1}{2}\left(7+\frac{17}{3}\right)=\frac{19}{3}$

When $y=\frac{1}{2}$, $x=\frac{1}{2}(7-1)=3$

Solutions are

$x=\frac{19}{3}$, $y=-\frac{17}{6}$ or $x=3$, $y=\frac{1}{2}$

b Rearrange $x + y = 9$ to give:

$x=9-y$

Substitute into $x^2-3xy+2y^2=0$:

$(9-y)^2-3y(9-y)+2y^2=0$

$81-18y+y^2-27y+3y^2+2y^2=0$

$6y^2-45y+81=0$

Divide by 3:

$2y^2-15y+27=0$

$(2y-9)(y-3)=0$

$y=\frac{9}{2}$ or $y=3$

Substitute into $x=9-y$:

When $y=\frac{9}{2}$, $x=9-\frac{9}{2}=\frac{9}{2}$

When $y=3$, $x=9-3=6$

Solutions are $x=\frac{9}{2}$, $y=\frac{9}{2}$ or $x=6$, $y=3$

2 c Rearrange $5y - 4x = 1$ to give:

$5y=4x+1$

$y=\frac{4}{5}x+\frac{1}{5}$

Substitute $y=\frac{4}{5}x+\frac{1}{5}$ into

$x^2 - y^2 + 5x = 41$:

$x^2-(\frac{4}{5}x+\frac{1}{5})^2+5x=41$

$x^2-\frac{16}{25}x^2-\frac{8}{25}x-\frac{1}{25}+5x=41$

$25x^2-16x^2-8x-1+125x=1025$

$9x^2+117x-1026=0$

$x^2+13x-114=0$

$(x+19)(x-6)=0$

So $x = -19$ or $x = 6$

Substitute into $y=\frac{4}{5}x+\frac{1}{5}$

Solutions are $x = -19$, $y = -15$

or $x = 6$, $y = 5$

3 a Rearrange $x - y = 6$ to give:

$x=6+y$

Substitute into $xy = 4$:

$y(6+y)=4$

$6y+y^2=4$

$y^2+6y-4=0$

Use the quadratic formula:

$a=1, b=6, c=-4$

$$y=\frac{-6\pm\sqrt{36+16}}{2}$$

$$=\frac{-6\pm\sqrt{52}}{2}$$

$$=\frac{-6\pm\sqrt{4\times 13}}{2}$$

$$=-3\pm\sqrt{13}$$

Substitute into $x = 6 + y$:

When $y=-3-\sqrt{13}$,

$x=6-3-\sqrt{13}=3-\sqrt{13}$

When $y=-3+\sqrt{13}$,

$x=6-3+\sqrt{13}=3+\sqrt{13}$

Solutions are

$x=3-\sqrt{13}$, $y=-3-\sqrt{13}$

or $x=3+\sqrt{13}$, $y=-3+\sqrt{13}$

3 b Rearrange $2x + 3y = 13$ to give:
$2x = 13 - 3y$
$x = \frac{1}{2}(13-3y)$
Substitute into $x^2 + y^2 = 78$:
$\left(\frac{1}{2}(13-3y)\right)^2 + y^2 = 78$
$\frac{1}{4}(13-3y)^2 + y^2 = 78$
Multiply by 4:
$(13-3y)^2 + 4y^2 = 312$
$169 - 78y + 9y^2 + 4y^2 = 312$
$13y^2 - 78y - 143 = 0$
Divide by 13:
$y^2 - 6y - 11 = 0$

Use the quadratic formula:
$a = 1,\ b = -6,\ c = -11$
$$y = \frac{6 \pm \sqrt{36+44}}{2} = \frac{6 \pm \sqrt{80}}{2} = \frac{6 \pm 4\sqrt{5}}{2} = 3 \pm 2\sqrt{5}$$

Substitute into $x = \frac{1}{2}(13-3y)$:
When $y = 3 - 2\sqrt{5}$,
$$x = \frac{1}{2}\left(13 - 3\left(3 - 2\sqrt{5}\right)\right) = \frac{1}{2}\left(13 - 9 + 6\sqrt{5}\right) = 2 + 3\sqrt{5}$$
When $y = 3 + 2\sqrt{5}$,
$$x = \frac{1}{2}\left(13 - 3\left(3 + 2\sqrt{5}\right)\right) = \frac{1}{2}\left(13 - 9 - 6\sqrt{5}\right) = 2 - 3\sqrt{5}$$
Solutions are
$x = 2 - 3\sqrt{5},\ y = 3 + 2\sqrt{5}$
or $x = 2 + 3\sqrt{5},\ y = 3 - 2\sqrt{5}$

4 Rearrange $x + y = 3$ to give:
$y = 3 - x$
Substitute into $x^2 - 3y = 1$:
$x^2 - 3(3 - x) = 1$
$x^2 + 3x - 10 = 0$
$(x + 5)(x - 2) = 0$
So $x = -5$ or $x = 2$
Solutions are $x = -5, y = 8$ or $x = 2, y = 1$

5 a $3x^2 + x(2 - 4x) + 11 = 0$
$3x^2 + 2x - 4x^2 + 11 = 0$
$-x^2 + 2x + 11 = 0$
$x^2 - 2x - 11 = 0$

b Use the quadratic formula.
$a = 1,\ b = -2,\ c = -11$
$$x = \frac{2 \pm \sqrt{4+44}}{2} = \frac{2 \pm \sqrt{48}}{2} = \frac{2 \pm 4\sqrt{3}}{2}$$
So $x = 1 \pm 2\sqrt{3}$
Substitute into $y = 2 - 4x$:
$x = 1 + 2\sqrt{3},\ y = -2 - 8\sqrt{3}$
or $x = 1 - 2\sqrt{3},\ y = -2 + 8\sqrt{3}$

6 a At the point $(1, p)$, $x = 1$ and $y = p$.
Substituting these values into the first equation gives:
$p = k - 5$ **(1)**
Substituting these values into the second equation gives:
$4 - p = 6$ **(2)**
$p = -2$
When $p = -2$, $k = 3$
$k = 3$, $p = -2$

b Substitute for k into $y = kx - 5$:
$y = 3x - 5$
Substitute into $4x^2 - xy = 6$:
$4x^2 - x(3x - 5) = 6$
$4x^2 - 3x^2 + 5x - 6 = 0$
$x^2 + 5x - 6 = 0$
$(x - 1)(x + 6) = 0$
$x = 1$ or $x = -6$
When $x = -6$, $y = -23$ and $x = 1$, $y = -2$
The solutions are $x = -6$, $y = -23$
or $x = 1$, $y = -2$

Challenge

Rearrange $y - x = k$ to give:
$y = x + k$
Substitute into $x^2 + y^2 = 4$
$x^2 + (x + k)^2 = 4$
$x^2 + x^2 + 2kx + k^2 - 4 = 0$
$2x^2 + 2kx + k^2 - 4 = 0$
Using the discriminant for one pair of solutions,
$b^2 - 4ac = 0$
$(2k)^2 - 4(2)(k^2 - 4) = 0$
$4k^2 - 8k^2 + 32 = 0$
$-4k^2 = -32$
$k^2 = 8$
$k = \pm\sqrt{8}$
$= \pm 2\sqrt{2}$