C1 June 2011 (MA)

Q1a) $25^{1/2} = \sqrt{25}$ $\boxed{=5}$

b) $25^{-\frac{3}{2}} = \dfrac{1}{25^{3/2}} = \dfrac{1}{(\sqrt[2]{25})^3} = \boxed{\dfrac{1}{125}}$

Q2) $y = 2x^5 + 7 + \dfrac{1}{x^3}$, $x \neq 0$

$y = 2x^5 + 7 + x^{-3}$

a) $\dfrac{dy}{dx} = 10x^4 - 3x^{-4}$

$= \boxed{10x^4 - \dfrac{3}{x^4}}$

b) $\int (2x^5 + 7 + x^{-3})\,dx = \dfrac{2x^6}{6} + 7x + \dfrac{x^{-2}}{-2} + c$

$= \dfrac{x^6}{3} + 7x - \dfrac{x^{-2}}{2} + c$

$= \boxed{\dfrac{x^6}{3} + 7x - \dfrac{1}{2x^2} + c}$

Q3) Gradient $PQ = \dfrac{y_2 - y_1}{x_2 - x_1}$

$= \dfrac{0-6}{9-(-1)}$

$= -\dfrac{3}{5}$

$\therefore$ the gradient of the line perpendicular to PQ is $\frac{5}{3}$

Midpoint $PQ = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

$= \left(\frac{-1+9}{2}, \frac{6+0}{2}\right)$

$= (4,3)$

Equation of perpendicular line:

$y - y_1 = m(x - x_1)$

Using Midpoint $PQ(4,3)$ and $m = \frac{5}{3}$:

$y - 3 = \frac{5}{3}(x-4)$

$3(y-3) = 5(x-4)$

$3y - 9 = 5x - 20$

$\boxed{5x - 3y - 11 = 0}$

Q4) $x+y=2 \Rightarrow x=2-y$ ①

$4y^2-x^2=11 \Rightarrow 4y^2-x^2=11$ ②

Substitute ① into ②:

$4y^2-(2-y)^2=11$

$4y^2-(2-y)(2-y)=11$

$4y^2-(4-4y+y^2)=11$

$4y^2-4+4y-y^2-11=0$

$3y^2+4y-15=0$

$(3y-5)(y+3)=0$

Either $y=\frac{5}{3}$ or $y=-3$

Substitute into ① for x:

When $y=\frac{5}{3}$, $x=2-\frac{5}{3}=\underline{\frac{1}{3}}$

When $y=-3$, $x=2-(-3)\underline{=5}$

Solution set: $\boxed{x=\frac{1}{3},\ y=\frac{5}{3} \text{ and } x=5,\ y=-3}$

Q5) $a_1 = k$

$a_{n+1} = 5a_n + 3, \; n \geq 1$

a) $a_2 = 5a_1 + 3$

$\boxed{a_2 = 5k + 3}$

b) $a_3 = 5a_2 + 3$

$a_3 = 5(5k+3)+3$

$a_3 = 25k + 15 + 3$

$\boxed{a_3 = 25k + 18}$

c i) $\sum_{r=1}^{4} a_r = a_1 + a_2 + a_3 + a_4$

$a_4 = 5a_3 + 3$

$a_4 = 5(25k + 18) + 3$

$a_4 = 125k + 90 + 3$

$\underline{a_4 = 125k + 93}$

$\therefore \sum_{r=1}^{4} a_r = k + 5k + 3 + 25k + 18 + 125k + 93$

$= \boxed{156k + 114}$

ii) $\sum_{r=1}^{4} a_r = \boxed{6(26k + 19)}$

$\therefore$ divisible by 6

Q6a) $\dfrac{6x+3x^{\frac{5}{2}}}{\sqrt{x}} = \dfrac{6x+3x^{\frac{5}{2}}}{x^{1/2}}$

$= \dfrac{6x}{x^{1/2}} + \dfrac{3x^{5/2}}{x^{1/2}}$

$= 6x^{\frac{1}{2}} + 3x^2$

$\therefore \boxed{p = \frac{1}{2} \quad \text{and} \quad q = 2}$

b) $\dfrac{dy}{dx} = \dfrac{6x+3x^{\frac{5}{2}}}{\sqrt{x}}$

$\dfrac{dy}{dx} = 6x^{\frac{1}{2}} + 3x^2$

$y = \int (6x^{\frac{1}{2}} + 3x^2)\, dx$

$y = \dfrac{6x^{\frac{3}{2}}}{\frac{3}{2}} + \dfrac{3x^3}{3} + c$

$y = 4x^{3/2} + x^3 + c$

$y = 4x\sqrt{x} + x^3 + c$

Since $y = 90$ when $x = 4$, substitute in:

$90 = 4(4)(\sqrt{4}) + 4^3 + c$

$90 = 4(4)(2) + 64 + c$

$90 = 96 + c$

$\therefore \underline{c = -6}$

$\boxed{y = 4x\sqrt{x} + x^3 - 6}$

Q7a) $f(x) = x^2 + (k+3)x + k$

$b^2 - 4ac = (k+3)^2 - (4)(1)(k)$

$= (k+3)(k+3) - 4k$

$= k^2 + 6k + 9 - 4k$

$= \boxed{k^2 + 2k + 9}$

b) $k^2 + 2k + 9 \equiv (k+1)^2 - 1^2 + 9$

$= (k+1)^2 - 1 + 9$

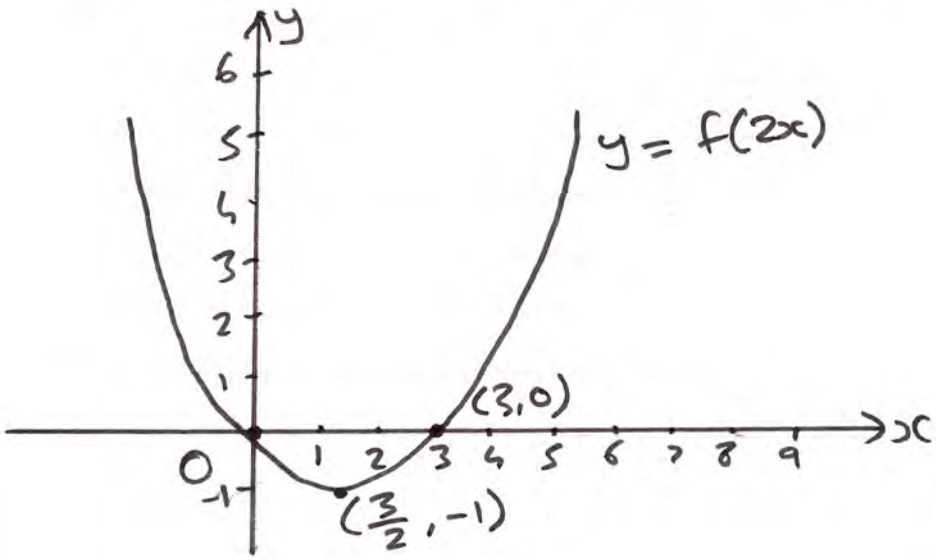
$= \boxed{(k+1)^2 + 8}$

c) For a quadratic to have real roots, the discriminant must be greater than or equal to zero.

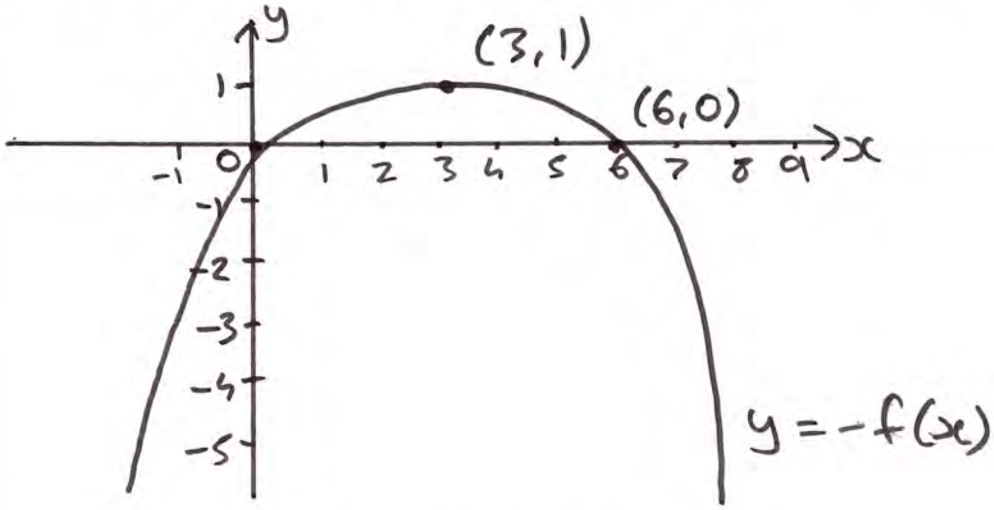
For this discriminant, $(k+1)^2 + 8$, the value inside the brackets is squared, so will always be positive. When you add 8, it will still be positive.

$\therefore f(x) = 0$ has real roots.

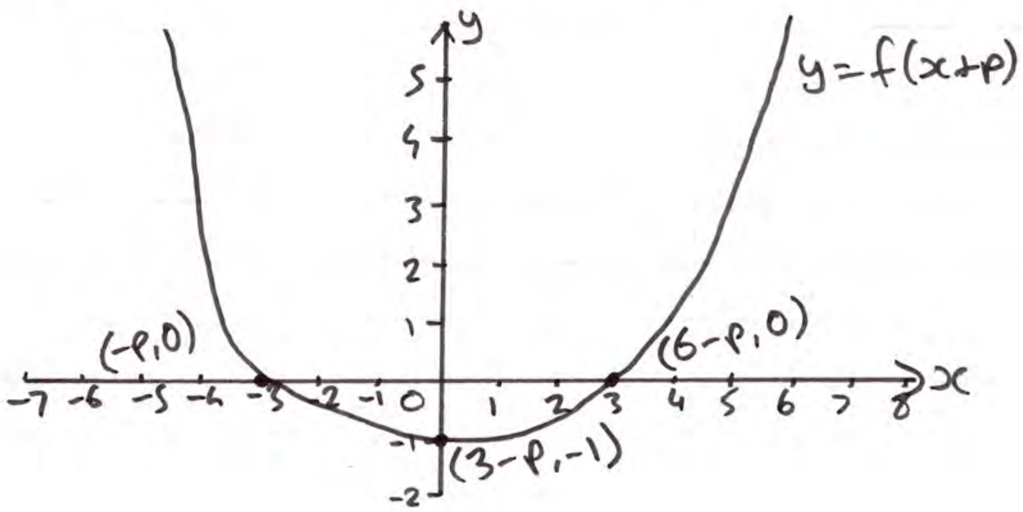
Q8a) $y=f(2x)$ - halve the x-coordinates:



b) $y=-f(x)$ - reflection in the x-axis:



c) $y=f(x+p)$ - transformation horizontally of '$-p$'



Q9a) $2+4+6+\ldots+100$

First term, $a=2$
Common difference, $d=2$

There are 50 even numbers between 1 and 100, so use S_{50}.

$$S_n = \frac{n}{2}(2a+(n-1)d)$$

$$S_{50} = \frac{50}{2}(2(2)+(50-1)2)$$

$$S_{50} = 25(4+(49\times 2))$$

$$S_{50} = 25(4+98)$$

$$S_{50} = 25\times 102$$

$$\boxed{S_{50} = 2550}$$

$\therefore\ 2+4+6+\ldots+100 = 2550$

b) $k+2k+3k+\ldots+100$

i) $U_n = a+(n-1)d$

First term, $a=k$
Common difference, $d=k$

$$100 = k+(n-1)k$$

$$100 = k+kn-k$$

$$100 = kn$$

$$\therefore \boxed{n = \frac{100}{k}}$$

ii) $S_n = \frac{n}{2}(2a + (n-1)d)$

$$S_{\frac{100}{k}} = \frac{\frac{100}{k}}{2}\left(2k + \left(\frac{100}{k} - 1\right)k\right)$$

$$S_{\frac{100}{k}} = \frac{100}{2k}\left(2k + \frac{100k}{k} - k\right)$$

$$S_{\frac{100}{k}} = \frac{100}{2k}\left(k + 100\right)$$

$$S_{\frac{100}{k}} = \frac{100k}{2k} + \frac{10000}{2k}$$

$$\boxed{S_{\frac{100}{k}} = 50 + \frac{5000}{k}}$$

c) $(2k+1), (4k+4), (6k+7), \ldots,$

First term, $a = (2k+1)$
Common difference, $d = (2k+3)$

$U_n = a + (n-1)d$

$U_{50} = (2k+1) + (50-1)(2k+3)$

$U_{50} = (2k+1) + (49)(2k+3)$

$U_{50} = 2k + 1 + 98k + 147$

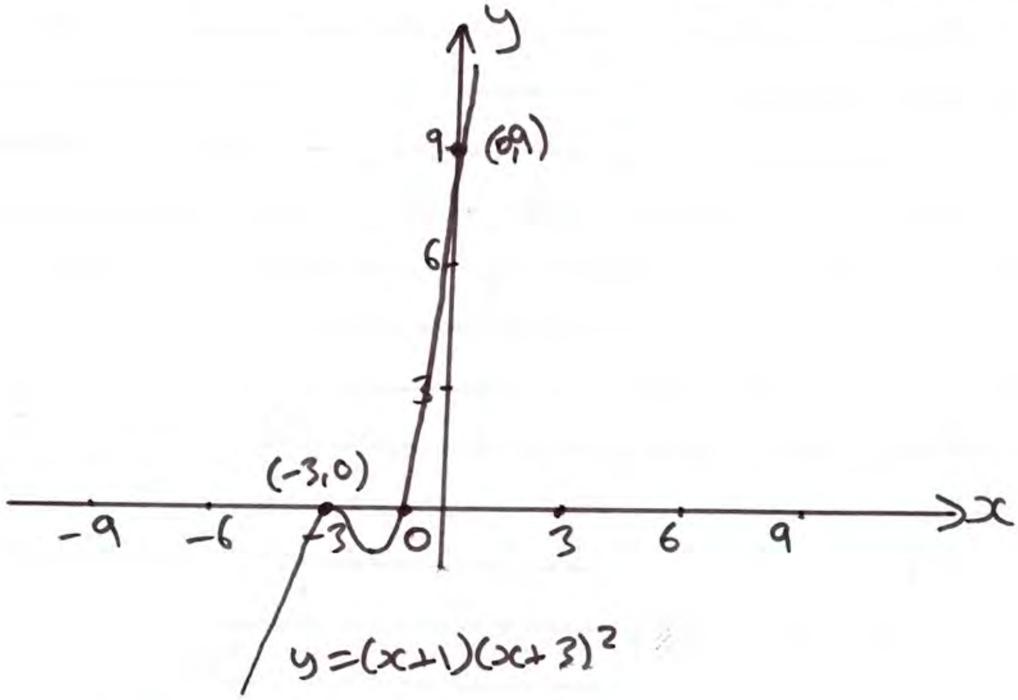
$\boxed{U_{50} = 100k + 148}$

Q10a) $y = (x+1)(x+3)^2$

When $x=0$, $y = (0+1)(0+3)^2 \underline{= 9}$

When $y=0$, $(x+1)(x+3)^2 = 0$

$\underline{\text{Either } x=-1 \text{ or } x=-3 \text{ or } x=-3}$



b) $\frac{dy}{dx} = \frac{dy}{dx}\left((x+1)(x+3)^2\right)$

$y = (x+1)(x+3)^2$

$y = (x+1)(x+3)(x+3)$

$y = (x+1)(x^2+6x+9)$

$y = x^3 + 7x^2 + 15x + 9$

$$\boxed{\frac{dy}{dx} = 3x^2 + 14x + 15}$$

c) When $x = -5$, $\frac{dy}{dx} = 3(-5)^2 + 14(-5) + 15$

$= 3(25) - 70 + 15$

$= 75 - 70 + 15$

$\underline{= 20}$

$\therefore$ the gradient of the tangent at A is $\underline{20}$

When $x = -5$, $y = (-5)^3 + 7(-5)^2 + 15(-5) + 9$

$y = -125 + 7(25) - 75 + 9$

$y = -125 + 175 - 75 + 9$

$\underline{y = -16}$

Equation of tangent at A: $y - y_1 = m(x - x_1)$

Using $A(-5, -16)$ and $m = 20$ $\longrightarrow$ $y - (-16) = 20(x - (-5))$

$y + 16 = 20(x + 5)$

$y + 16 = 20x + 100$

$$\boxed{y = 20x + 84}$$

d) If the tangents at A and B are parallel, then their gradients are both equal.

$$\therefore \frac{dy}{dx} = 20 = 3x^2 + 14x + 15$$

$$\therefore 3x^2 + 14x + 15 = 20$$

$$3x^2 + 14x - 5 = 0$$

$$(3x - 1)(x + 5) = 0$$

Either $x = \frac{1}{3}$ or $x = -5$

We already know the coordinates when $x = -5$, which is $A(-5, -16)$.

So, the point $\boxed{B \text{ has } x\text{-coordinate } \frac{1}{3}.}$