C1 June 2007 (MA)

Q1) $(3+\sqrt{5})(3-\sqrt{5}) = 9 - 3\sqrt{5} + 3\sqrt{5} - 5$

$\boxed{=4}$

Q2a) $8^{\frac{4}{3}} = (\sqrt[3]{8})^4 = 2^4$ $\boxed{=16}$

b) $\dfrac{15x^{\frac{4}{3}}}{3x} = 5x^{\frac{1}{3}}$ $\boxed{= 5\sqrt[3]{x}}$

Q3) $y = 3x^2 + 4\sqrt{x}$, $x > 0$

$y = 3x^2 + 4x^{\frac{1}{2}}$

a) $\dfrac{dy}{dx} = 6x + 2x^{-\frac{1}{2}}$

$\boxed{= 6x + \dfrac{2}{\sqrt{x}}}$

b) $\dfrac{d^2y}{dx^2} = 6 - x^{-\frac{3}{2}}$

$\boxed{= 6 - \dfrac{1}{x^{3/2}}}$

c) $\int (3x^2 + 4x^{\frac{1}{2}})\, dx = \dfrac{3x^3}{3} + \dfrac{4x^{\frac{3}{2}}}{\frac{3}{2}} + c$

$= \boxed{x^3 + \dfrac{8x^{3/2}}{3} + c}$

Q4) First term, $a = 5p$.
Common difference, $d = 2p$

a) $U_n = a + (n-1)d$

$U_{200} = 5 + (200-1)2$

$U_{200} = 5 + (199 \times 2)$

$U_{200} = 398 + 5$

$\boxed{U_{200} = 403p}$

$\therefore$ She saves £4.03 in week 200.

b) $S_n = \frac{n}{2}(2a + (n-1)d)$

$S_{200} = \frac{200}{2}(2(5) + (200-1)2)$

$S_{200} = 100(10 + (199 \times 2))$

$S_{200} = 100(10 + 398)$

$S_{200} = 100 \times 408$

$\boxed{S_{200} = 40800p}$

$\therefore$ She saves £408 over the complete 200 week period

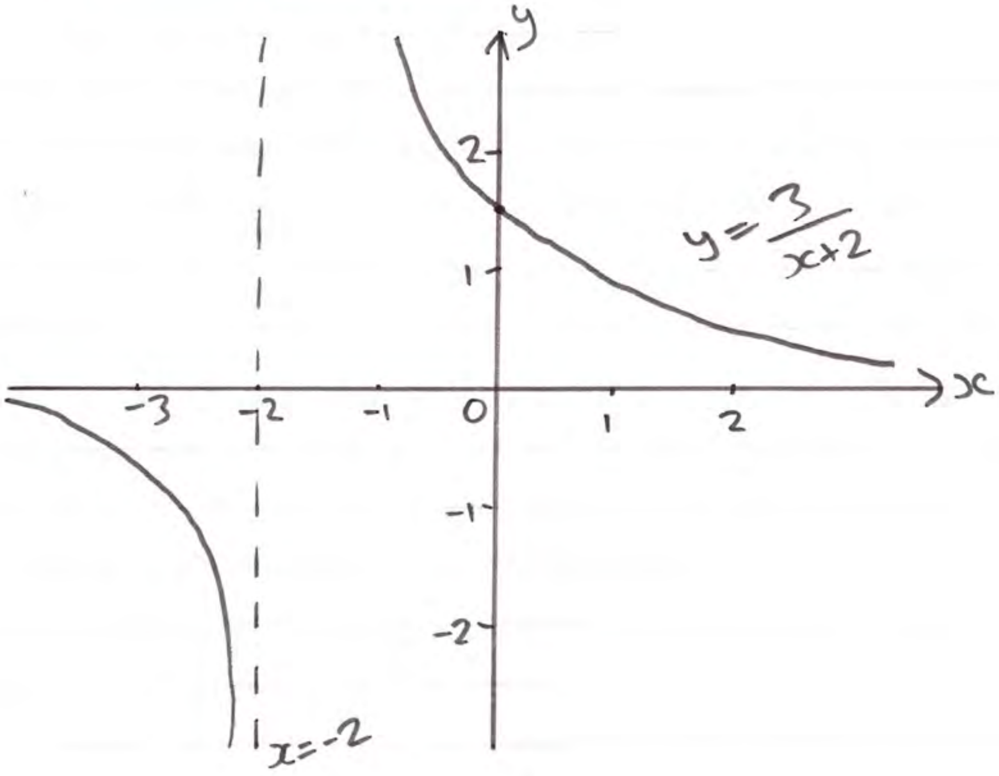
Q5a) Let $f(x) = \frac{3}{x}$, $x \neq 0$

Then $\frac{3}{x+2} \equiv f(x+2)$, which is a transformation of '-2' along the x-axis

When $y=0$, $\frac{3}{x+2} = 0$

$x \to \infty$

When $x=0$, $y = \frac{3}{2}$



b) Equations of the asymptotes for $y = \frac{3}{x+2}$ are:

$\boxed{x = -2 \quad \text{and} \quad y = 0}$

Q6a)

$y = x - 4$ ①
$2x^2 - xy = 8$ ②

Substitute ① into ②:

$2x^2 - x(x-4) = 8$

$2x^2 - x^2 + 4x - 8 = 0$

$$\boxed{x^2 + 4x - 8 = 0}$$

b) $x^2 + 4x - 8 = 0$

$(x+2)^2 - 2^2 - 8 = 0$

$(x+2)^2 - 4 - 8 = 0$

$(x+2)^2 = 12$

$x + 2 = \pm\sqrt{12}$

$x = -2 \pm \sqrt{4 \times 3}$

$x = -2 \pm \sqrt{4}\sqrt{3}$

$\underline{x = -2 \pm 2\sqrt{3}}$

Substitute into ① for y:

When $x = -2 \pm 2\sqrt{3}$, $y = x - 4 \Rightarrow \underline{y = -6 \pm 2\sqrt{3}}$

Solution set: $\boxed{x = -2 + 2\sqrt{3},\ y = -6 + 2\sqrt{3} \text{ and } x = -2 - 2\sqrt{3},\ y = -6 - 2\sqrt{3}}$

Q7) $x^2 + kx + (k+3) = 0$ has different real roots.

a) If a quadratic has different real roots, then the discriminant is greater than 0.

$\therefore b^2 - 4ac > 0$

$k^2 - (4)(1)(k+3) > 0$

$k^2 - 4(k+3) > 0$

$\therefore \boxed{k^2 - 4k - 12 > 0}$

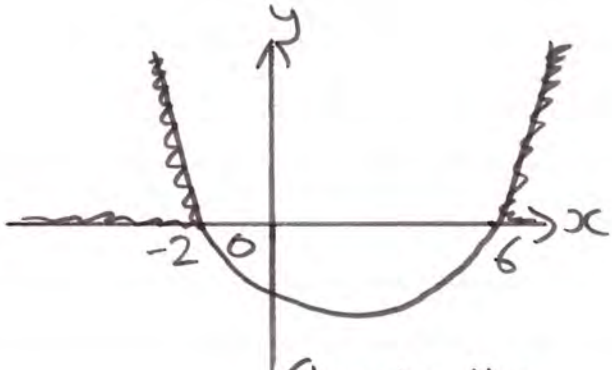
b) For $k^2 - 4k - 12 = 0$,

$(k-6)(k+2) = 0$

Either $k = 6$ or $k = -2$

Set of possible values of k:

$\boxed{k < -2 \text{ or } k > 6}$



Choosing the values 'above' the x-axis, since $k^2 - 4k - 12 > 0$

Q8) $a_1 = k$

$a_{n+1} = 3a_n + 5, \; n \geq 1$

a) $a_2 = 3a_1 + 5$

$\boxed{a_2 = 3k + 5}$

b) $a_3 = 3a_2 + 5$

$a_3 = 3(3k+5) + 5 \Rightarrow \boxed{a_3 = 9k + 20}$

c i) $\sum_{r=1}^{4} a_r = a_1 + a_2 + a_3 + a_4$

$a_4 = 3a_3 + 5$

$a_4 = 3(9k+20) + 5$

$\underline{a_4 = 27k + 65}$

$\therefore \sum_{r=1}^{4} a_r = k + 6k+5) + (9k+20) + (27k+65)$

$= k + 3k + 5 + 9k + 20 + 27k + 65$

$= 40k + 90$

$\boxed{= 10(4k+9)}$

ii) $\dfrac{40k+90}{10} = 4k+9$, which means $40k+90$ is divisible by 10.

Q 9a) $f'(x) = 6x^2 - 10x - 12$

$f(x) = \dfrac{6x^3}{3} - \dfrac{10x^2}{2} - 12x + c$

$\underline{f(x) = 2x^3 - 5x^2 - 12x + c}$

Since $f(x)$ passes through $(5, 65)$, substitute in $x=5$ and $y=65$:

$65 = 2(5)^3 - 5(5)^2 - 12(5) + c$

$65 = 2(125) - 5(25) - 60 + c$

$65 = 250 - 125 - 60 + c$

$65 = 65 + c$

$\therefore \underline{c = 0}$

$\boxed{f(x) = 2x^3 - 5x^2 - 12x}$

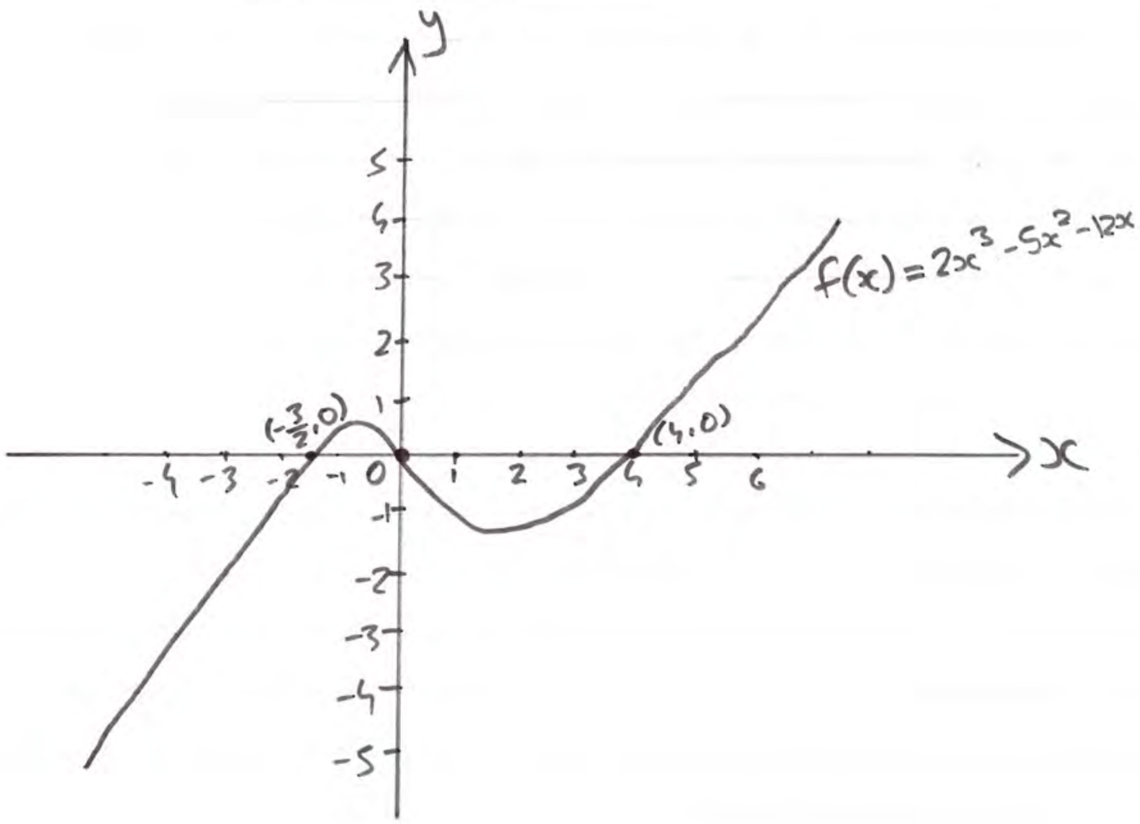
b) $f(x) = x(2x^2 - 5x - 12)$

$\boxed{f(x) = x(2x+3)(x-4)}$

c) When $x=0$, $f(x) = 0(2(0)+3)(0-4) \underline{= 0}$

When $f(x)=0$, $x(2x+3)(x-4)=0$

Either $x=0$ or $x=-\frac{3}{2}$ or $x=4$



Q10) $y = x^2(x-6) + \frac{4}{x}$, $x > 0$

a) When $x = 1$, $y = 1^2(1-6) + \frac{4}{1}$

$y = 1(-5) + 4$

$y = -1$

$\therefore$ P is at $(1, -1)$

When $x = 2$, $y = 2^2(2-6) + \frac{4}{2}$

$y = 4(-4) + 2$

$y = -14$

$\therefore$ Q is at $(2, -14)$

Length PQ $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{(2-1)^2 + (-14-(-1))^2}$

$= \sqrt{1 + (-13)^2}$

$= \boxed{\sqrt{170} \text{ units}}$

b) $y = x^2(x-6) + \frac{4}{x}, \quad x > 0$

$y = x^3 - 6x^2 + 4x^{-1}$

$\frac{dy}{dx} = 3x^2 - 12x - 4x^{-2}$

$= 3x^2 - 12x - \frac{4}{x^2}$

At $P(1,-1)$, $\frac{dy}{dx} = 3(1)^2 - 12(1) - \frac{4}{1^2}$

$= 3 - 12 - 4$

$= -13$

$\therefore$ the gradient of the tangent at P is -13.

At $Q(2,-14)$, $\frac{dy}{dx} = 3(2)^2 - 12(2) - \frac{4}{2^2}$

$= 12 - 24 - 1$

$= -13$

$\therefore$ the gradient of ~~th~~gent at Q is -13.

Since the tangents at P and Q are both gradients of -13, they are parallel.

c) If the gradient of the tangent at P is -13, then the gradient of the normal at P is $\frac{1}{13}$.

Equation of normal at P: $y - y_1 = m(x - x_1)$

Using $P(1, -1)$ and $m = \frac{1}{13}$ $\longrightarrow$ $y - (-1) = \frac{1}{13}(x - 1)$

$$13(y+1) = 1(x-1)$$

$$13y + 13 = x - 1$$

$$\boxed{x - 13y - 14 = 0}$$

Q11) $l_1: y = 3x + 2$ $\quad$ $l_2: 3x + 2y - 8 = 0$

a) Gradient of l_2: $2y = -3x + 8$

$$y = -\frac{3}{2}x + 4$$

$\therefore$ $\boxed{\text{Gradient of } l_2 = -\frac{3}{2}}$

b) To find coordinates of the point of intersection, solve simultaneous equations:

$y = 3x + 2$ ①
$3x + 2y - 8 = 0$ ②

Substitute ① into ②

$$3x+2(3x+2)-8=0$$

$$3x+6x+4-8=0$$

$$9x-4=0$$

$$9x=4$$

$$x=\frac{4}{9}$$

Substitute into ① for y:

$$y=3x+2$$

When $x=\frac{4}{9}$, $y=3\left(\frac{4}{9}\right)+2$

$$y=\frac{12}{9}+2$$

$$y=\frac{4}{3}+2$$

$$y=\frac{10}{3}$$

$\therefore$ the coordinates of the point of intersection, P are $\boxed{\left(\frac{4}{9},\frac{10}{3}\right)}$

c) For l_1, when $y=1$, $1=3x+2$

$$-1=3x$$

$$x=-\frac{1}{3}$$

$\therefore$ A is at $\left(-\frac{1}{3}, 1\right)$

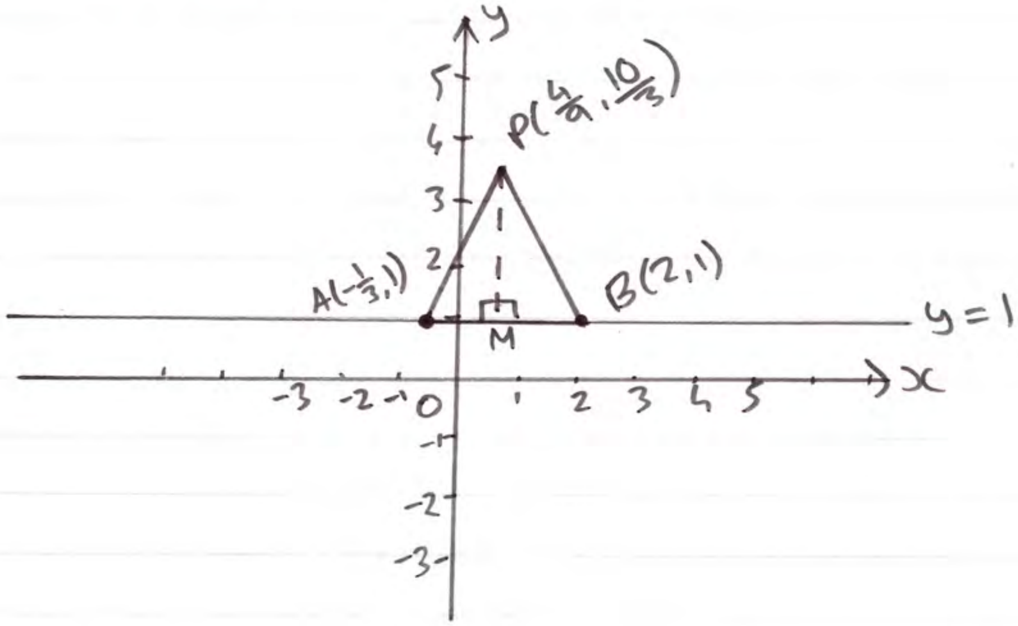
For l_2, when $y=1$, $3x+2(1)-8=0$

$$3x+2-8=0$$

$$3x=6$$

$$x=2$$

$\therefore$ B is at $(2,1)$



$$\text{Length } AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(2-(-\tfrac{1}{3}))^2 + (1-1)^2}$$

$$= \sqrt{(\tfrac{7}{3})^2 + 0^2}$$

$$= \sqrt{\tfrac{49}{9}}$$

$$= \frac{7}{3} \text{ units}$$

$$\text{Length } PM = \frac{10}{3} - 1 = \frac{7}{3} \text{ units}$$

$$\text{Area } \Delta ABP = \frac{1}{2}bh = \frac{1}{2}(AB)(PM)$$

$$= \frac{1}{2}\left(\frac{7}{3}\right)\left(\frac{7}{3}\right)$$

$$= \frac{1}{2}\left(\frac{49}{9}\right)$$

$$= \boxed{\frac{49}{18} \text{ units}^2}$$