AQA Computer Science A-Level

4.12.1 Functional programming paradigm

Advanced Notes

Specification:

4.12.1.1 Function type

Know that a function, f, has a function type f: A → B (where the type is A → B, A is the argument type, and B is the result type).

Know that A is called the domain and B is called the co-domain.

Know that the domain and co-domain are always subsets of objects in some data type.

Loosely speaking, a function is a rule that, for each element in some set A of inputs, assigns an output chosen from set B, but without necessarily using every member of B. For example, f: {a,b,c,…z} → {0,1,2,…,25} could use the rule that maps a to 0, b to 1, and so on, using all values which are members of set B.

The domain is a set from which the function's input values are chosen.

The co-domain is a set from which the function's output values are chosen. Not all of the codomain's members need to be outputs.

4.12.1.2 First-class object

Know that a function is a first-class object in functional programming languages and in imperative programming languages that support such objects.

This means that it can be an argument to another function as well as the result of a function call.

First-class objects (or values) are objects which may:

- appear in expressions
- be assigned to a variable
- be assigned as arguments
- be returned in function calls.

For example, integers, floating-point values, characters and strings are first class objects in many programming languages.

4.12.1.3 Function application

Know that function application means a function applied to its arguments.

The process of giving particular inputs to a function is called function application, for example add(3,4) represents the application of the function add to integer arguments 3 and 4. The type of the function is f: integer x integer → integer where integer x integer is the Cartesian product of the set integer with itself. Although we would say that function f takes two arguments, in fact it takes only one argument, which is a pair, for example (3,4).

4.12.1.4 Partial function application

Know what is meant by partial function application for one, two and three argument functions and be able to use the notations shown opposite. The function add takes two integers as arguments and gives an integer as a result. Viewed as follows in the partial function application scheme: add: integer → (integer → integer) add 4 returns a function which when applied to another integer adds 4 to that integer. The brackets may be dropped so function add becomes add: integer → integer → integer The function add is now viewed as taking one argument after another and returning a result of data type integer.

4.12.1.5 Composition of functions

Know what is meant by composition of functions. The operation functional composition combines two functions to get a new function. Given two functions f: A → B g: B → C function g ○ f, called the composition of g and f, is a function whose domain is A and co-domain is C. If the domain and co-domains of f and g are $\mathbb{R}$, and f(x) = (x + 2) and g(y) = y3 . Then g ○ f = (x + 2)3 f is applied first and then g is applied to the result returned by f.

Functions

According to the AQA specification, *a function is a rule that, for each element in some set A of inputs, assigns an output chosen from set B, but without necessarily using every member of B.*

An argument (a piece of data) is passed to a function, and the rule is applied to the argument, creating the return value (the output). An argument could be a number (0, 1, 3.4, -8 e.t.c), a character ("a", "D", "!" e.t.c) or any other data type. The argument could even be another function (if the language supports it). The function will specify what data type is required for the argument.

Function Example 1:

A function is called DoubleMe. This outputs the double of the input. The inputs (set A) could be the set of natural numbers {0, 1, 2, 3, 4….}, and therefore the output (set B) could be the set of even natural numbers {0, 2, 4, 6, 8 ….}.

Synoptic Link

The set of **Natural Numbers** is the set of positive whole numbers including 0. I.e. {0,1,2,3…}

Natural Numbers are covered in **Number Systems** under **Fundamentals of Data Representation**.

Synoptic Link

A **set** is an **abstract data type** which stores **unordered unique values**. A **subset** of a set **only** contains item from the set (it may be **equal** to the set); a proper subset **only** contains item from the set but must have **less items** than the set (cannot be equal to the set).

Subsets and proper subsets are covered in **Regular Languages** under **Theory of Computation**.

Of course, the set of even natural numbers is a proper subset of the set A. Therefore set B could otherwise be the set of natural numbers, although the odd items would never be outputted.

The function DoubleMe could be called with a value of 6. The output would be the double of 6, which is 12. 6 is called the argument, and 12 is the result. Here is the code for such an interaction.

```
DoubleMe 6
>> 12
```

Function Example 2:

A function is called HalfMe. This outputs half of the input. The input (Set A) could be the set of positive integers {0, 1, 2, 3, 4….} and the output (Set B) could be drawn from the set of real numbers. Most of the real numbers will never be

Synoptic Link

The set of **integers** is the set of **whole numbers** {...-1,0,1,2...}. The set of **reals** is the set of all **non-imaginary** numbers, including **negatives**, **irrationals**, and **decimals**.

Integers and reals are covered in **Number Systems** under **Fundamentals of Data Representation**.

outputted (e.g. 3.201 is not half of an integer), but the real numbers is still an appropriate set.

Below is an example of how HalfMe might be used.

```
HalfMe 200
>> 100

HalfMe 5
>> 2.5
```

Function Example 3:

A function is called LetterPosition. This outputs the position of each letter in the alphabet. The inputs - Set A - would be the English Alphabet {A, B, C, … Z} and the output - Set B - would be the set of whole numbers (integers) between 1 to 26 inclusive {1, 2, 3, … 26}.

Here is an example of how LetterPosition could be used.

```
LetterPosition N
>> 14
```

Function Types

All functions have a function type. If f is the function, A is the input and B is the output, the function type can be defined as the following:

$$f: A \rightarrow B$$

Argument

Parameters are the variables/items in the declaration of the function. The argument is the actual value passed into the parameter when the function is called.

A is known as the argument type, and B is the result type. This means function f maps A to B. In computer science, we describe the the set of inputs (A) as the domain, and the set of outputs (B) as the co-domain. Remember, not all members of the co-domain have to be used as outputs. The domain and co-domain are always subsets of objects in some data type, as further explained below.

Function Types Example 1:

The function f returns double the input. Hence, this function type could be described as the following:

$$f: \{0, 1, 2, 3, 4\ldots.\} \rightarrow \{0, 2, 4, 6, 8\ldots.\}$$

The set of natural numbers is the domain and the set of even natural numbers is the co-domain.

Domain — Co-domain

$$f: \{0, 1, 2, 3, 4\ldots.\} \rightarrow \{0, 2, 4, 6, 8\ldots.\}$$

Natural Numbers — Even Natural Numbers

In this example, the programmer only allows positive integers to be doubled. This function could not be used to double a negative number or a decimal value.

The domain is a subset of the natural numbers, and a proper subset of the integers and reals. The co-domain is a proper subset of the naturals, integers and reals.

However, the functionality of f could be changed if it had a different function type.

$$f: real \rightarrow real$$

If f was declared with this function type any real number could be doubled including negatives and decimal values.

Synoptic Link

A **set** is an **abstract data type** which stores **unordered unique values**. A **subset** of a set **only** contains item from the set (it may be **equal** to the set); a proper subset **only** contains item from the set but must have **less items** than the set (cannot be equal to the set).

Subsets and proper subsets are covered in **Regular Languages** under **Theory of Computation.**

Function Types Example 2:

The function g returns half of the input. It's function type could be described as below:

$$g: \{0, 1, 2, 3, 4\ldots\} \rightarrow real$$

The set of natural numbers is the domain and the set of reals is the co-domain.

This function can only be used to half positive integers. Therefore not all of the reals could be used as outputs (e.g. negative numbers).

The domain is a subset of the natural numbers, and a proper subset of the integers and reals. The co-domain is a subset of the reals.

If the programmer wanted to only half integers, the domain could be the set of integers.

```
g: { … -2, -1, 0, 1, 2, 0 …} → real
```

,

If the programmer defined the function as

```
g: real → real
```

then any non-imaginary number could be halved by the function.

First-class object

Taken from the specification:

First-class objects (or values) are objects which may:

- *appear in expressions*
- *be assigned to a variable*
- *be assigned as arguments*
- *be returned in function calls*

In functional programming, functions are first-class objects. Some imperative languages also support functions as first-class objects. Hence, functions can be passed as arguments or returned as the result of another function. Examples of first-class objects include integers, floating-point values, characters and strings.

Synoptic Link

In **imperative programming**, you must specify **how** to achieve the desired result. In a **declarative language**, you simply **describe** the **final result**.

Imperative and declarative languages are covered in **Classification of Programming Languages** under **Fundamentals of Computer Systems**.

Function application

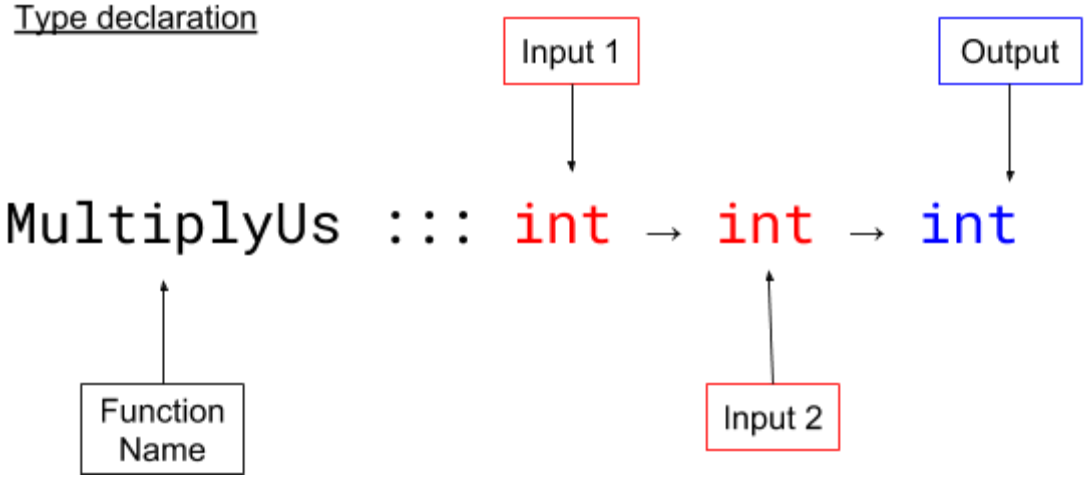
Function application is just a fancy term for applying the function rule to the arguments of the function.

Function Application Example 1:

MultiplyUs(x,y) returns the product of x and y. First, we need a function type declaration. In this instance we will only be multiplying integers - two integers multiplied together will always produce another integer.

```
MultiplyUs ::: int → int → int
```

The first two “int”s signify the inputs, and the last int is the output.



Next we need to specify what the function will be doing with the inputs.

```
MultiplyUs x y = x * y
```

The x and y represent the parameters - variables created when the function is declared, into which the arguments (data) is passed.

Now we are ready for function application. In this instance, the arguments are 3 and 5.

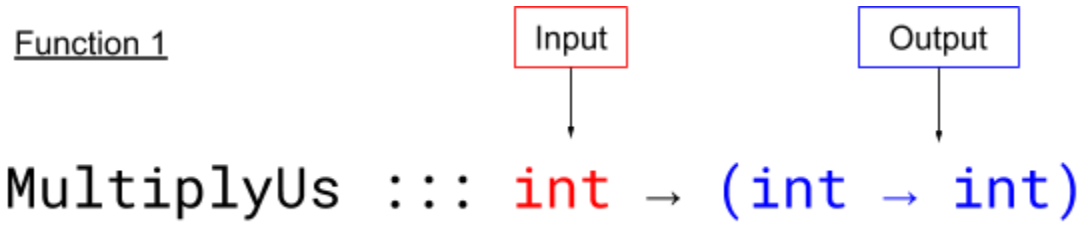
```
MultiplyUs 3 5

>> 15
```

Although it may look like the multiply function is taking two arguments, it is in fact only taking one - a pair. Every function in a functional programming language (e.g. Haskell) only takes one argument. How can this be true? If we look back at the type declaration, it clearly has two inputs. However, the type declaration can also be written as this:

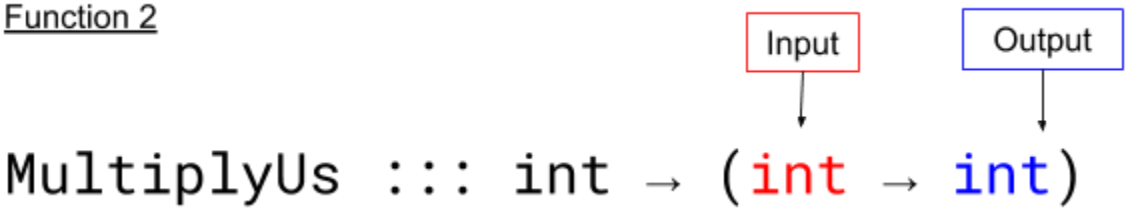
```
MultiplyUs ::: int → (int → int)
```

Now we have two functions, each with one input and one output.

Function 1



Function 2



Function 1 is called first.

```
MultiplyUs ::: int → (int → int)

MultiplyUs     3   → MultiplyYBy3 5
```

As you can see, the first function has created a new function based off the input. The output of MultiplyUs 3 is MultiplyYBy3 5 (remember in this example y = 5). This new function is called.

```
MultiplyUs ::: int → (int → int)

MultiplyYBy3                5   →   15
```

The output is 15. All inputs have been dealt with so MultiplyUs 3 5 returns 15.

Partial function application

Partial function application takes advantage of the inability of a function to take more than one input. In partial function application, one of the arguments is fixed, leading to a more restricted, specialised function.

Partial function application example:

The function Add3Numbers should add the three numbers given as arguments and output the total. Add3Numbers could be created as thus:

```
Add3Numbers ::: real -> real -> real -> real

Add3Numbers x y z = x + y + z
```

In this case, the inputs and outputs are real. This function can take decimals and negative numbers as arguments. Add3Numbers could be called as thus:

```
Add3Numbers   5   7   3.2

>> 15.2
```

However, we could use partial function application to bind one of the arguments. Consider the new function Add3ToTwoNumbers. This new function should take a pair of arguments, add them together and then add 3, before returning the output. We could write a new definition for this function, or we could use partial function application to modify Add3Numbers.

```
Add3ToTwoNumbers ::: real -> real -> real

Add3ToTwoNumbers = Add3Numbers 3
```

Calling Add3Numbers 3 would ordinarily cause an error, as there are not enough inputs. However, by defining Add3ToTwoNumbers in terms of Add3Numbers, Add3ToTwoNumbers can be called with two additional inputs.

```
Add3ToTwoNumbers   4  1

>> 8
```

Composition of functions

Functional composition is the act of combining two functions to create a new function. The benefit of this is that the user is able to use the functions both separately, and in conjunction. Any two functions can be combined as long as the domain of one of the functions is the same as the co-domain of the other. The symbol ○ indicates that two functions are being combined e.g. Add2 ○ DoubleMe. The function works from the inside out. Hence in the previous example, the input would be doubled first, then 2 would be added to it.

Composition of functions Example:

Function f takes an argument pair. It adds them together, and then doubles the answer.

```
f ::: real -> real -> real

f( x, y ) = (x + y) * 2
```

The function g squares its input.

```
g ::: real -> real

g( x ) = x * x
```

How can f and g be combined?

```
f ::: real -> real -> real

g ::: real -> real
```

To be combined, the domain of one function must be the same as the co-domain of the other function.

The domain of f is a pair of reals; the co-domain of g is one real. Therefore they cannot be combined as f ○ g.

The domain of g is a real; the co-domain of f is also a real. Therefore they can be combined as g ○ f.

The domain of g ○ f is the domain of f (a pair of reals), and the co-domain of g ○ f is the co-domain of g (a single real).

Calling g ○ f with input (3,5) would produce the following:

```
g ○ f (3,4)
```

Step 1
Function f is applied on the inputs

```
f (3,4) = (3 + 4) * 2 = 14
```

Step 2
The output of function f is the argument passed to g

```
g (14) = 14 * 14 = 196
```

Step 3
Answer is outputted

```
>> 196
```