Cambridge Assessment
Admissions Testing

TEST OF MATHEMATICS FOR UNIVERSITY ADMISSION

D513/01

PAPER 1

November 2021

75 minutes

0456100783

Additional materials: Answer sheet

INSTRUCTIONS TO CANDIDATES

Please read these instructions carefully, but do not open the question paper until you are told that you may do so.

A separate answer sheet is provided for this paper. Please check you have one. You also require a soft pencil and an eraser.

Please complete the answer sheet with your candidate number, centre number, date of birth, and full name.

This paper is the first of two papers.

There are 20 questions on this paper. For each question, choose the one answer you consider correct and record your choice on the separate answer sheet. If you make a mistake, erase thoroughly and try again.

There are no penalties for incorrect responses, only marks for correct answers, so you should attempt **all** 20 questions. Each question is worth one mark.

You can use the question paper for rough working or notes, but **no extra paper** is allowed.

You **must** complete the answer sheet within the time limit.

Calculators and dictionaries are NOT permitted.

There is no formulae booklet for this test.

Please wait to be told you may begin before turning this page.

PV2

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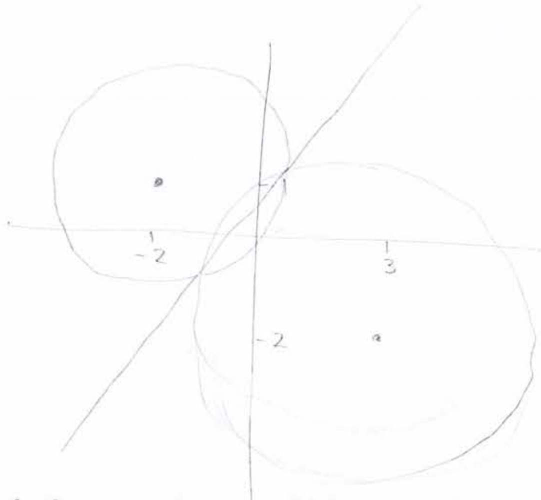
1 Two circles have the same radius.

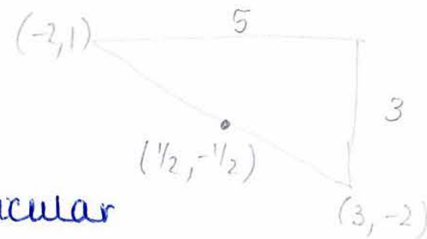


The centre of one circle is (–2, 1).

The centre of the other circle is (3, –2).

The circles intersect at two distinct points.

What is the equation of the straight line through the two points at which the circles intersect?



A $3x - 5y = 4$

B $3x + 5y = -1$

C $5x - 3y = -4$

D $5x - 3y = -1$

E $5x - 3y = 1$

F $5x - 3y = 4$

G $5x + 3y = 1$

Gradient = $-\frac{3}{5}$

$\therefore$ gradient of perpendicular bisector = $5/3$

Midpoint = $(1/2, -1/2)$

$y = mx + c$

$y = \frac{5}{3}x + c$

$-1/2 = 5/3 \times 1/2 + c$

$-1/2 = 5/6 + c$

$c = -\frac{1}{2} - \frac{5}{6} = \frac{-3}{6} - \frac{5}{6} = \frac{-8}{6} = -\frac{4}{3}$

so $y = \frac{5}{3}x - \frac{4}{3}$

$3y = 5x - 4$

$\underline{5x - 3y = 4}$

2 The curve $y = x^3 - 6x + 3$ has turning points at $x = \alpha$ and $x = \beta$, where $\beta > \alpha$.

Find

$$\int_{\alpha}^{\beta} x^3 - 6x + 3 \,\mathrm{d}x$$

A $-8\sqrt{2}$

B -10

C $-10 + 6\sqrt{2}$

D 0

E $12 - 8\sqrt{2}$

F $6\sqrt{2}$

G 12

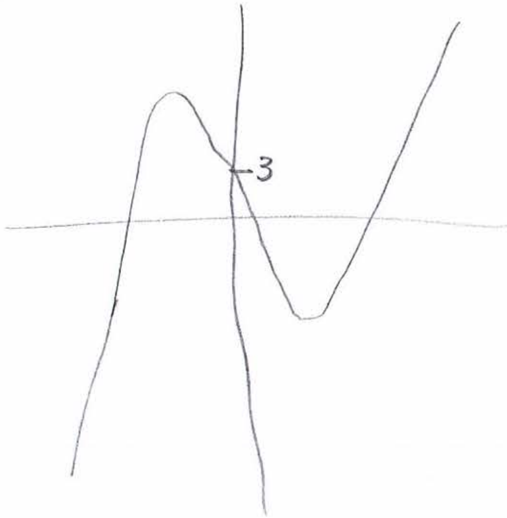
$\frac{dy}{dx} = 3x^2 - 6$

$= 0$ at turning points

$3x^2 = 6$

$x^2 = 2$

$x = \pm\sqrt{2}$



so $\beta = \sqrt{2}$, $\alpha = -\sqrt{2}$

$$\int_{\alpha}^{\beta} x^3 - 6x + 3 \, dx = \frac{x^4}{4} - \frac{6x^2}{2} + 3x \Big|_{-\sqrt{2}}^{\sqrt{2}}$$

$$= \left(\frac{(\sqrt{2})^4}{2} - 3(\sqrt{2})^2 + 3\sqrt{2}\right) - \left(\frac{(-\sqrt{2})^4}{4} - 3(\sqrt{2})^2 - 3\sqrt{2}\right)$$

$$= \frac{4}{2} - 3 \times 2 + 3\sqrt{2} - \left(\frac{4}{2} - 3 \times 2 - 3\sqrt{2}\right)$$

$$= \underline{\underline{6\sqrt{2}}}$$

3 An arithmetic progression and a convergent geometric progression each have first term $\frac{1}{2}$

The sum of the second terms of the two progressions is $\frac{1}{2}$ $\frac{1}{2}+d+\frac{1}{2}r=\frac{1}{2}$ ①

The sum of the third terms of the two progressions is $\frac{1}{8}$ $\frac{1}{2}+2d+\frac{1}{2}r^2=\frac{1}{8}$ ②

What is the sum to infinity of the geometric progression?

A -2

B -1

C $-\frac{1}{2}$

D $-\frac{1}{3}$

E $\frac{1}{3}$

F $\frac{1}{2}$

G 1

H 2

AP : $\frac{1}{2}, \frac{1}{2}+d, \frac{1}{2}+2d, \ldots$

GP : $\frac{1}{2}, \frac{1}{2}r, \frac{1}{2}r^2, \ldots$

① $\frac{1}{2}+d+\frac{1}{2}r=\frac{1}{2}$

$d+\frac{1}{2}r=0$

$\boxed{r=-2d}$

② $\frac{1}{2}+2d+\frac{1}{2}r^2=\frac{1}{8}$

$\boxed{\frac{1}{2}r^2+2d=-\frac{3}{8}}$

then $\frac{1}{2}r^2-r=-\frac{3}{8}$

$r^2-2r=-\frac{3}{4}$

$r^2-2r+\frac{3}{4}=0$

$r=\frac{2\pm\sqrt{4-3}}{2}=\frac{2\pm1}{2}=\cancel{\frac{3}{2}}$ or $\frac{1}{2}$

so $S_\infty=\frac{r}{1-r}=\frac{\frac{1}{2}}{1-\frac{1}{2}}=\frac{1}{2}$

4 Find the minimum value of the function

$$2^{2x} - 2^{x+3} + 4$$

$= (2^x)^2 - 2^x 2^3 + 2^2$

$= (2^x)^2 - 8 \times 2^x + 4$

$= (2^x - 4)^2 - 16 + 4$

$= (2^x - 4)^2 - 12$

$\therefore$ minimum is $\underline{\underline{-12}}$

A -16

B -12

C -8

D 0

E 4

F 20

5 The function f is such that

$$f(mn) = \begin{cases} f(m)f(n) & \text{if } mn \text{ is a multiple of 3} \\ mn & \text{if } mn \text{ is not a multiple of 3} \end{cases}$$

for all positive integers m and n.

Given that $f(9) + f(16) - f(24) = 0$, what is the value of $f(3)$?

A $\frac{8}{3}$

B $2\sqrt{2}$

C 3

D $\frac{16}{5}$

E $3\sqrt{2}$

F 4

$f(9) = f(3 \times 3) = f(3)f(3) = [f(3)]^2$

16 isn't a multiple of 3 so $f(16) = 16$

$f(24) = f(8 \times 3) = f(8)f(3) = 8f(3)$

$f(9) + f(16) - f(24) = [f(3)]^2 + 16 - 8f(3) = 0$

let $y = f(3)$ then $y^2 - 8y + 16 = 0$

$(y-4)^2 = 0$

$y = 4$

so $\underline{f(3) = 4}$

6 The function f is given by

$$f(x) = \frac{\cos x + 3}{7 + 5\cos x - \sin^2 x}$$

Find the positive difference between the maximum and the minimum values of $f(x)$.

A 0

B $\frac{1}{3}$

C $\frac{1}{2}$

D $\frac{2}{3}$

E 1

F 2

$$f(x) = \frac{\cos x + 3}{7 + 5\cos x - 1 + \cos^2 x}$$

$$= \frac{\cos x + 3}{6 + \cos^2 x + 5\cos x}$$

$$= \frac{\cancel{\cos x + 3}}{(\cancel{\cos x + 3})(\cos x + 2)}$$

$$= \frac{1}{\cos x + 2}$$

minimum value when $\cos x = 1$, ie $1/3$

maximum $= -1$ 1

$\therefore$ difference between min & max $= \underline{\underline{2/3}}$

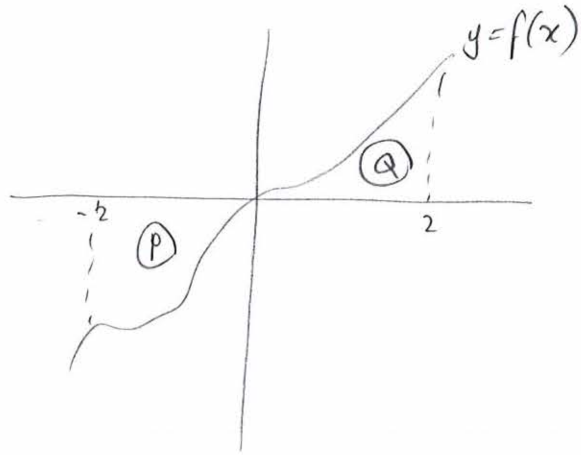
7 The function f is such that $f(0) = 0$, and $xf(x) > 0$ for all non-zero values of x.

It is given that

$$\int_{-2}^{2} f(x)\, dx = 4$$

and

$$\int_{-2}^{2} |f(x)|\, dx = 8$$



Evaluate

$$\int_{-2}^{0} f(|x|)\, dx$$

A -8

B -6

C -4

D -2

E 2

F 4

G 6

H 8

$$\int_{-2}^{2} f(x)\, dx = Q - P$$
$$= 4$$

$$\int_{-2}^{2} |f(x)|\, dx = Q + P$$
$$= 8$$

So $2Q = 8 + 4 = 12$ so $Q = 6$ $\therefore$ $P = 2$

Then $$\int_{-2}^{0} f(|x|)\, dx = \int_{0}^{2} f(x)\, dx = Q = \underline{\underline{6}}$$

8 The line $y = 2x + 3$ meets the curve $y = x^2 + bx + c$ at exactly one point. (1)

The line $y = 4x - 2$ also meets the curve $y = x^2 + bx + c$ at exactly one point. (2)

What is the value of $b - c$?

A -9

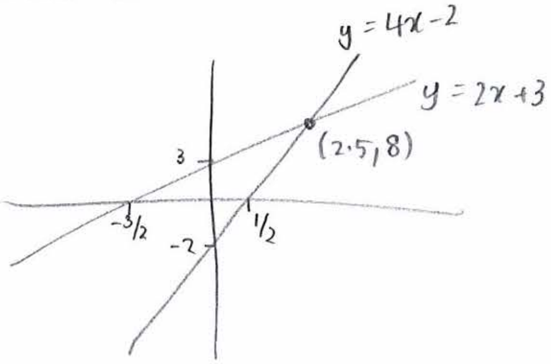
B -5.5

C -1

D 5

E 6

F 14



(1) so $x^2 + bx + c = 2x + 3$

$x^2 + (b-2)x + (c-3) = 0$

must have one soln ∴ discriminant = $\boxed{(b-2)^2 - 4(c-3) = 0}$ (3)

(2) $x^2 + bx + c = 4x - 2$

$x^2 + (b-4)x + (c+2) = 0$

also must have one soln ∴ $\boxed{(b-4)^2 - 4(c+2) = 0}$ (4)

(3) - (4) gives $(b-2)^2 - (b-4)^2 - 4(c-3) + 4(c+2) = 0$

$b^2 - 4b + 4 - (b^2 - 8b + 16) - 4c + 12 + 4c + 8 = 0$

$4b - 12 + 20 = 0$

$4b + 8 = 0$

$4b = -8$

$b = -2$

sub into (3) and $(-4)^2 - 4(c-3) = 0$

$16 - 4c + 12 = 0$

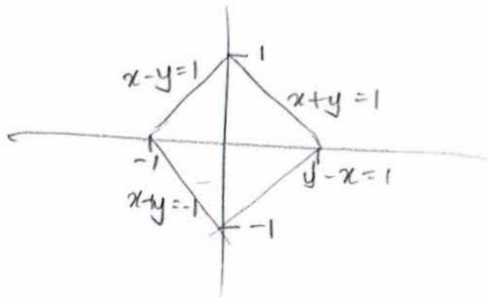
$4c = 28$

$c = 7$

so $b - c = -2 - 7 = \underline{\underline{-9}}$

9 Find the area enclosed by the graph of

$$|x| + |y| = 1$$



A $\frac{1}{2}$

B 1

C 2

D 4

E $\frac{1}{2}\sqrt{2}$

F $\sqrt{2}$

G $2\sqrt{2}$

$x \geq 0$ & $y \geq 0$ eq^n is $x+y=1$

$x \leq 0$ & $y \geq 0$ $\quad y-x=1$

$x \leq 0$ & $y \leq 0$ $\quad x+y=-1$

$x \geq 0$ & $y \leq 0$ $\quad x-y=1$

square side length $\sqrt{1^2+1^2} = \sqrt{2}$

$\therefore$ area enclosed $= \sqrt{2} \times \sqrt{2} = \underline{\underline{2}}$

Trapezium rule $\int_{x_0}^{x_n} f(x)\,dx = \frac{1}{2}h\left[(y_0+y_n)+2(y_1+y_2+\dots+y_{n-1})\right]$

10 Use the trapezium rule with 3 strips to estimate

$$\int_{\frac{1}{2}}^{2} 2\log_{10} x\,\mathrm{d}x$$ Strips width $1/2$

A $\log_{10}\dfrac{\sqrt{6}}{2}$

B $\log_{10}\dfrac{3}{2}$

C $\log_{10}\dfrac{9}{4}$

D $\log_{10} 3$

E $\log_{10}\dfrac{81}{16}$

F $\log_{10}\dfrac{\sqrt{23}}{2}$

So $\int_{1/2}^{2} 2\log_{10}x\,dx \approx \frac{1}{2}\times\frac{1}{2}\left[(2\log_{10}1/2 + 2\log_{10}2) + 2(2\log_{10}1 + 2\log_{10}3/2)\right]$

$= \frac{1}{4}\left[(2\log_{10}2^{-1} + 2\log_{10}2) + 2(0 + 2\log_{10}3/2)\right]$

$= \frac{1}{4}\left[-2\log_{10}2 + 2\log_{10}2 + 4\log_{10}3/2\right]$

$= \frac{1}{4}\left[4\log_{10}3/2\right]$

$= \underline{\underline{\log_{10}3/2}}$

11 The function f is given by

$$f(x) = x^{\frac{1}{7}}(x^2 - x + 1)$$
$$= x^{15/7} - x^{8/7} + x^{1/7}$$

Find the fraction of the interval $0 < x < 1$ for which $f(x)$ is decreasing.

A $\frac{2}{15}$

B $\frac{1}{5}$

C $\frac{1}{3}$

D $\frac{1}{2}$

E $\frac{2}{3}$

F $\frac{4}{5}$

G $\frac{13}{15}$

$f(x)$ decreasing where $f'(x) < 0$

$$f'(x) = \frac{15}{7}x^{8/7} - \frac{8}{7}x^{1/7} + \frac{1}{7}x^{-6/7}$$

$$= \frac{1}{7}\left(15x^{8/7} - 8x^{1/7} + x^{-6/7}\right)$$

$$= \frac{x^{-6/7}}{7}(15x^2 - 8x + 1)$$

$$= \frac{x^{-6/7}}{7}(3x-1)(5x-1)$$

$$= \underbrace{\frac{x^{-6/7}}{7 \times 15}}_{>0}(x - 1/5)(x - 1/3)$$

Need $(x - 1/5)(x - 1/3) \leq 0$ ie $1/5 < x < 1/3$

$$\frac{1}{3} - \frac{1}{5} = \underline{\underline{\frac{2}{15}}}$$

12 The minimum value of the function $x^4 - p^2x^2$ is -9

p is a real number.

Find the minimum value of the function $x^2 - px + 6$

A -3

B $6 - \frac{3\sqrt{2}}{2}$

C $\frac{3}{2}$

D 3

E $\frac{9}{2}$

F $6 + \frac{3\sqrt{2}}{2}$

$x^4 - p^2x^2 = \left(x^2 - \frac{p^2}{2}\right)^2 - \frac{p^4}{4}$

min value when $\left(x^2 - \frac{p^2}{2}\right)^2 = 0$

ie. $-\frac{p^4}{4} = -9$

$p^4 = 36$

$p^2 = 6$

$x^2 - px + 6 = \left(x - \frac{p}{2}\right)^2 - \frac{p^2}{4} + 6$

min. value of this is $-\frac{p^2}{4} + 6 = \frac{-6}{4} + 6 = 4\frac{1}{2} = \underline{\underline{\frac{9}{2}}}$

13 The function f is such that, for every integer n

$$\int_n^{n+1} f(x)\,dx = n+1$$

$$= \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + \ldots + \int_{n-1}^{n} f(x)\,dx$$

Evaluate

$$\sum_{r=1}^{8}\left(\int_0^r f(x)\,dx\right) = 1+2+3+\ldots+n$$

$$= \frac{1}{2}n(n+1)$$

A 36

B 84

C 120

D 165

E 204

F 288

Either: $\int_0^1 f(x)\,dx + \int_0^2 f(x)\,dx + \ldots + \int_0^8 f(x)\,dx$

$= 8\int_0^1 f(x)\,dx + 7\int_1^2 f(x)\,dx + \ldots + \int_7^8 f(x)\,dx$

$= 8\times1 + 7\times2 + 6\times3 + \ldots + 2\times7 + 1\times8$

$= 8+14+18+20+20+18+14+8$

$= \underline{\underline{120}}$

or: $\sum_{r=1}^{8}\left(\int_0^r f(x)\,dx\right) = \sum_{r=1}^{8}\left(\frac{1}{2}r(r+1)\right)$

$= \frac{1}{2}(1+1) + (2+1) + \ldots + \frac{7}{2}\times 8 + 4\times 9$

$= 1+3+\ldots+28+36$

$= \underline{\underline{120}}$

14 *This question uses radians.*

Find the number of distinct values of x that satisfy the equation

$$(x+1)(3-x) = 2(1-\cos(\pi x))$$

A 2

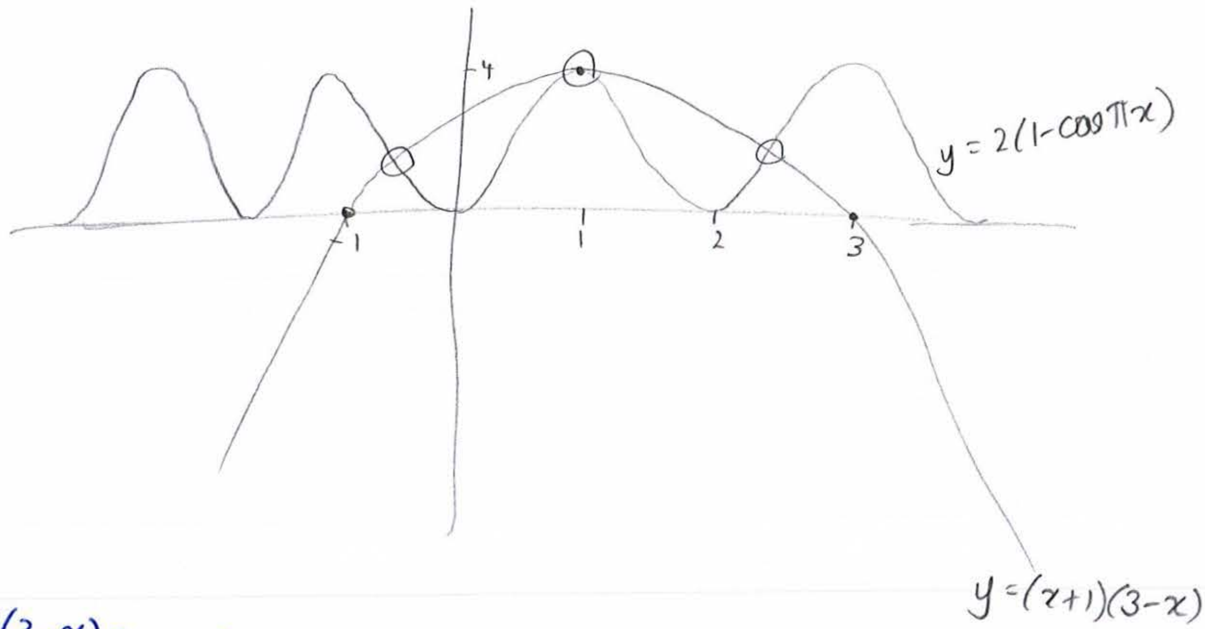
B 3

C 4

D 5

E 6

F 7



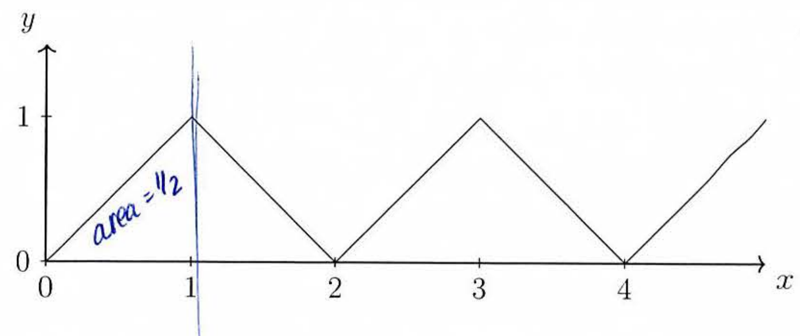
$(x+1)(3-x) = -x^2 + 2x + 3$

$\frac{d}{dx} = -2x+2 = 0$ at turning point

$2x = 2$

$x = 1$, $y = 4$

15 The diagram shows the graph of $y = \mathrm{f}(x)$.



The graph consists of alternating straight-line segments of gradient 1 and -1 and continues in this way for all values of x.

$2^9 = 2^4 2^5 = 16 \times 32 = 512$

The function g is defined as

$$\mathrm{g}(x) = \sum_{r=1}^{10} f\left(2^{r-1}x\right)$$

Find the value of

$= f(x) + f(2x) + f(4x) + f(8x) + \ldots + f(512x)$

$$\int_0^1 \mathrm{g}(x)\,\mathrm{d}x$$

A $\dfrac{1023}{1024}$

B $\dfrac{1023}{512}$

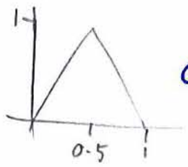
C 5

D 10

E $\dfrac{55}{2}$

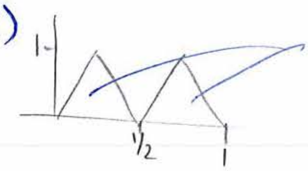
F 55

For $y = f(2x)$



area = 1/2

$y = f(4x)$



total area = 1/4 + 1/4 = 1/2

so each part of $\int_0^1 g(x)$ consists of n triangles of area 1/2n with total area 1/2

$10 \times 1/2 = \underline{5}$

16 Consider the expansion of

$$(a+bx)^n$$

The third term, in **ascending** powers of x, is $105x^2$

The fourth term, in **ascending** powers of x, is $210x^3$

The fourth term, in **descending** powers of x, is $210x^3$

Find the value of $\left(\frac{a}{b}\right)^2$

A $\frac{1}{4}$

B $\frac{4}{9}$

C $\frac{25}{36}$

D $\frac{5}{6}$

E 1

4th term in ascending & descending is of x^3

$\therefore$ must be 7 terms so $n = 6$

$(a+bx)^6 = a^6 + 6a^5bx + 15a^4b^2x^2 + 20a^3b^3x^3 + 15a^2b^4x^4 + 6ab^5x^5 + b^6x^6$

$105 = 15a^4b^2$

$7 = a^4b^2$

$20a^3b^3 = 210$

$a^3b^3 = 10.5 = \frac{21}{2}$

so $\frac{a}{b} = \frac{a^4b^2}{a^3b^3} = \frac{7}{21/2} = \frac{14}{21} = \frac{2}{3}$ so $\left(\frac{a}{b}\right)^2 = \frac{4}{9}$

Sol^{n}s are in $x^2+y^2=\frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \ldots$

'pairs' that are closer to one another than to the next pair

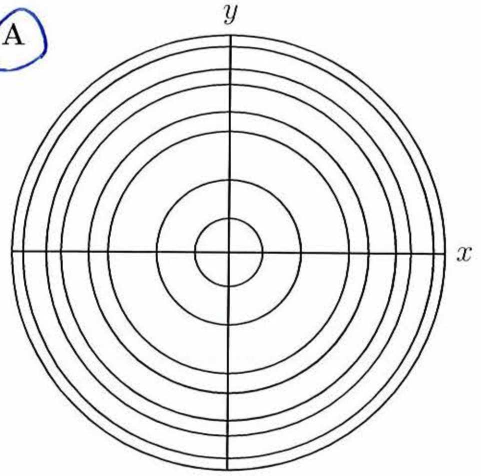
Square rooting RHS causes circles to get progressively closer

17 Which of the following sketches shows the graph of

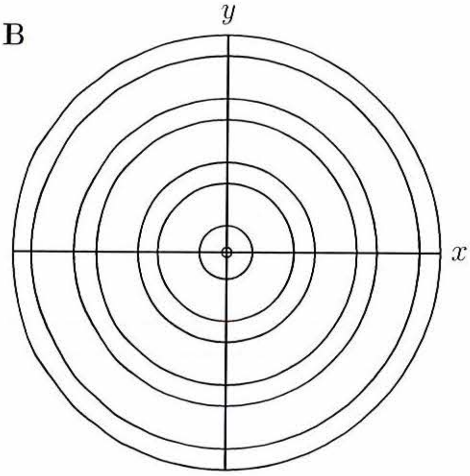
$$\sin(x^2+y^2)=\frac{1}{2}$$

where $x^2+y^2 \le 8\pi$?

A



B



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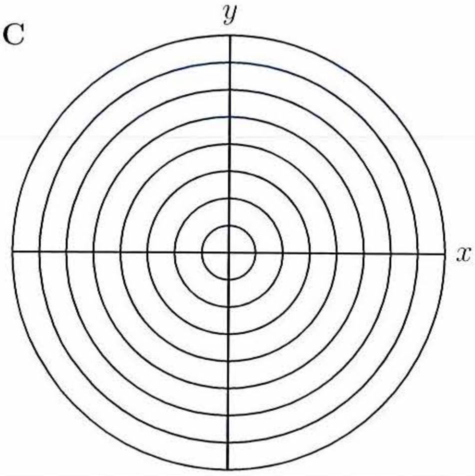
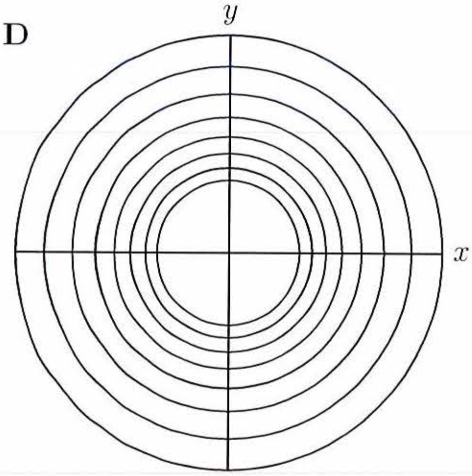
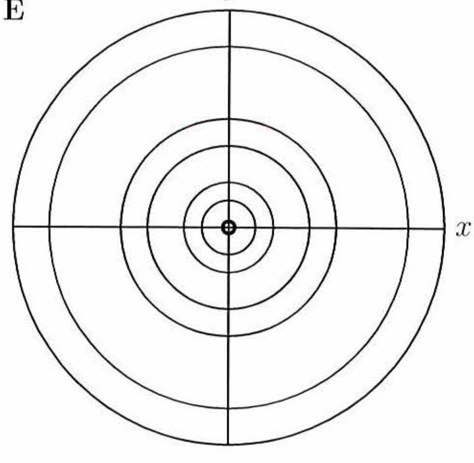


D



E



18 The curve with equation

$$x = y^2 - 6y + 11 = (y-3)^2 + 2$$

is rotated 90° clockwise about the point P to give the curve C.

P has x-coordinate -2 and y-coordinate 3.

What is the equation of C ?

A $y = -x^2 - 4x - 3$

B $y = -x^2 - 4x - 5$

C $y = -x^2 - 6x - 7$

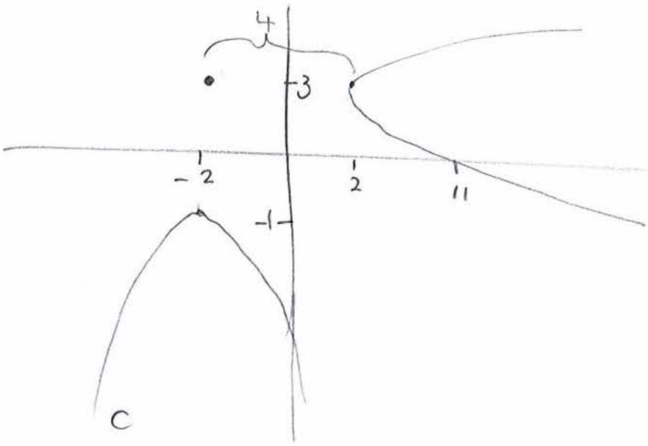
D $y = -x^2 - 6x - 11$

E $y = x^2 - 4x + 5$

F $y = x^2 + 4x + 3$

G $y = x^2 - 6x + 11$

H $y = x^2 + 6x + 7$



eq^n of C is $y = -(x+2)^2 - 1$

$= \underline{\underline{-x^2 - 4x - 5}}$

19 The equation

$$\sin^2\left(4^{\cos\theta} \times 60^\circ\right) = \frac{3}{4}$$

has exactly three solutions in the range $0^\circ \leq \theta \leq x^\circ$

What is the range of all possible values of x?

A $90 \leq x < 120$

B $90 \leq x < 270$

C $120 \leq x < 240$

D $270 \leq x < 300$

E $300 \leq x < 360$

F $450 \leq x < 630$

$\sin(4^{\cos\theta} \times 60^\circ) = \pm\frac{\sqrt{3}}{2}$

$4^{\cos\theta} \times 60^\circ = 60^\circ, 120^\circ, 240^\circ, \ldots$

$4^{\cos\theta} = 1, 2, 4$

so $\cos\theta = 0$, $\cos\theta = 1/2$, $\cos\theta = 1$

$\cos\theta = 0 \Rightarrow \theta = 90^\circ, 270^\circ, 540^\circ, \ldots$

$\cos\theta = 1/2 \Rightarrow \theta = 60^\circ, 300^\circ, 420^\circ, \ldots$

$\cos\theta = 1 \Rightarrow \theta = 0^\circ, 360^\circ, 720^\circ, \ldots$

So $\underline{90^\circ \leq x < 270^\circ}$

20 Find the length of the curve with equation

$$2\log_{10}(x-y) = \log_{10}(2-2x) + \log_{10}(y+5)$$

A 5

B 10

C 15

D 3π

E 9π

F 12π

$\log((x-y)^2) = \log((2-2x)(y+5))$

$(x-y)^2 = (2-2x)(y+5)$

$x^2 - 2xy + y^2 = 2y + 10 - 2xy - 10x$

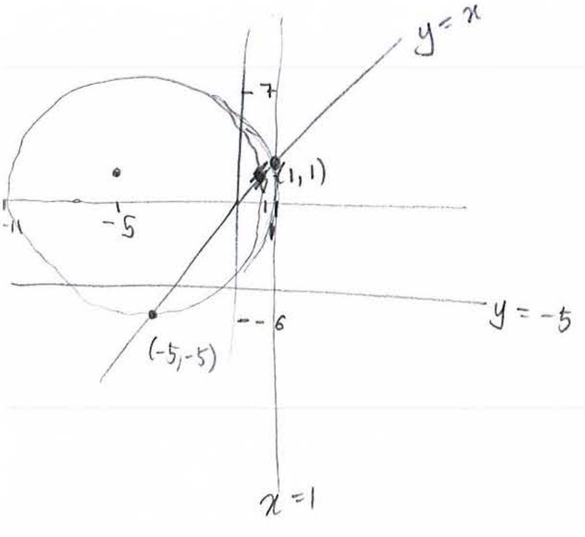
$x^2 + y^2 = 2y + 10 - 10x$

$x^2 + 10x + y^2 - 2y = 10$

$(x+5)^2 - 25 + (y-1)^2 - 1 = 10$

$(x+5)^2 + (y-1)^2 = 36 = 6^2$

circle centre $(-5,1)$ & radius 6



As can only take logs of +ve numbers we need

$x - y > 0$ $\quad$ $2 - 2x > 0$ $\quad$ $y + 5 > 0$

$y < x$ $\quad$ $x < 1$ ✓ $\quad$ $y > -5$ ✓

$(-5,-5)$ to $(1,1)$ is $\frac{1}{4}$ of the circle

circumference $= 2\pi r = 2\pi \times 6 = 12\pi$

$\therefore$ length of curve $= \frac{12\pi}{4} = \underline{\underline{3\pi}}$

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