Cambridge Assessment Admissions Testing

TMUA/CTMUA **D513/02**

Model Solutions

PAPER 2

November 2020 **75 minutes**

Additional materials: Answer sheet

0955972523

INSTRUCTIONS TO CANDIDATES

Please read these instructions carefully, but do not open the question paper until you are told that you may do so.

A separate answer sheet is provided for this paper. Please check you have one. You also require a soft pencil and an eraser.

Please complete the answer sheet with your candidate number, centre number, date of birth, and full name.

This paper is the second of two papers.

There are 20 questions on this paper. For each question, choose the one answer you consider correct and record your choice on the separate answer sheet. If you make a mistake, erase thoroughly and try again.

There are no penalties for incorrect responses, only marks for correct answers, so you should attempt **all** 20 questions. Each question is worth one mark.

You can use the question paper for rough working or notes, but **no extra paper** is allowed.

You **must** complete the answer sheet within the time limit.

Calculators and dictionaries are NOT permitted.

There is no formulae booklet for this test.

Please wait to be told you may begin before turning this page.

This question paper consists of 21 printed pages and 3 blank pages.

PV2

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1 Find the complete set of values of k for which the line $y = x - 2$ crosses or touches the curve $y = x^2 + kx + 2$

A $-1 \leq k \leq 3$

B $-3 \leq k \leq 5$

C $-4 \leq k \leq 4$

D $k \leq -1$ or $k \geq 3$

E $k \leq -3$ or $k \geq 5$

F $k \leq -4$ or $k \geq 4$

$\Rightarrow x - 2 = x^2 + kx + 2$

$\Rightarrow x^2 - x(k-1) + 4 = 0$

Then $b^2 - 4ac \geqslant 0$

$\Rightarrow (k-1)^2 - 4(1)(4) \geqslant 0$

$\Rightarrow k^2 - 2k - 15 \geqslant 0$

$\Rightarrow (k+3)(k-5) \geqslant 0$

$\Rightarrow \underline{k \geqslant -3}$ or $\underline{k \geqslant 5}$

2 Given that $\tan\theta = 2$ and $180° < \theta < 360°$, find the value of $\cos\theta$

A $\sqrt{3}$

B $-\sqrt{3}$

C $\dfrac{\sqrt{3}}{2}$

D $-\dfrac{\sqrt{3}}{2}$

E $\dfrac{\sqrt{5}}{5}$

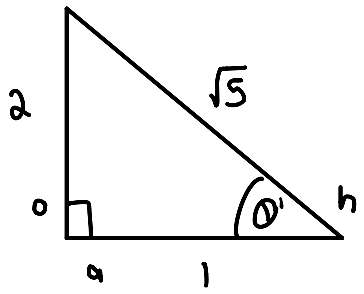
F $-\dfrac{\sqrt{5}}{5}$

G $\dfrac{2\sqrt{5}}{5}$

H $-\dfrac{2\sqrt{5}}{5}$

$\tan\theta = 2$ then $\dfrac{\sin\theta}{\cos\theta} = \tan\theta = \dfrac{2}{1}$

$h = \sqrt{2^2 + 1^2} = \sqrt{5}$



$\Rightarrow \cos\theta = \dfrac{a}{h} = \dfrac{1}{\sqrt{5}} = \dfrac{\sqrt{5}}{5}$

then since $180° < \theta < 360°$, we must negate it $\Rightarrow -\dfrac{\sqrt{5}}{5}$.

3 A student makes the following claim:

For all integers n, the expression $4\left(\dfrac{9n+1}{2} - \dfrac{3n-1}{2}\right)$ is divisible by 3.

Here is the student's argument:

$$\begin{aligned}
4\left(\frac{9n+1}{2} - \frac{3n-1}{2}\right) &= 2\left(2\left(\frac{9n+1}{2} - \frac{3n-1}{2}\right)\right) && \text{(I)}\\
&= 2(9n+1-3n-1) && \text{(II)}\\
&= 2(6n) && \text{(III)}\\
&= 12n && \text{(IV)}\\
&= 3(4n) && \text{(V)}\\
&\text{which is always a multiple of 3.} && \text{(VI)}
\end{aligned}$$

So the expression $4\left(\dfrac{9n+1}{2} - \dfrac{3n-1}{2}\right)$ is always divisible by 3.

Which one of the following is true?

A The argument is correct.

B The argument is incorrect, and the first error occurs on line (I).

C The argument is incorrect, and the first error occurs on line (II).

D The argument is incorrect, and the first error occurs on line (III).

E The argument is incorrect, and the first error occurs on line (IV).

F The argument is incorrect, and the first error occurs on line (V).

G The argument is incorrect, and the first error occurs on line (VI).

$(9n+1) - (3n-1) = 6n+2$, not $6n$.

4 Consider the following statement:

> Every positive integer N that is greater than 6 can be written as the sum of two non-prime integers that are greater than 1.

Which of the following is/are **counterexample(s)** to this statement?

I $N = 5$

II $N = 7$

III $N = 9$

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

$N = 5$ is irrelevant since $5 \leq 6$.

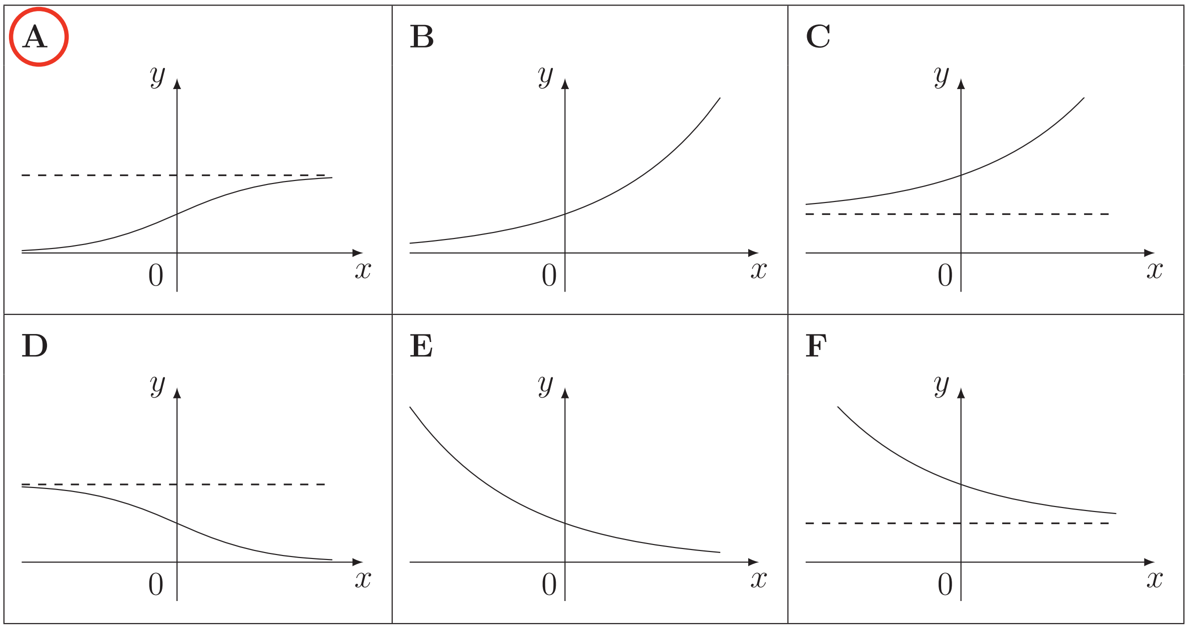
$N = 7 \Rightarrow$ $6 + 1$ (1 is not greater than 1)
$5 + 2$ (2 is prime)
$4 + 3$ (3 is prime)
$\Rightarrow$ Counter example

$N = 9 \Rightarrow$ $8 + 1$ (1 is not greater than 1).
$7 + 2$ (2 is prime)
$6 + 3$ (3 is prime)
$5 + 4$ (5 is prime)
$\Rightarrow$ Counter Example

5 Which one of the following shows the graph of

$$y = \frac{2^x}{1 + 2^x}$$

(Dotted lines indicate asymptotes.)



$y = \dfrac{2^x}{1+2^x}$ as $x \to 0$, $\dfrac{2^x}{1+2^x} \to \frac{1}{2}$

as $x \to +\infty$ (take x to be large ie 100)

$\dfrac{2^{100}}{1+2^{100}}$ which will tend to 1 as x gets large

$x \to -\infty$ (take $x = -100$) $\dfrac{2^{-100}}{1+2^{-100}}$ as $2^{-100} \approx 0$ thus the fraction $= 0$ as $x \to -\infty$ so this fits with the shape of A

6 The function $\mathrm{f}(x)$ is defined for all real values of x.

Which of the following conditions on $\mathrm{f}(x)$ is/are **necessary** to ensure that

$$(*) \quad \int_{-5}^{0} \mathrm{f}(x)\,\mathrm{d}x = \int_{0}^{5} \mathrm{f}(x)\,\mathrm{d}x$$

Condition I: $\mathrm{f}(x) = \mathrm{f}(-x)$ for $-5 \le x \le 5$

Condition II: $\mathrm{f}(x) = c$ for $-5 \le x \le 5$, where c is a constant

Condition III: $\mathrm{f}(x) = -\mathrm{f}(-x)$ for $-5 \le x \le 5$

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

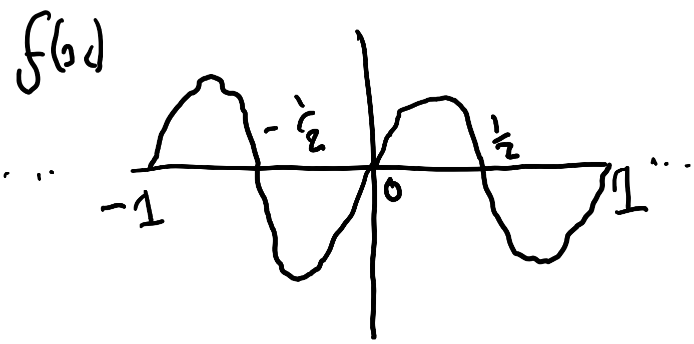
G II and III only

H I, II and III

Need to seek counter examples

Consider $f(x) = x^2$. This fits condition $(*)$ but does not fit condition II or III

Now consider $f(x) = \sin 2\pi x$



This fits condition (A) from looking at the graph but does not fit condition I

7 Consider the following conditions on a **parallelogram** $PQRS$, labelled anticlockwise:

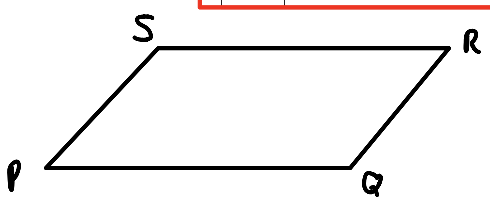
I length of PQ = length of QR

II The diagonal PR intersects the diagonal QS at right angles

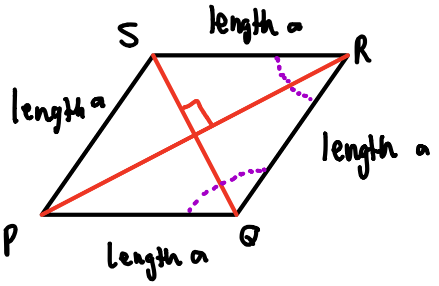
III $\angle PQR = \angle QRS$

Which of these conditions is/are individually **sufficient** for the parallelogram $PQRS$ to be a square?

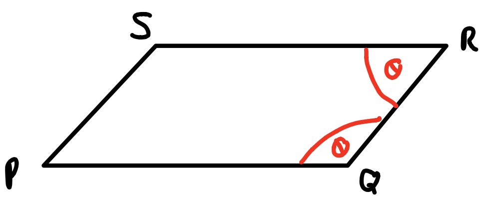
	Condition I is sufficient	**Condition II is sufficient**	**Condition III is sufficient**
A	yes	yes	yes
B	yes	yes	no
C	yes	no	yes
D	yes	no	no
E	no	yes	yes
F	no	yes	no
G	no	no	yes
H	no	no	no



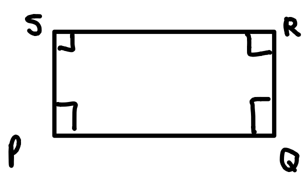
I: the length of PQ and QR do not mean we neccessarily have a square as angles aren't all 90°, shown in second diagram.



II: Again, this doesn't mean we must have a square; see red annotations on second diagram.



then θ = θ =>



III: No matter where you label your shape, ∠PQR and ∠QRS will always be 'next' to each other. If these are equal then we would have a rectangle, but this does not neccessarily mean we have a square

=> option H.

8 A student is asked to prove whether the following statement $(*)$ is true or false:

$(*)$ For all real numbers a and b, $|a+b| < |a| + |b|$

The student's proof is as follows:

Statement $(*)$ is **false**. A counterexample is $a = 3$, $b = 4$, as $|3+4| = 7$ and $|3| + |4| = 7$, but $7 < 7$ is false.

Which of the following best describes the student's proof?

A The statement $(*)$ is true, and the student's proof is not correct.

B The statement $(*)$ is false, but the student's proof is not correct: the counterexample is not valid.

C The statement $(*)$ is false, but the student's proof is not correct: the student needs to give *all* the values of a and b where $|a+b| < |a| + |b|$ is false.

D The statement $(*)$ is false, but the student's proof is not correct: the student should have instead stated that for all real numbers a and b, $|a+b| \leq |a| + |b|$.

E The statement $(*)$ is false, and the student's proof is fully correct.

The Student has proven the statement to be false Using a counter example.

=> This is a Valid and Correct method of proof.

9 A student wishes to evaluate the function $f(x) = x \sin x$, where x is in radians, but has a calculator that only works in degrees.

What could the student type into their calculator to correctly evaluate $f(4)$?

$\Rightarrow a$ radians $= \frac{a \times 180}{\pi}°$

A $(\pi \times 4 \div 180) \times \sin(4)$

B $(\pi \times 4 \div 180) \times \sin(\pi \times 4 \div 180)$

C $4 \times \sin(\pi \times 4 \div 180)$

$\Rightarrow f(4) = 4\sin\left(\frac{4 \times 180}{\pi}\right)$

D $(180 \times 4 \div \pi) \times \sin(4)$

E $(180 \times 4 \div \pi) \times \sin(180 \times 4 \div \pi)$

F $4 \times \sin(180 \times 4 \div \pi)$

10 The real numbers a, b, c and d satisfy both

$$\circledast \quad 0 < a + b < c + d$$

and

$$\circledast\circledast \quad 0 < a + c < b + d$$

Which of the following inequalities **must** be true?

I $\quad a < d$

II $\quad b < c$

III $\quad a + b + c + d > 0$

I: $\circledast + \circledast\circledast \Rightarrow 2a + b + c < c + b + 2d$

$\Rightarrow 2a < 2d$

$\Rightarrow a < d$ so I is true

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

II: take $a = 1$ $b = 3$, $c = 1$ and $d = 4$

i) $0 < 4 < 5 \Rightarrow$ true

ii) $0 < 2 < 7 \Rightarrow$ true

but $b = 3 < 1 = c$ is false, hence this must not neccessarily be true.

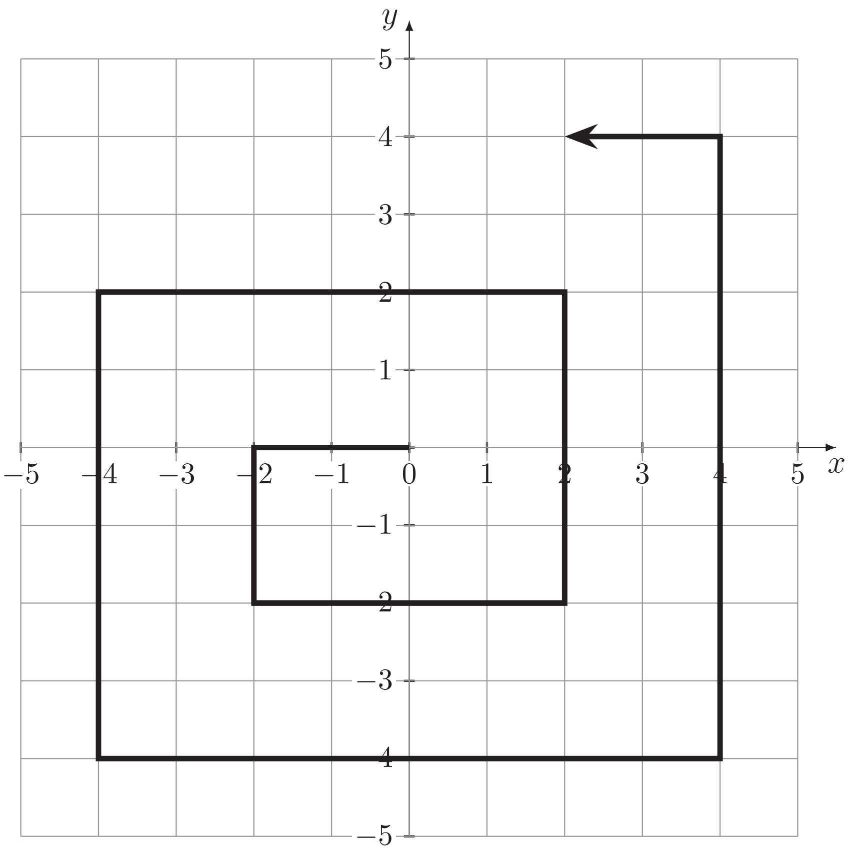
III:

$0 < a + b$ and $0 < a + c$

$0 < c + d$ and $0 < b + d$

hence $a + b + c + d > 0$

11 A spiral line is drawn as shown.



This spiral pattern continues indefinitely.

Which one of the following points is **not** on the spiral line?

A $(99, 100)$

B $(99, -100)$

C $(-99, 100)$

D $(-99, -100)$

E $(100, 99)$

F $(100, -99)$

G $(-100, 99)$

H $(-100, -99)$

For simplicity we can change the 99, 100 to 3. and 4 preserving the odd and eveness. This shows that (-4, 3) will not be on the spiral by the diagram which corresponds to (-100, 99)

12 Which one of **A–F** correctly completes the following statement?

Given that $a < b$, and $f(x) > 0$ for all x with $a < x < b$, the trapezium rule produces an overestimate for $\int_a^b f(x)\,dx$ ...

A ... **if** $f'(x) > 0$ **and** $f''(x) < 0$ for all x with $a < x < b$

B ... **only if** $f'(x) > 0$ **and** $f''(x) < 0$ for all x with $a < x < b$

C ... **if and only if** $f'(x) > 0$ **and** $f''(x) < 0$ for all x with $a < x < b$

D ... **if** $f'(x) < 0$ **and** $f''(x) > 0$ for all x with $a < x < b$

E ... **only if** $f'(x) < 0$ **and** $f''(x) > 0$ for all x with $a < x < b$

F ... **if and only if** $f'(x) < 0$ **and** $f''(x) > 0$ for all x with $a < x < b$

The trapezium method gives an overestimation for a concave up curve, and we know this occurs when we have a minimum, i.e. $f''(x) > 0$.

$f'(x) < 0$ means f will be decreasing on our interval, but this doesn't strictly matter (it could be increasing) hence the correct statement in this question must be D as it is true.

13 f(x) is a function for which

$$\int_0^3 (\mathrm{f}(x))^2 \, \mathrm{d}x + \int_0^3 \mathrm{f}(x) \, \mathrm{d}x = \int_0^1 \mathrm{f}(x) \, \mathrm{d}x$$

Which of the following claims about f(x) is/are **necessarily** true?

I f$(x) \leq 0$ for some x with $1 \leq x \leq 3$

II $\int_0^3 \mathrm{f}(x) \, \mathrm{d}x \leq \int_0^1 \mathrm{f}(x) \, \mathrm{d}x$

A neither of them

B I only

C II only

D I and II

II must be true. let

$a = \int_0^3 (f(x))^2 \, dx$, $b = \int_0^3 f(x) \, dx$, $c = \int_0^1 f(x) \, dx$

=> we know $a + b = c$

and $a \geq 0$ since its f(x) square and has non-negative limits.

hence $a + b = c \Rightarrow b \leq c$

$\Rightarrow \int_0^3 f(x) \, dx \leq \int_0^1 f(x) \, dx$

is true.

I must be true as we otherwise wouldn't be able to split the integral in such a way.

The fact $f(x) \leq 0$ allows us to do this.

14 An arithmetic sequence T has first term a and common difference d, where a and d are non-zero integers.

Property P is:

> **For some** positive integer m, the sum of the first m terms of the sequence is equal to the sum of the first $2m$ terms of the sequence.

For example, when $a = 11$ and $d = -2$, the sequence T has property P, because

$$11 + 9 + 7 + 5 = 11 + 9 + 7 + 5 + 3 + 1 + (-1) + (-3)$$

i.e. the sum of the first 4 terms equals the sum of the first 8 terms.

Which of the following statements is/are **true**?

I For T to have property P, it is **sufficient** that $ad < 0$.

II For T to have property P, it is **necessary** that d is even.

$\boxed{\textbf{A} \text{ neither of them}}$

B I only

C II only

D I and II

I : False as this condition is neccessary for the property P to hold, if $ad > 0$ then there will never be the introduction of 'oppositley' signed terms.

II : False by counter example using odd d :

$d = 3 \qquad a = -3$

$m = 1 \Rightarrow 3$
$m = 2 \Rightarrow 3 + 0 = 3,$

hence d is not neccessarily even.

15 Which one of the following is a **necessary and sufficient** condition for

$$\sum_{k=1}^{n} \sin\left(\frac{k\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

to be true?

A $n = 1$

B n is a multiple of 3

C n is a multiple of 6

$\boxed{\textbf{D} \quad n \text{ is 1 more than a multiple of 3}}$

E n is 1 more than a multiple of 6

F n is 1 more than a multiple of 6 or n is 2 more than a multiple of 6

$n = 1 \Rightarrow \sum_{k=1}^{n} \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \Rightarrow$ Statement is true (Sufficient but not neccessary).

$n = 3k \Rightarrow \sin\left(\frac{3k\pi}{3}\right) = 0$ for all a in $0, 1, 2, \ldots$

$n = 6k \Rightarrow \sin\left(\frac{3k\pi}{3}\right) = 0$ for all a in $0, 1, 2, \ldots$

$n = 3k+1 \Rightarrow \sum_{k=1}^{3k+1} \sin\left(\frac{k\pi}{3}\right) \Rightarrow$ For $k = 1 \Rightarrow \sin\left(\frac{\pi}{3}\right) + \sin\left(\frac{2\pi}{3}\right) + \sin\left(\frac{3\pi}{3}\right) + \sin\left(\frac{4\pi}{3}\right)$

$= \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + 0 - \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$

and this patter repeats for $k = 0, 1, 2, \ldots$

hence D is neccessary and sufficient.

16 The Fundamental Theorem of Calculus (FTC) tells us that for any polynomial f :

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^x \mathrm{f}(t)\,\mathrm{d}t\right) = \mathrm{f}(x)$$

A student calculates $\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_x^{2x} t^2\,\mathrm{d}t\right)$ as follows:

(I) $\int_x^{2x} t^2\,\mathrm{d}t = \int_0^{2x} t^2\,\mathrm{d}t - \int_0^x t^2\,\mathrm{d}t$ ✓

(II) By FTC, $\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^x t^2\,\mathrm{d}t\right) = x^2$ ✓

(III) By FTC, $\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^{2x} t^2\,\mathrm{d}t\right) = (2x)^2 = 4x^2$

$\int_0^{2x} t^2\,dt = \left[\frac{t^3}{3}\right]_0^{2x} = \frac{8x^3}{3}$ then $\frac{d}{dx}\left(\frac{8x^3}{3}\right) = 8x^2$

(IV) So $\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_x^{2x} t^2\,\mathrm{d}t\right) = 4x^2 - x^2$

$\Rightarrow$ line III is wrong.

(V) giving $\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_x^{2x} t^2\,\mathrm{d}t\right) = 3x^2$

$\Rightarrow$ D is the correct answer.

Which of the following best describes the student's calculation?

A The calculation is completely correct.

B The calculation is incorrect, and the first error occurs on line (I).

C The calculation is incorrect, and the first error occurs on line (II).

D The calculation is incorrect, and the first error occurs on line (III).

E The calculation is incorrect, and the first error occurs on line (IV).

F The calculation is incorrect, and the first error occurs on line (V).

17 A set of six **distinct** integers is split into two sets of three.

The first set of three integers has a mean of 10 and a median of 8.

The second set of three integers has a mean of 12 and a median of 9.

What is the smallest possible range of the set of all six integers?

A 8

B 10

C 11

D 12

$\boxed{\text{E} \quad 14}$

F 15

We have six distinct integers: a, b, c, d, e, f.
let set 1 be a, b, c and set 2 be d, e, f.
Then set 1 median is 8 ⇒ b = 8 assuming a < b < c and set 2 median is 9 ⇒ e = 9 assumming d < e < f.

⇒

a	b	c	d	e	f
6	8	16	7	9	20

Then:

Set 1: mean = 10

$\Rightarrow \frac{a+b+c}{3} = 10 \Rightarrow a+c = 22$

⇒ 22

0 22
1 21
2 20
3 19
4 18
5 17
6 16
~~7 15~~ 7 used already
~~8 14~~ 8 used already
~~9 13~~ 9 used already
~~10 12~~ Both > median

$a = 6$
$c = 16$

Set 2:

$\frac{d+e+f}{3} = 12 \Rightarrow d+f = 27$

⇒ 27

0 27
1 26
2 25
3 24
4 23
5 22
6 21
7 20
~~8 19~~
~~9 18~~
~~10 17~~
~~11 16~~
~~12 15~~
~~13 14~~
Both > median

$d = 7$
$f = 20$

⇒ Range = $20 - 6 = \underline{14}$

18 In this question, $\mathrm{f}(x) = ax^3 + bx^2 + cx + d$ and $\mathrm{g}(x) = px^3 + qx^2 + rx + s$ are cubic polynomials.

If $f(x) - g(x) > 0$ for every real x, which of the following is/are **necessarily** true?

I $a > p$

II **if** $b = q$ **then** $c = r$

III $d > s$

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

I : let $f(x) = x^3 + 2$ and $g(x) = x^3$

then $f(x) - g(x) = x^3 + 2 - x^3 = 2 > 0$

but $a = p = 1$, hence I is not necessarily true by counter example.

II : if $b = q$ then $f(x) - g(x) = (c - r)x + (d - s)$ which to be positive would require $c - r = 0$

$\Rightarrow c = r$ as required

III : let us assume that $d \leq s$

then we know for any x

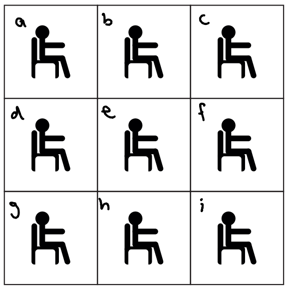
that $f(x) - g(x) > 0$, so

for $x = 0$, we have $d - s > 0$.

hence $d > s$ which is a contradiction.

Hence $d > s$ and III is true.

19 Nine people are sitting in the squares of a 3 by 3 grid, one in each square, as shown. Two people are called *neighbours* if they are sitting in squares that share a side. (People in diagonally adjacent squares, which only have a point in common, are not called neighbours.)



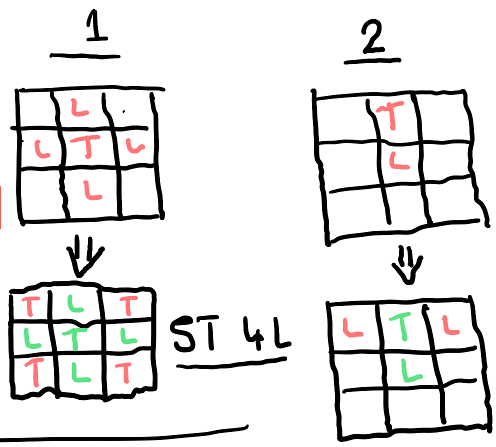
Each of the nine people in the grid is either a truth-teller who **always** tells the truth, or a liar who **always** lies.

Every person in the grid says: ‘My neighbours are all liars’.

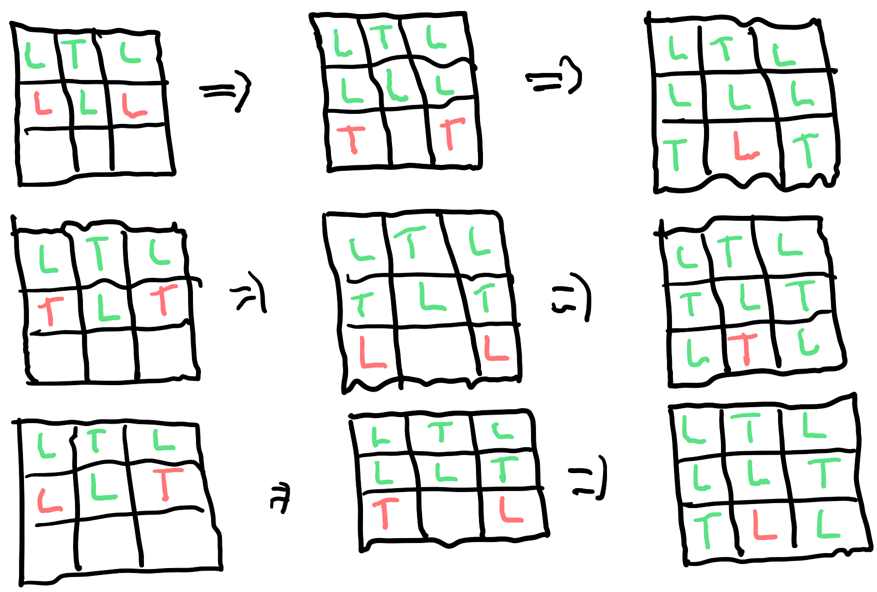
Given only this information, what are the **smallest** number and the **largest** number of people who could be telling the truth?

Two initial cases, person in centre is either a truth teller or liar

	smallest	largest
A	1	4
B	2	4
C	2	5
D	3	4
E	3	5
F	4	4
G	4	5
H	5	5



Case ②



3T 6L

③

4T 5L

④

3T 6L

So the least number is 3 and the max is 5

20 x is a real number and f is a function.

Given that **exactly one** of the following statements is true, which one is it?

A $x \geq 0$ **only if** $f(x) < 0$

B $x < 0$ **if** $f(x) \geq 0$

$\boxed{\textbf{C} \quad x \geq 0 \textbf{ only if } f(x) \geq 0}$

D $f(x) < 0$ **if** $x < 0$

E $f(x) \geq 0$ **only if** $x \geq 0$

F $f(x) \geq 0$ **if and only if** $x < 0$

A : $f(x) = x^2$, hence $x \geqslant 0$ when $f(x) \geqslant 0$ => False

B : $f(x) = x^2$ and $x = 2$ then $x \geqslant 0$ and $f(x) = 4 \geqslant 0$ => False.

D : let $f(x) = x^2$ then for any x, $f(x) \geqslant 0$ => False.

E : let $f(x) = x^2$ and $x = -2$, then $f(-2) = 4 \geqslant 0$ but $x = -2 < 0$ => False.

F : Counter Example : $f(x) = 2x$, but if $x = 3$ then $f(3) = 6 \geqslant 0$ but $x \geqslant 0$ => False.

=> C is therefore true.

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