Cambridge Assessment
Admissions Testing

TEST OF MATHEMATICS FOR UNIVERSITY ADMISSION

D513/11

model answers

PAPER 1

Wednesday 31 October 2018

Time: 75 minutes

Additional materials: Answer sheet

INSTRUCTIONS TO CANDIDATES

Please read these instructions carefully, but do not open the question paper until you are told that you may do so.

A separate answer sheet is provided for this paper. Please check you have one. You also require a soft pencil and an eraser.

This paper is the first of two papers.

There are 20 questions on this paper. For each question, choose the one answer you consider correct and record your choice on the separate answer sheet. If you make a mistake, erase thoroughly and try again.

There are no penalties for incorrect responses, only marks for correct answers, so you should attempt **all** 20 questions. Each question is worth one mark.

Any rough work should be done on this question paper. No extra paper is allowed.

Please complete the answer sheet with your candidate number, centre number, date of birth, and full name.

Calculators and dictionaries must **NOT** be used.

There is no formulae booklet for this test.

Please wait to be told you may begin before turning this page.

This question paper consists of 21 printed pages and 3 blank pages.

PV1

$$\frac{3-2x}{x\sqrt{x}} = \frac{x\sqrt{x}(3-2x)}{x^2 x} = \frac{x\sqrt{x}(3-2x)}{x^3} = \frac{\sqrt{x}(3-2x)}{x^2} = \frac{x^{1/2}(3-2x)}{x^2}$$

$$= \frac{3x^{1/2} - 2x^{3/2}}{x^2}$$

$$= 3x^{-3/2} - 2x^{-1/2}$$

1 Find the value of

$$\int_1^4 \frac{3-2x}{x\sqrt{x}}\,\mathrm{d}x$$

A $-\frac{13}{2}$

B $-\frac{85}{16}$

C $-\frac{13}{8}$

D -1

E $-\frac{1}{4}$

F $\frac{7}{4}$

G 7

$$= \int_1^4 3x^{-3/2} - 2x^{-1/2}\,dx = \frac{-3x^{-1/2}}{1/2} - \frac{2x^{1/2}}{1/2}\Big|_1^4$$

$$= -6x^{-1/2} - 4x^{1/2}\Big|_1^4$$

$$= \left(-6\left(\frac{1}{\sqrt{4}}\right) - 4\sqrt{4}\right) - \left(-6\left(\frac{1}{\sqrt{1}}\right) - 4\sqrt{1}\right)$$

$$= (-3-8) - (-6-4)$$

$$= -11 - -10$$

$$= -1$$

Sum of an AP $= \frac{n}{2}(2a + (n-1)d)$

we have a and d, & are told $S_5 = S_8$

2 An arithmetic progression has first term a and common difference d.

The sum of the first 5 terms is equal to the sum of the first 8 terms.

Which one of the following expresses the relationship between a and d?

A $a = -\frac{38}{3}d$

B $a = -7d$

C $a = -6d$

D $a = 6d$

E $a = 7d$

F $a = \frac{38}{3}d$

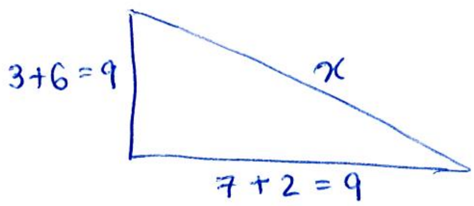
$S_5 = \frac{5}{2}(2a + 4d) = 2.5(2a + 4d) = 5a + 10d$

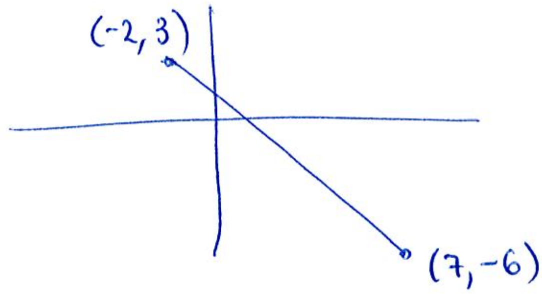
$S_8 = \frac{8}{2}(2a + 7d) = 4(2a + 7d) = 8a + 28d$

Since $S_5 = S_8$ we get $5a + 10d = 8a + 28d$

$-3a = 18d$

$a = -6d$





3 Find the shortest distance between the two circles with equations:

$$(x+2)^2 + (y-3)^2 = 18$$ centre $(-2, 3)$ & radius $3\sqrt{2}$

$$(x-7)^2 + (y+6)^2 = 2$$ centre $(7, -6)$ & radius $\sqrt{2}$

A 0

B 4

C 16

D $2\sqrt{2}$

E $5\sqrt{2}$

$x^2 = 9^2 + 9^2 = 162 \Rightarrow x = \sqrt{162} = 9\sqrt{2}$

Sum of radii of both circles $= 3\sqrt{2} + \sqrt{2} = 4\sqrt{2}$

$\therefore$ shortest distance between circumferences is

$9\sqrt{2} - 4\sqrt{2} = 5\sqrt{2}$

$3x^2 + 2xy = 4$

$3x^2 + 2ax - 2x^2 = 4$

$x^2 + 2ax = 4$

$x^2 + 2ax - 4 = 0$

4 Consider the simultaneous equations

$$3x^2 + 2xy = 4$$

$$x + y = a \Rightarrow y = a - x$$

where a is a real constant.

Find the complete set of values of a for which the equations have two distinct real solutions for x.

A There are no values of a.

B $-2 < a < 2$

C $-1 < a < 1$

D $a = 0$

E $a < -1$ or $a > 1$

F $a < -2$ or $a > 2$

G All real values of a

Discriminant $= 4a^2 + 16 \geqslant 0$

$\therefore$ 2 distinct solutions & is OK for all A

5 The function f is defined by $f(x) = x^3 + ax^2 + bx + c$.

a, b and c take the values 1, 2 and 3 with no two of them being equal and not necessarily in this order.

The remainder when $f(x)$ is divided by $(x + 2)$ is R.

The remainder when $f(x)$ is divided by $(x + 3)$ is S.

What is the largest possible value of $R - S$?

A -26

B 5

C 7

D 17

E 29

$$f(-2) = (-2)^3 + a(-2)^2 + b(-2) + c$$
$$= -8 + 4a - 2b + c$$
$$= R$$

$$f(-3) = (-3)^3 + a(-3)^2 + b(-3) + c$$
$$= -27 + 9a - 3b + c$$
$$= S$$

$$R - S = -8 + 4a - 2b + c + 27 - 9a + 3b - c$$
$$= 19 - 5a + b$$

maximise $19 - 5a + b$ requires a as small as possible from 1, 2, 3
b big

ie $19 - 5 + 3 = 17$

$x \sin 2x = \cos 2x$

$x \tan 2x = 1$

$\tan 2x = \frac{1}{x}$

6 Find the number of solutions of the equation

$$x \sin 2x = \cos 2x$$

with $0 \leq x \leq 2\pi$.

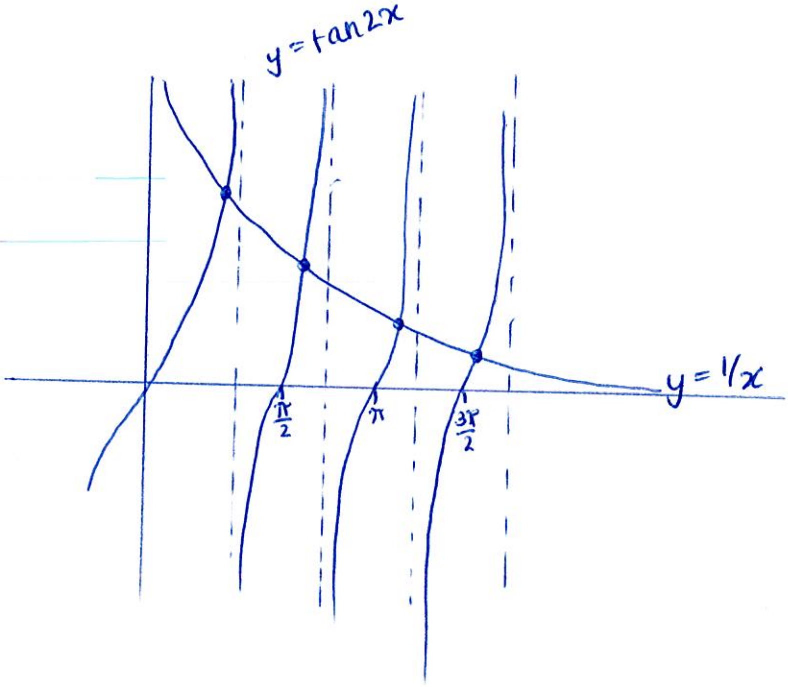
A 0

B 1

C 2

D 3

E 4



4 intersections

7 The non-zero constant k is chosen so that the coefficients of x^6 in the expansions of $(1+kx^2)^7$ and $(k+x)^{10}$ are equal.

What is the value of k?

A $\frac{1}{6}$

B 6

C $\frac{\sqrt{6}}{6}$

D $\sqrt{6}$

E $\frac{\sqrt{30}}{30}$

F $\sqrt{30}$

$$(1+kx)^7 = \ldots + \binom{7}{3} 1^4 (kx)^3 + \ldots$$

$$= \ldots + \frac{7\times 6\times 5}{3\times 2\times 1} k^3 x^3 + \ldots$$

$$= \ldots + 35k^3x^3 + \ldots$$

$$(k+x)^{10} = \ldots + \binom{10}{6} k^4 x^6 + \ldots$$

$$= \ldots + \frac{10\times 9\times 8\times 7}{4\times 3\times 2\times 1} k^4 x^6 + \ldots$$

$$= \ldots + 210k^4x^6 + \ldots$$

So $35k^3 = 210k^4$

$35 = 210k$

$k = \frac{35}{210} = \frac{5}{30} = \frac{1}{6}$

$$S = \frac{a}{1-r} = 6 \implies a = 6(1-r)$$
$$a = 6-6r$$
$$r = 1 - \frac{a}{6} \quad (*)$$

8 The sum to infinity of a geometric progression is 6.

The sum to infinity of the squares of each term in the progression is 12.

Find the sum to infinity of the cubes of each term in the progression.

A 8

B 18

C 24

D $\frac{216}{7}$

E 72

F 216

$$S_{sq.} = \frac{a^2}{1-r^2} = 12$$
$$= \frac{a^2}{(1-r)(1+r)}$$

Since $2S = S_{sq.}$ we get:

$$\frac{2a}{1-r} = \frac{a^2}{(1-r)(1+r)}$$
$$2\cancel{a}\cancel{(1-r)}(1+r) = a^{\cancel{2}}\cancel{(1-r)}$$
$$2(1+r) = a$$
$$1+r = \frac{a}{2}$$
$$r = \frac{a}{2} - 1$$

Sub this into $(*)$ to get $1 - \frac{a}{6} = \frac{a}{2} - 1$

$$6 - a = 3a - 6$$
$$12 = 4a$$
$$a = 3 \implies r = 1 - \frac{a}{6} = 1 - \frac{3}{6} = \frac{1}{2}$$

$$S_{cub.} = \frac{a^3}{1-r^3} = \frac{3^3}{1-(1/2)^3} = \frac{27}{1-1/8} = 27 \times \frac{8}{7} = \frac{216}{7}$$

$\frac{dy}{dx} = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x-2)(x+1)$

stationary points at $x = 2$ and $x = -1$

9 Find the complete set of values of the constant c for which the cubic equation

$$2x^3 - 3x^2 - 12x + c = 0$$

has three distinct real solutions.

A $-20 < c < 7$

B $-7 < c < 20$

C $c > 7$

D $c > -7$

E $c < 20$

F $c < -20$

$x = 2 \Rightarrow 2 \times 2^3 - 3 \times 2^2 - 12 \times 2 + c = y$

$2 \times 8 - 3 \times 4 - 24 + c = y$

$16 - 12 - 24 + c = y$

$c - 20 = y$

ie. $(2, c-20)$

$x = \cancel{1}-1 \Rightarrow 2 \times -1 - 3 \times 1 + 12 \times 1 + c = y$

$-2 - 3 + 12 + c = y$

$c + 7 = y$

ie. $(-1, c+7)$

Need these points on opposite sides of the axis, ie. one is +ve & one is -ve.

$c - 20 < c + 7$ so $c - 20 < 0$ and $c + 7 > 0$

$c < 20$ $c > -7$

$\therefore -7 < c < 20$

$|2-x| \leq 6$ so $2-x \leq 6$ and $-(2-x) \leq 6$

$-4 \leq x$

$-2+x \leq 6$

$x \leq 8$

combining those to get $-4 \leq x \leq 8$

10 x and y satisfy $|2-x| \leq 6$ and $|y+2| \leq 4$.

What is the greatest possible value of $|xy|$?

$|y+2| \leq 4$ so:

$y+2 \leq 4$ and $-(y+2) \leq 4$

$y \leq 2$

$-y-2 \leq 4$

$-6 \leq y$

combine & get $-6 \leq y \leq 2$

A 16

B 24

C 32

D 40

E 48

F There is no greatest possible value.

$|xy| = |x||y|$

maximising this by $|8| \times |-6| = 8 \times 6 = 48$

Intersect at (p, q) ie $q = 10 - p^2$ so point is $(p, 10 - p^2)$

$q = mp + c$ $(p, mp + c)$

Gradient of curve by $\frac{dy}{dx} = -2x$

11 The line $y = mx + 5$, where $m > 0$, is normal to the curve $y = 10 - x^2$ at the point (p, q).

What is the value of p?

A $\frac{\sqrt{2}}{6}$

B $-\frac{\sqrt{2}}{6}$

C $\frac{3\sqrt{2}}{2}$

D $-\frac{3\sqrt{2}}{2}$

E $\sqrt{5}$

F $-\sqrt{5}$

Gradient at (p, q) is $-2p$ for curve & $\frac{1}{2p}$ for normal.

Equating curve & normal at $(p,q) = (p, 10-p^2)$ gives

$10 - p^2 = pm + c$

$m = \frac{10 - p^2 - c}{p}$

For $y = mx + c$ at (p, q) we have $y = \frac{x}{2p} + c$

Equating gradients, m, gets $\frac{1}{2p} = \frac{10 - p^2 - c}{p}$

$p = 2p(10 - p^2 - c)$

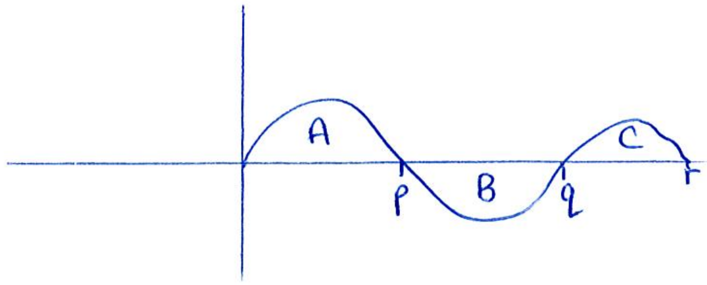
$p = 20p - 2p^3 - 2pc$

$2c = 19 - 2p^2$

$c = \frac{19}{2} - p^2$ so $y = \frac{x}{2p} + \frac{19}{2} - p^2$

$\therefore\ 5 = \frac{19}{2} - p^2$ so $p^2 = \frac{9}{2} \Rightarrow p = \frac{\pm 3}{\sqrt{2}} = \frac{\pm 3\sqrt{2}}{2}$

Since $m = \frac{1}{2p} > 0$ then $p > 0$ so $p = \frac{3\sqrt{2}}{2}$



12 A curve has equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = x(x-p)(x-q)(r-x)$$

with $0 < p < q < r$.

You are given that:

$$\int_0^r \mathrm{f}(x)\,\mathrm{d}x = 0 = A - B + C$$

$$\int_0^q \mathrm{f}(x)\,\mathrm{d}x = -2 = A - B$$

$$\int_p^r \mathrm{f}(x)\,\mathrm{d}x = -3 = -B + C$$

What is the total area enclosed by the curve and the x-axis for $0 \le x \le r$?

A 0

B 1

C 4

D 5

E 6

F 10

$A - B + C = 0$

$A - B = -2$

$-B + C = -3$ put into $A - B + C = 0$

$A - 3 = 0$

$A = 3$

so $A - B = -2$ then $C = -3 + B$

$3 - B = -2$ $= -3 + 5$

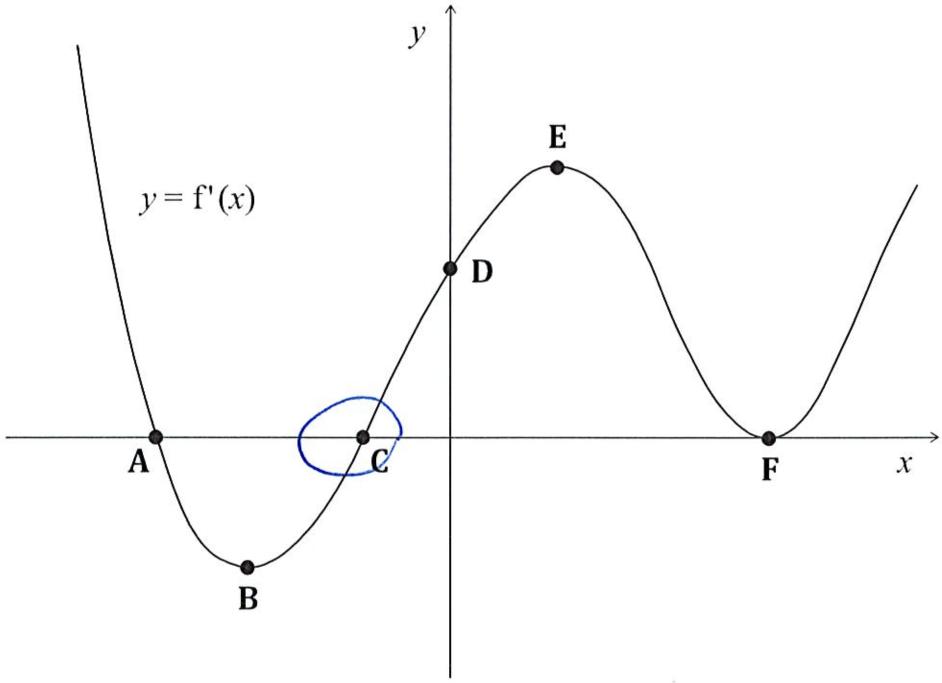
$B = 5$ $= 2$

$\therefore$ total area $= A + B + C = 3 + 5 + 2 = 10$

13 The function $\mathrm{f}(x)$ has derivative $\mathrm{f}'(x)$.

The diagram below shows the graph of $y = \mathrm{f}'(x)$.

Which point corresponds to a local minimum of $\mathrm{f}(x)$?



Local minimum requires $f'(x)=0$ so A, C, or F

AND has to be -ve to the left & +ve to the right

so can only be C

14 The line $y = mx + 4$ passes through the points $(3, \log_2 p)$ and $(\log_2 p, 4)$.

What are the possible values of p?

A $p = 1$ and $p = 4$

B $p = 1$ and $p = 16$

C $p = \frac{1}{4}$ and $p = 4$

D $p = \frac{1}{4}$ and $p = 64$

E $p = \frac{1}{64}$ and $p = 4$

F $p = \frac{1}{64}$ and $p = 16$

$(3, \log_2 p) \Rightarrow \log_2 p = 3m + 4$

$(\log_2 p, 4) \Rightarrow 4 = (\log_2 p)m + 4$

$0 = (\log_2 p)m$

so either $m = 0$ or $p = 1$

$(\text{as} \Rightarrow 2^0 = p)$

For $m = 0$, $\log_2 p = 4 \Rightarrow 2^4 = p = 16$

so $p = 1$ or 16

15 Find the sum of the real solutions of the equation:

$$3^x - (\sqrt{3})^{x+4} + 20 = 0$$

A 1

B 4

C 9

D $\log_3 20$

E $2\log_3 20$

F $4\log_3 20$

$3^x - (\sqrt{3})^x(\sqrt{3})^4 + 20 = 0$

$3^x - 9(\sqrt{3})^x + 20 = 0$

Let $y = (\sqrt{3})^x = 3^{x/2}$ then

$y^2 - 9y + 20 = 0 = (y-4)(y-5)$ so $y = 4$ or $y = 5$

then $4 = 3^{x/2}$ or $5 = 3^{x/2}$

$\log_3 4 = \frac{x}{2}\log_3 3$ $\qquad$ $\log_3 5 = \frac{x}{2}\log_3 3$

$\frac{x}{2} = \log_3 4$ $\qquad$ $\frac{x}{2} = \log_3 5$

$x = 2\log_3 4$ $\qquad$ $x = 2\log_3 5$

sum $= 2\log_3 4 + 2\log_3 5 = 2(\log_3 4 + \log_3 5) = 2\log_3 20$

Stationary point when $\frac{dy}{dx} = 0$ i.e. $0 = 2x + b$

$x = -\frac{b}{2}$

This is at $y = (-b/2)^2 + b(-b/2) + 2 = \frac{b^2}{4} - \frac{b^2}{2} + 2 = -\frac{b^2}{4} + 2$

16 The curve C has equation $y = x^2 + bx + 2$, where $b \geq 0$.

Find the value of b that minimises the distance between the origin and the stationary point of the curve C.

Stationary point is $\therefore$ $(-b/2, -b^2/4 + 2)$

A $b = 0$

B $b = 1$

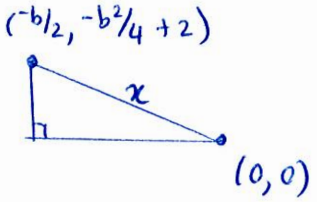
C $b = 2$

D $b = \frac{\sqrt{6}}{2}$

E $b = \sqrt{2}$

F $b = \sqrt{6}$

$-\frac{b}{2} \leq 0$ as $b \geq 0$ so the points look like



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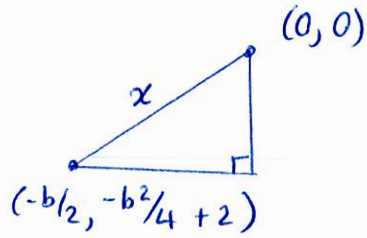


Distance from origin is x and $x^2 = (-b/2)^2 + (-b/4 + 2)^2$

$x^2 = \frac{b^2}{4} + \left(-\frac{b^2}{4} + 2\right)\left(-\frac{b^2}{4} + 2\right) = \frac{b^2}{4} + \frac{b^4}{16} - \frac{2b^2}{4} - \frac{2b^2}{4} + 4$

$= \frac{b^4}{16} - \frac{3b^2}{4} + 4$

Looking to minimise the above (as minimising x^2 also minimises x)

$x^2 = \frac{1}{16}(b^4 - 12b^2) + 4$

Differentiate $b^4 - 12b^2$ to get $4b^3 - 24b$

If $4b^3 - 24b = 0$

$b^3 = 6b$

$b = \sqrt{6}$

call the sum of the first set A There are n numbers

second B m

everything C n+m

A = 15

B = 20 So 15n + 20m = C

17 There are two sets of data: the mean of the first set is 15, and the mean of the second set is 20.

One of the pieces of data from the first set is exchanged with one of the pieces of data from the second set.

As a result, the mean of the first set of data increases from 15 to 16, and the mean of the second set of data decreases from 20 to 17.

What is the mean of the set made by combining all the data?

A $16\frac{1}{4}$

B $16\frac{1}{3}$

C $16\frac{1}{2}$

D $16\frac{2}{3}$

E $16\frac{3}{4}$

So exchanging numbers results in A = 16
B = 17

The overall sum, C, won't change so:

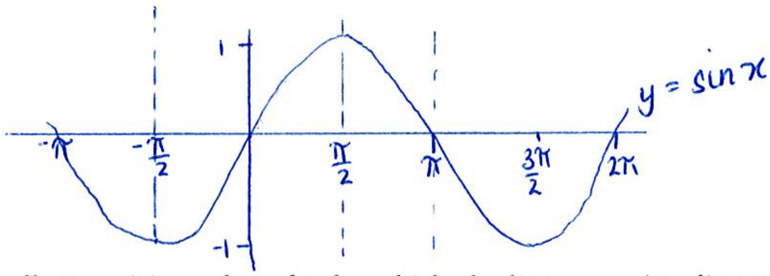
$15n + 20m = 16n + 17m$

$3m = n$

$\therefore$ total $= 15n + 20m = 15(3m) + 20m) = 45m + 20m = 65m$

overall mean $= \frac{65m}{n+m} = \frac{65m}{3m+m} = \frac{65m}{4m} = \frac{65}{4} = 16\frac{1}{4}$

Lines of symmetry when $y = \pm 1$



18 What is the smallest positive value of a for which the line $x = a$ is a line of symmetry of the graph of $y = \sin\left(2x - \frac{4\pi}{3}\right)$?

A $\frac{\pi}{12}$

B $\frac{5\pi}{12}$

C $\frac{7\pi}{12}$

D $\frac{11\pi}{12}$

E $\frac{19\pi}{12}$

Need $\sin\left(2x - \frac{4}{3}\pi\right) = \pm 1$

(1) $\sin\theta = 1$ when $\theta = \frac{\pi}{2} + 2n\pi$

$\hookrightarrow \frac{\pi}{2} + 2n\pi = 2x - \frac{4}{3}\pi$

$2x = \frac{\pi}{2} + \frac{4}{3}\pi + 2n\pi$

$2x = \pi\left(\frac{11}{6} + 2n\right)$

$x = \pi\left(\frac{11}{12} + n\right)$

minimised when $n = 0$ ie $x = \frac{11}{12}\pi$

(2) $\sin\theta = -1$ when $-\frac{\pi}{2} + 2n\pi = \theta$

$\hookrightarrow -\frac{\pi}{2} + 2n\pi = 2x - \frac{4}{3}\pi$

$2x = -\frac{\pi}{2} + 2n\pi + \frac{4}{3}\pi$

$2x = \pi\left(\frac{5}{6} + 2n\right)$

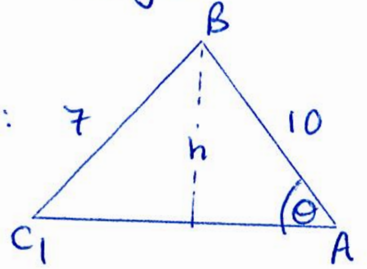
$x = \pi\left(\frac{5}{12} + n\right)$

minimised when $n = 0$ ie $x = \frac{5}{12}\pi$

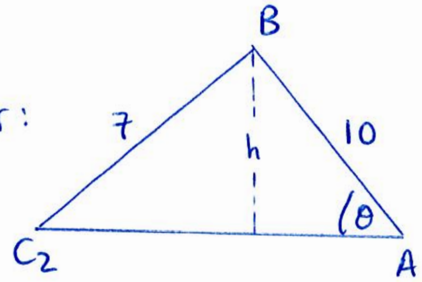
Smallest of those values of x is $\frac{5}{12}\pi$

Possible triangles

smaller:



larger:



19 A triangle ABC is to be drawn with $AB = 10$cm, $BC = 7$cm and the angle at A equal to θ, where θ is a certain specified angle.

Of the two possible triangles that could be drawn, the larger triangle has three times the area of the smaller one.

What is the value of $\cos\theta$?

A $\frac{5}{7}$

B $\frac{151}{200}$

C $\frac{2\sqrt{2}}{5}$

D $\frac{\sqrt{17}}{5}$

E $\frac{\sqrt{51}}{8}$

F $\frac{\sqrt{34}}{8}$

Area $= \frac{1}{2}bh = \frac{1}{2}(AC_1)h$ or $\frac{1}{2}(AC_2)h$

we're told $\frac{3(AC_1)h}{2} = \frac{(AC_2)h}{2} \Rightarrow 3(AC_1) = (AC_2)$

Rename $AC_1 = x$ and $AC_2 = y$ so $3x = y$

Using the cos rule for the smaller triangle:

$7^2 = x^2 + 10^2 + 2\times 10x\cos\theta$

$49 = x^2 + 100 + 20\cos\theta$

$0 = x^2 + (20\cos\theta)x + 51$ (1)

For the larger triangle this is (2) $0 = y^2 + (20\cos\theta)y + 51$

As $3x = y$, sub into (2) to get $0 = (3x)^2 + (20\cos\theta)(3x) + 51$

$0 = 9x^2 + (60\cos\theta)x + 51$ (3)

Equate (3) & (1): $x^2 + (20\cos\theta)x + 51 = 9x^2 + (60\cos\theta)x + 51$

$-(40\cos\theta)x = 8x^2$

$8x = -40\cos\theta$

$x = -5\cos\theta$

substitute into (1) then $0 = (-5\cos\theta)^2 + (-5\cos\theta)(20\cos\theta) + 51$

$0 = 25\cos^2\theta - 100\cos^2\theta + 51$

$0 = -75\cos^2\theta + 51$

$75\cos^2\theta = 51$

$\cos^2\theta = \frac{51}{75} = \frac{17}{25} \Rightarrow \cos\theta = \frac{\sqrt{17}}{5}$

20 Find the value of

$S = \sin^2 0° + \sin^2 1° + \sin^2 2° + \sin^2 3° + \cdots + \sin^2 87° + \sin^2 88° + \sin^2 89° + \sin^2 90°$

A 0.5

B 1

C 1.5

D 45

E 45.5

F 46

Also $S = \sin^2 90° + \sin^2 89° + \ldots + \sin^2 1° + \sin^2 0°$

then $2S = (\sin^2 0° + \sin^2 90°) + \ldots + (\sin^2 90° + \sin^2 0°)$

Since $\sin^2\theta + \cos^2\theta = 1$ and $\sin(90° - \theta) = \cos\theta$

$$2S = (\sin^2 0° + \sin^2 90°) + (\sin^2 1° + \sin^2 89°) + \ldots + (\sin^2 90° + \sin^2 0°)$$
$$= (\sin^2 0° + \cos^2 0°) + (\sin^2 1° + \cos^2 1°) + \ldots + (\sin^2 90° + \cos^2 90°)$$
$$= 1 + 1 + \ldots + 1$$
$$= 91$$

$\therefore S = \frac{91}{2} = 45.5$

END OF TEST