Cambridge Assessment Admissions Testing

Sixth Term Examination Paper [STEP]

Mathematics 3 [9475]

2022

Examiners' Report

Mark Scheme

STEP MATHEMATICS 3

2022

Examiners' Report

STEP 3 Introduction

One question was attempted by well over 90% of the candidates two others by about 90%, and a fourth by over 80%. Two questions were attempted by about half the candidates and a further three questions by about a third of the candidates. Even the other three received attempts from a sixth of the candidates or more, meaning that even the least popular questions were markedly more popular than their counterparts in previous years.

Nearly 90% of candidates attempted no more than 7 questions.

Question 1

This was the most popular question with 94% attempting it, and it was also the most successful with a mean mark of nearly 14/20. Apart from very occasional inaccuracies, part (i) was always successfully done.

The first summation result in part (ii) was usually successfully done, though there was some poor summation notation which let some candidates down. The second summation was completed successfully by virtue of some heavy algebra or, more efficiently, by seeing the connection to the first result dividing the quartic by t^4 and comparing the quartic for the reciprocals with that in part (i). Some candidates were penalised for not justifying their result, having clearly worked backwards from part (iii).

Part (iii) was well done except when candidates disregarded their result from part (ii).

A lot of candidates managed to correctly interpret the implication of the curves touching at two distinct points in terms of the roots for t and the consequent result for the product of the four roots, but then struggled to reach the required result by algebra or poorly justified geometric arguments.

Question 2

The fifth most popular question, being attempted by just a little over half the candidates, it was the fourth most successful with a mean score of 9/20.

Whilst the algebra associated with this question was not difficult, the logic and communication required was certainly too much for many students.

In part (i) it required some justification that a had to be even. Contradiction or infinite descent could be used but either way the argument had to be made clear. Claiming "this can be continued forever" or moduli were always decreasing would eventually get to zero was not good enough. Successful candidates were able to explain why the integer nature of the solutions was vital to reach a contradiction.

In part (ii) many candidates were able to see that this was a similar problem to the first one, and most observed that divisibility by three was now the key idea.

In part (iii), many candidates were able to consider the remainders when divided by 3, but again many struggled to communicate clearly an argument leading to the final contradiction.

By part (iv) most candidates were expecting to recreate the original equation again and the fact that this did not happen meant some came to a dead halt. Other were either oblivious to the issue or were bluffing their way through as a slightly more subtle argument was now required.

Question 3

This was very popular, being attempted by over 90%, but not very successfully, with a mean score of about 5.5/20. In part (i), candidates generally obtained a correct equation for x or y, but then failed to properly justify the manipulation of the inequality. Whilst the quartic was frequently correctly obtained in part (ii), there were a number of different incorrect assumptions or assertions made regarding the two stationary points being repeated roots or the value of the quartic having different signs at the two stationary points. It was also common that the case when c is negative was not considered. Whilst it was not uncommon for candidates to argue incorrectly for part (iii) that the three equations were equivalent to the curve C_2 in part (ii) having one stationary point, (often using $\frac{dy}{dx} = \frac{0}{0}$), in contrast, a pleasing number of candidates who made little progress in (ii) past obtaining the quartic, approached part (iii) by simply attempting to solve the equations by elimination, earning full or close to full marks.

Question 4

This was the fourth most popular question being attempted by more than four fifths of the candidates, with a moderate degree of success scoring a mean of 9/20.

Part (i) suffered from incorrect flows of logic in the inductive and base cases, as well as failure to mention anything about not dividing by zero.

In part (ii) many ignored the instruction to use the Maclaurin series, and used de L'Hopital's Rule to their cost, and some ignored the higher order terms.

Part (iii) was generally well done, though the most common error was not justifying the evaluation of the product using a geometric series in the exponent.

For part (iv), the best attempted route was to use an imaginary substitution which led to mostly successful solutions. Some candidates attempted to prove an analogous trigonometric identity using similar arguments to the previous parts, however losing marks for not sufficiently fleshing out the details, and some attempted to use Osborn's Rule, often with insufficient justification or stating that it was being used. Once the identity was achieved, the calculation was generally done well if the candidate progressed this far.

Question 5

This was only very slightly less popular than question 3, but it was the third most successful with a mean of just under half marks.

Part (i) was well done, with a variety of methods used, the most common being by a substitution of e^x. In this part, the most frequent errors were showing insufficient working to fully justify the given result, not spotting how to simplify $\frac{1+e^a}{1+e^{-a}}$, and incorrectly integrating $\int \frac{1}{1+e^x} dx$ to get $\ln(1+e^x)$.

Part (ii) was generally found to be the hardest. There was a range of responses to the first requirement from concise use of the Fundamental theorem of Calculus to long, often imprecise, paragraphs of text. Candidates attempting proof by contradiction tended to be more successful if they used a sketch to back up their argument. The second result saw many different methods used. The most common mistakes were not showing enough working when using a *u=-x* substitution, not showing that the argument can be reversed, and using an incorrect argument such as $\int_{-a}^{a} g(x) = 0 \rightarrow g(x) = 0$ (to which $g(x) = sin\ x$ is a counterexample). Many candidates did not see the link with the first requirement of the part. However, the final result of this part was usually done well.

Candidates found part (iii) easier than part (ii), the most common mistake was not realising that $h(x) = h(-x)$ holds, even though this was stated in the question.

Part (iv) was generally done well, with the most common mistakes being neglecting to show that the functions satisfied the conditions in part 3 or omitting a factor of 2. A few candidates did not use the results from the previous parts, instead using other methods, which, as the question stated "hence", gained no credit for this part.

Question 6

About half the candidates attempted this, but it was one of the least successful with a mean score of one quarter marks.

Many candidates managed the opening 'show that' in part (i) but the limit attempt had varying levels of success, and a common error was division by a quantity that was not necessarily nonzero.

In part (ii), diagrams were regularly lacking, often being drawn extremely small with the most salient details omitted.

In part (a), very few indicated from where the second term in the expression for x arose. Most attempts appealed to a diagram but did not indicate the pertinent angles.

Many formed the correct equation in (b), but a large number forgot to account for the periodicity; those that remembered to do so largely did so correctly.

Many who got to (c), erroneously evaluated a 0/0 limit and then argued that the cotangent was the answer they wanted. However, pleasingly others did spot the zeros and manipulated the trigonometry effectively.

Question 7

More than a third attempted this, marginally more successfully than questions 3 and 6.

Many attempts were restricted to part (i). The first result was generally achieved, and whilst the second result was often obtained, quite a few had difficulties doing so because they overlooked that **n** was a unit vector and what this implied. Far fewer correctly drew and labelled the diagram required in part (i) because they failed to appreciate the magnitudes of the three vectors and that two were perpendicular.

Parts (ii) and (iii), when attempted, saw candidates fall into two camps. A small number could see what both transformations were and using the considerations suggested in (ii) in part (iii) as well, could justify their answers. However, a larger number had some idea what the transformations might be, but often failed to define them precisely, and likewise failed to justify their conclusions, even given the approach to use in (ii).

Question 8

This was the least popular Pure question, being attempted by marginally fewer than question 7, but by more than any of the Mechanics or Probability and Statistics. The mean mark was 6/20.

Generally, part (i) was done well and candidates used binomial expansions accurately, manipulating their results to find the two required expressions. A few did not gain full credit through providing insufficient working for the result given in the question.

More than half the candidates progressed no further than attempting part (ii) and, of those who did attempt it, often stopped part of the way through, although there were some very well-reasoned attempts. Most candidates attempting part (ii) substituted $a = \sec(\theta)$ into their sin expansion but found it difficult to complete the argument to explain why k had to be even. Of those who got further and successfully managed to show the given results, often the relevance of those results was not appreciated, and some candidates attempted to prove irrationality by quoting the irrationality of π, despite the fact that the question stated θ was measured in degrees. Very few candidates gained full credit for this part. Those candidates who gained full credit in part (ii) also did well in (iii).

Question 9

The most popular of the Applied questions, with a third of candidates attempting it; it was the second-best scoring question on the paper with a mean score of just above half marks.

The question relied mainly on the use of conservation of momentum and Newton's experimental law of impact. Most candidates made a very good start with several scoring full or close to full marks in the first part. The difficulties arose later when dealing with the three-particle situation in part (ii). Very few candidates were able to take a step back and see how this problem linked to part (i), resulting in long pages of algebraic manipulation which were inefficient and rarely correct. A good diagram would have made the link so much more obvious!

Question 10

Along with question 12, this was the least popular question on the paper with a sixth of candidates trying it, and scoring one third marks, on average.

Part (i) was done well, demonstrating good use of Hooke's law, and resolving forces. It was failing to think about right angled triangle trigonometry that created most problems.

In part (ii), many candidates got the signs of their potential energies wrong. Of those candidates who got to the correct expression for p most were able to find the maximum value correctly but very few were able to explain why the physical situation resulted in a restricted domain for the function. Showing the value of p must be 0.7 to one significant figure was rarely done well as many candidates used known approximations to the given surds without justifying the accuracy of these approximations.

Question 11

A quarter of the candidates attempted this, scoring a mean of one third marks. Of these, about a quarter made little or no progress. However, there were also several very good attempts achieving most or all of the marks available.

In part (i)(a) the majority of candidates noticed the symmetry of the distribution and were therefore able to answer this part well, although errors such as omitting the binomial coefficients and forgetting that X could take the value 0 were made in some cases.

In part (i)(b) most candidates were able to see that the modulus sign in the sum effectively meant that the calculation of δ should be split into two sums. However, in several cases candidates simply observed that the given result followed from the two sums by symmetry without sufficient justification to earn the marks.

Almost all candidates who attempted part (i)(c) were able to show the first result by applying the definition of $\binom{2n}{r}$ and then cancelling terms. A small number of candidates argued the result by viewing the two expressions as representing different ways of counting the same total number of things. For the next part of (i)(c) the majority of candidates split the sum and then applied the previous result to the second term. Many candidates, however, did not pay sufficient attention to the case where $r = 0$ and ended up with an incorrect term in the sum. Many candidates jumped straight to the given answer at this point and therefore did not show sufficient detail to earn the remaining marks for this part. Many of the candidates who progressed further with this part dealt with the two sums separately, but some used the fact that $\binom{2n}{r} = \binom{2n-1}{r-1} + \binom{2n-1}{r}$ to rearrange into a sum of differences, most of which then cancelled out.

Candidates who had completed part (i)(c) well were able to apply the same methods to the case in part (ii) and this part was generally completed well, although a small number of candidates failed to notice that the expression for the mean in terms of n had changed.

Question 12

The least popular question on the paper, it was also the least successful with a mean score of just under one quarter marks. Many attempts did not make much progress beyond the first part. Candidates with a good understanding of how to calculate the expectation of a function of a random variable generally made very good progress.

In part (i) many candidates were able to calculate the length of the chord, although many used the cosine rule on an isosceles triangle to reach $a\sqrt{2 - 2\cos 2\theta}$, making that integration a little harder. A significant number of candidates who attempted this part omitted to include the probability density function when integrating to calculate the expected value. A small number of candidates chose to consider the length of the chord as a random variable and calculated its probability density function, from which they could then calculate the expected value. While this approach was in general successful it was a significantly more complicated approach.

In part (ii), many candidates were able to work out the probability density function. Several candidates struggled to find an expression for the length of the chord and so failed to make any further progress from this point. Those that did were often able to complete the calculation of the expected value correctly. In a small number of cases, candidates attempted to calculate the probability density function for the length of the chord in order to calculate the expected value. In this case care needs to be taken with the limits of the integration as the shortest possible length for such chords needs to be calculated. A good number of candidates were able to rearrange the expected value in part (ii) into the requested form and many were then able to complete part (iii) successfully, although a number of attempts again omitted the probability density function and other attempts multiplied the function by t before integrating.

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1. (i) At intersections

$$(ct-a)^2+\left(\frac{c}{t}-b\right)^2=r^2$$

M1

Expanding brackets, collecting like terms and multiplying through by t^2 $(t \neq 0)$ gives

$$c^2t^4-2ac\,t^3+(\,a^2+\,b^2-\,r^2)\,t^2-2bct+\,c^2=0$$

as required. ***A1 (2)**

(ii)

$$\sum_{i=1}^{4}{t_i}^2=\left(\sum_{i=1}^{4}t_i\right)^2-2\sum_{i=1}^{3}\sum_{j>i}^{4}t_i\,t_j\ =\left(\frac{2ac}{c^2}\right)^2-2\frac{(\,a^2+\,b^2-\,r^2)}{c^2}=\frac{2}{c^2}(a^2-\,b^2+\,r^2)$$

M1 A1 **dM1 A1** ***A1 (5)**

as required.

Dividing the equation (*) by t^4 (again $t \neq 0$) gives

$$c^2-\frac{2ac}{t}+(\,a^2+\,b^2-\,r^2)\,-\frac{2bc}{t^3}+\frac{c^2}{t^4}=0$$

which has roots t_i and thus **M1**

$$c^2t^4-2bc\,t^3+(\,a^2+\,b^2-\,r^2)\,t^2-2act+\,c^2=0$$

M1 A1

has roots $\frac{1}{t_i}$, which is just (*) with a and b interchanged. **E1**

Thus

$$\sum_{i=1}^{4}\frac{1}{{t_i}^2}=\frac{2}{c^2}(b^2-\,a^2+\,r^2)$$

from the first result of (ii). **A1 (5)**

Alternative:-

$$\sum_{i=1}^{4}\frac{1}{{t_i}^2}=\frac{{t_1}^2{t_2}^2{t_3}^2+{t_2}^2{t_3}^2{t_4}^2+{t_3}^2{t_4}^2{t_1}^2+{t_4}^2{t_1}^2{t_2}^2}{{t_1}^2{t_2}^2{t_3}^2{t_4}^2}$$

M1

$$=\frac{(t_1t_2t_3+t_2t_3t_4+t_3t_4t_1+t_4t_1t_2)^2-2t_1t_2t_3t_4(t_1t_2+t_1t_3+t_1t_4+t_2t_3+t_2t_4+t_3t_4)}{(t_1t_2t_3t_4)^2}$$

M1 A1 A1

$$=\frac{2}{c^2}(b^2-\,a^2+\,r^2)$$

A1

(iii)

$$\sum_{i=1}^{4} OP_i{}^2 = \sum_{i=1}^{4}\left(c^2 t_i{}^2 + \frac{c^2}{t_i{}^2}\right) = c^2\left(\frac{2}{c^2}(a^2 - b^2 + r^2) + \frac{2}{c^2}(b^2 - a^2 + r^2)\right) = 4r^2$$

as required. **M1 *A1 (2)**

(iv) Touching at two distinct points implies the roots of (*) are two pairs of coincident roots.

WLOG say $t_1 = t_3$ and $t_2 = t_4$. **E1**

then as the product of the four roots is 1 (from (*)), $t_1{}^2 t_2{}^2 = 1$ **B1** and therefore $t_1 t_2 = \pm 1$.

P_1 is $\left(ct_1, \frac{c}{t_1}\right)$ and P_2 is $\left(ct_2, \frac{c}{t_2}\right) = \pm\left(\frac{c}{t_1}, ct_1\right)$ **B1** which are reflections of one another in $y = \pm x$ respectively, and these are the mediators of the pairs of points. **E1** The centre of the circle C_2 lies on the mediator of P_1 and P_2 **E1** which we have shown is $y = \pm x$. **E1 (6)**

Alternative E1 B1 as before

$$t_1 + t_2 + t_3 + t_4 = \frac{2a}{c} \Rightarrow t_1 + t_2 = \frac{a}{c}$$

$$t_1 t_2 t_3 + t_2 t_3 t_4 + t_3 t_4 t_1 + t_4 t_1 t_2 = \frac{2b}{c} \Rightarrow t_1{}^2 t_2 + t_2{}^2 t_1 + t_1{}^2 t_2 + t_2{}^2 t_1 = \frac{2b}{c}$$

$$\Rightarrow (t_1 + t_2) t_1 t_2 = \frac{b}{c}$$

M1 A1

So $t_1 t_2 = \frac{b}{a} = \pm 1$ and hence $a = \pm b$ and so the centre of C_2 is $(a, \pm a)$ as required.

M1 A1

Alternative E1 B1 as before

$$t_1{}^2 + t_2{}^2 + t_3{}^2 + t_4{}^2 = 2(t_1{}^2 + t_2{}^2) = 2\left(\frac{1}{t_2{}^2} + \frac{1}{t_1{}^2}\right) = \frac{1}{t_1{}^2} + \frac{1}{t_2{}^2} + \frac{1}{t_3{}^2} + \frac{1}{t_4{}^2}$$

M1 A1

and thus $\frac{2}{c^2}(a^2 - b^2 + r^2) = \frac{2}{c^2}(b^2 - a^2 + r^2)$ so $a^2 = b^2$ and $a = \pm b$

M1 A1

2. (i)

If

$$a^3 + 2b^3 + 4c^3 = 0$$

then

$$a^3 = 0 - 2b^3 - 4c^3 = 2(-b^3 - 4c^3)$$

which is even. If a were odd, then a^3 would be odd. So, a is even.

Thus $\exists p$ where $2p = a$, with p an integer and $|p| < |a|$ **E1**

Substituting for a in the original equation, $8p^3 + 2b^3 + 4c^3 = 0$. Dividing by 2 and rearranging gives $b^3 + 2c^3 + 4p^3 = 0$ which is the original equation with a,b,c replaced by b,c p.

So we may repeat the argument with, say, $2q = b$ and then having done so repeat the whole argument with $2r = c$. **E1**

Thus $\exists p, q, r$ integers with $2r = c$, $|r| < |c|$ and $p^3 + 2q^3 + 4r^3 = 0$.

So if there were to be a set of such integers a, b, c, there would be a set of such integers p, q, r with smaller modulus satisfying the same result. This argument may be repeated ad infinitum leading to the conclusion that there is no least modulus set of integers which is not possible as an infinitely decreasing sequence of positive integers cannot exist being bounded by 1. (alternatively, assume a, b, c to be smallest modulus, then we have a contradiction) **E1** Hence no such a, b, c exist. ***B1 (4)**

(ii) If $9a^3 + 10b^3 + 6c^3 = 0$, then $10b^3 = -9a^3 - 6c^3 = 3(-3a^3 - 2c^3)$

Thus $10b^3$ is a multiple of 3 and so, b^3 is a multiple of 3 and thus, b is a multiple of 3.

Thus $\exists q$ where $3q = b$, with q an integer and $|q| < |b|$ **M1 A1** and $9a^3 + 270q^3 + 6c^3 = 0$ which can be divided by 3 to give $3a^3 + 90q^3 + 2c^3 = 0$.

It would follow that $2c^3 = -3a^3 - 90q^3 = 3(-a^3 - 30q^3)$ and so $\exists r$ where $3r = c$, with r an integer and $|r| < |c|$.

Substituting for c , $3a^3 + 90q^3 + 54r^3 = 0$ leading to $a^3 + 30q^3 + 18r^3 = 0$.

We may repeat the argument with $3p = a$ leading to $27p^3 + 30q^3 + 18r^3 = 0$ which on division by 3 gives $9p^3 + 10q^3 + 6r^3 = 0$, the original equation with a, b, c replaced by p, q, r. **A1**

So the conclusion can be drawn in the same way as in part (i). ('by descent') **E1 (4)**

(iii) $(3n \pm 1)^2 = 9n^2 \pm 6n + 1 = 3(3n^2 \pm 2n) + 1$ **B1** Every integer may be written as $3n - 1$, $3n$ or $3n + 1$. We have shown that the square of an integer which is not a multiple of 3 is one more than a multiple of 3, and if an integer is a multiple of 3 then it can be written $3n$ and

$(3n)^2 = 9n^2 = 3(3n^2)$ which is a multiple of 3. Thus the sum of two integers can only be either a multiple of 3, one more than a multiple of 3, or two more than a multiple of 3 depending on whether the two integers are multiples of 3, exactly one is a multiple of 3 or neither is a multiple of 3 respectively. Hence the result that the sum of two squares can only be a multiple of three if each of the integers is a multiple of three. **E1**

If $a^2 + b^2 = 3c^2$, by the result just deduced,

$\exists p, q$ where $3p = a$, $3q = b$ and $|p| < |a|$, $|q| < |b|$ **M1**

Substituting for a and b, $9p^2 + 9q^2 = 3c^2$ so $c^2 = 3(3p^2 + 3q^2)$ meaning that c^2 is a multiple of 3 and hence c is a multiple of 3.

So $\exists r$ where $3r = c$, with r an integer and $|r| < |c|$, and substituting for c and dividing by 9 ,

$p^2 + q^2 = 3r^2$ which is the original with a, b ,c replaced by p, q, r. As in (i) and (ii), the required result follows by descent. **E1 (4)**

(iv) $(4n \pm 1)^2 = 16n^2 \pm 8n + 1 = 4(4n^2 \pm 2n) + 1$ so, the square of an odd integer is one more than a multiple of four. **M1** $(2n)^2 = 4n^2$ so the square of an even integer is a multiple of four. **M1**

Thus, the sum of the squares of three non-zero integers must be 0, 1, 2 or 3 more than a multiple of four as the integers are all even, two even and one odd, one even and two odd, or all odd respectively. **A1**

Thus if $a^2 + b^2 + c^2 = 4abc$, a, b , and c must all be even. **B1**

Thus $\exists p, q, r$ integers with $2p = a$, $2q = b$, $2r = c$, and $|p| < |a|$, $|q| < |b|$, $|r| < |c|$. **M1**

So, if $a^2 + b^2 + c^2 = 4abc$, $4p^2 + 4q^2 + 4r^2 = 32pqr$, which simplifies to

$p^2 + q^2 + r^2 = 8pqr$. (Alternatively, $a^2 + b^2 + c^2 = 2^n abc$, $a^2 + b^2 + c^2 = 2^{n+1} abc$)

M1

The argument can be repeated with p, q , and r all being even integers with the multiple of the RHS being a power of two greater than 4. **E1** Thus the result follows by descent. **E1 (8)**

3. (i) $ax^2 + bxy + cy^2 = 1$

Differentiating with respect to x ,

$$2ax + by + bx\frac{dy}{dx} + 2cy\frac{dy}{dx} = 0$$

M1

For stationary points, $\frac{dy}{dx} = 0$, so $2ax + by = 0$

Multiplying the original equation by b^2

$$ab^2x^2 + b^3xy + b^2cy^2 = b^2$$

Thus as $by = -2ax$, $ab^2x^2 - 2ab^2x^2 + 4a^2cx^2 = b^2$ **M1**

$$a(4ac - b^2)x^2 = b^2$$

A1

We require two stationary points and as $abc \neq 0$, $b \neq 0$ and as $a > 0$,

$$4ac - b^2 > 0$$

giving

$$b^2 < 4ac$$

as required.

(Alternatively, as $2ax = -by$, $(-by)^2 + 2b(-by)y + 4acy^2 = 4a$, $(4ac - b^2)y^2 = 4a$ for **M1A1**)

E1 (4)

(ii) $ay^3 + bx^2y + cx = 1$

Differentiating with respect to x ,

$$3ay^2\frac{dy}{dx} + 2bxy + bx^2\frac{dy}{dx} + c = 0$$

M1

For stationary points, $\frac{dy}{dx} = 0$, so $2bxy + c = 0$

Multiplying $ay^3 + bx^2y + cx = 1$ by $8b^3x^3$,

$$8ab^3x^3y^3 + 8b^4x^5y + 8b^3cx^4 = 8b^3x^3$$

So substituting for $2bxy$,

$$-ac^3 - 4b^3x^4c + 8b^3cx^4 = 8b^3x^3$$

M1

which simplifies to

$$4b^3cx^4 - 8b^3x^3 - ac^3 = 0$$

***A1**

Consider the curve,

$$y = 4b^3cx^4 - 8b^3x^3 - ac^3$$

This has stationary points given by

$$\frac{dy}{dx} = 16b^3cx^3 - 24b^3x^2 = 0$$

M1

i.e. $8b^3x^2(2cx - 3) = 0$ so, there are only two stationary points on this quartic, which are $(0, -ac^3)$, **A1** which is a point of inflection on the y axis, **E1** and

$$\left(\frac{3}{2c}, \frac{81b^3}{4c^3} - \frac{27b^3}{c^3} - ac^3\right)$$

A1

which is a turning point.

So for $4b^3cx^4 - 8b^3x^3 - ac^3 = 0$ to have two solutions, if $c > 0$, the turning point needs to be a minimum below the x axis and so $\frac{81b^3}{4c^3} - \frac{27b^3}{c^3} - ac^3 < 0$ **E1** and if $c < 0$, the turning point needs to be a maximum above the x axis and so $\frac{81b^3}{4c^3} - \frac{27b^3}{c^3} - ac^3 > 0$. **E1** Thus, in either case multiplication by $4c^3$ yields

$$81b^3 - 108b^3 - 4ac^6 < 0$$

which simplifies to

$$4ac^6 + 27b^3 > 0$$

as required. **E1 (10)**

(iii) These are three simultaneous equations in two unknowns so we may solve for two of them and substitute into the third. The first equation rules out $x = 0$ as the third equation would imply

$y = 0$, given that $abc \neq 0$ and thus $ay^3 + bx^2y + cx \neq 1$ as required.

If we consider $2bxy + c = 0$ and $3ay^2 + bx^2 = 0$, the second if these implies that as $b > 0$, then $a < 0$. **E1**

Multiplying the second of these by $4by^2$, $12aby^4 + 4b^2x^2y^2 = 0$ and substituting from the first of these two equations,

$$12aby^4 + c^2 = 0$$

M1

Thus

$$y = \pm\sqrt[4]{\frac{-c^2}{12ab}}$$

A1

and so

$$x = \mp\frac{c}{2b}\sqrt[4]{\frac{12ab}{-c^2}} = \mp\sqrt[4]{\frac{-3ac^2}{4b^3}}$$

A1

Substituting these in $ay^3 + bx^2y + cx = 1$, having first multiplied it by y ,

that is $ay^4 + bx^2y^2 + cxy = y$

gives

$$\frac{-c^2}{12b} + \frac{c^2}{4b} - \frac{c^2}{2b} = \pm\sqrt[4]{\frac{-c^2}{12ab}}$$

which simplifies to

$$-\frac{c^2}{3b} = \pm\sqrt[4]{\frac{-c^2}{12ab}}$$

M1

Raising this to the power four,

$$\frac{c^8}{81b^4} = \frac{-c^2}{12ab}$$

and thus

$$4ac^6 + 27b^3 = 0$$

as required. ***A1 (6)**

(Alternative: The first two equations were combined to give $4b^3cx^4 - 8b^3x^3 - ac^3 = 0$ in part (ii). **M1**

The second and third can be combined to give $4b^3x^4 + 3ac^2 = 0$ **M1**

So, $8b^3x^3 + 4ac^3 = 0$

Thus $x = -\frac{c}{b}\sqrt[3]{\frac{a}{2}}$ **A1**

and $y = \frac{1}{\sqrt[3]{4a}}$ **A1**

So, to have a solution we require

$$3a\left(\frac{1}{\sqrt[3]{4a}}\right)^2 + b\left(-\frac{c}{b}\sqrt[3]{\frac{a}{2}}\right)^2 = 0$$

which simplifies to the required result. **M1A1**)

4. (i) Suppose

$$2^k \cosh\frac{x}{2}\cosh\frac{x}{4}\cdots\cosh\frac{x}{2^k}\sinh\frac{x}{2^k} = \sinh x$$

for some integer k . **E1**

Then

$$2^{k+1}\cosh\frac{x}{2}\cosh\frac{x}{4}\cdots\cosh\frac{x}{2^{k+1}}\sinh\frac{x}{2^{k+1}} = 2\sinh x\frac{\cosh\frac{x}{2^{k+1}}\sinh\frac{x}{2^{k+1}}}{\sinh\frac{x}{2^k}}$$

(which is legitimate because $x \neq 0$ and hence $\sinh\frac{x}{2^k} \neq 0$)

$$= \sinh x\frac{2\sinh\frac{x}{2^{k+1}}\cosh\frac{x}{2^{k+1}}}{\sinh\frac{x}{2^k}} = \sinh x\frac{\sinh\frac{x}{2^k}}{\sinh\frac{x}{2^k}} = \sinh x$$

which is the desired result for $k+1$. **M1**

$$2\cosh\frac{x}{2}\sinh\frac{x}{2} = \sinh x$$

B1

so, the result is true for $n = 1$.

Hence by the principle of mathematical induction,

$$\sinh x = 2^n\cosh\frac{x}{2}\cosh\frac{x}{4}\cdots\cosh\frac{x}{2^n}\sinh\frac{x}{2^n}$$

for all positive integer n .

Thus

$$\frac{\sinh x}{x}\frac{\frac{x}{2^n}}{\sinh\frac{x}{2^n}} = 2^n\cosh\frac{x}{2}\cosh\frac{x}{4}\cdots\cosh\frac{x}{2^n}\sinh\frac{x}{2^n}\frac{x}{x}\frac{1}{2^n}\frac{1}{\sinh\frac{x}{2^n}} = \cosh\frac{x}{2}\cosh\frac{x}{4}\cdots\cosh\frac{x}{2^n}$$

as required. This working is permissible as $x \neq 0$, and so $\sinh\frac{x}{2^k} \neq 0$. **E1(4)**

(ii)

$$\frac{y}{\sinh y} = \frac{y}{y+\frac{y^3}{3!}+\frac{y^5}{5!}+\cdots} = \frac{1}{1+\frac{y^2}{3!}+\frac{y^4}{5!}+\cdots} \to 1$$

as $y \to 0$. **E1**

As, from (i),

$$\frac{\sinh x}{x}\frac{\frac{x}{2^n}}{\sinh\frac{x}{2^n}} = \cosh\frac{x}{2}\cosh\frac{x}{4}\cdots\cosh\frac{x}{2^n}$$

letting $n \to \infty$,

and using the result shown from the use of the Maclaurin series that

$$\frac{\frac{x}{2^n}}{\sinh\frac{x}{2^n}} \to 1$$

we have

$$\frac{\sinh x}{x} = \cosh\frac{x}{2}\cosh\frac{x}{4}\cdots\cosh\frac{x}{2^n}\cdots$$

as required. **E1 (2)**

(iii) Letting $x = \ln 2$, $\sinh x = \frac{2-\frac{1}{2}}{2} = \frac{3}{4}$, $\cosh\frac{x}{2} = \frac{\sqrt{2}+\frac{1}{\sqrt{2}}}{2} = \frac{3}{2\sqrt{2}}$, $\cosh\frac{x}{4} = \frac{\sqrt{\sqrt{2}}+\frac{1}{\sqrt{\sqrt{2}}}}{2} = \frac{\sqrt{2}+1}{2\sqrt{\sqrt{2}}}$

M1

etc.

Thus

$$\frac{\frac{3}{4}}{\ln 2} = \frac{3}{2\sqrt{2}} \times \frac{\sqrt{2}+1}{2\sqrt{\sqrt{2}}} \times \frac{\sqrt{\sqrt{2}}+1}{2\sqrt{\sqrt{\sqrt{2}}}}\cdots$$

M1 A1

and

$$\frac{1}{\ln 2} = \frac{4}{2\sqrt{2}} \times \frac{\sqrt{2}+1}{2\sqrt{\sqrt{2}}} \times \frac{\sqrt{\sqrt{2}}+1}{2\sqrt{\sqrt{\sqrt{2}}}}\cdots = \frac{4}{2\sqrt{2}\sqrt{\sqrt{2}}\sqrt{\sqrt{\sqrt{2}}}\cdots} \times \frac{1+\sqrt{2}}{2} \times \frac{1+\sqrt{\sqrt{2}}}{2} \times \cdots$$

A1

The denominator of the first fraction is

$$2 \times 2^{\frac{1}{2}} \times 2^{\frac{1}{4}} \times \cdots = 2^{1+\frac{1}{2}+\frac{1}{4}+\cdots} = 2^2 = 4$$

E1

So

$$\frac{1}{\ln 2} = \frac{1+\sqrt{2}}{2} \times \frac{1+\sqrt{\sqrt{2}}}{2} \times \cdots$$

as required.

(iv) Substituting $x = \frac{i\pi}{2}$ in **M1**

$$\frac{\sinh x}{x} = \cosh\frac{x}{2}\cosh\frac{x}{4}\cdots\cosh\frac{x}{2^n}\cdots$$

and using $\sinh ix = i\sin x$, $\cosh ix = \cos x$, **M1**

$$\frac{\sinh\frac{i\pi}{2}}{\frac{i\pi}{2}} = \cosh\frac{i\pi}{4}\cosh\frac{i\pi}{8}\cdots\cosh\frac{i\pi}{2^{n+1}} = \frac{i}{\frac{i\pi}{2}} = \cos\frac{\pi}{4}\cos\frac{\pi}{8}\cdots\cos\frac{\pi}{2^{n+1}}\cdots$$

M1 A1 A1

$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, $\cos\frac{\pi}{4} = 2\cos^2\frac{\pi}{8} - 1$ and thus $\cos\frac{\pi}{8} = \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}} = \frac{\sqrt{2+\sqrt{2}}}{2}$

and similarly, $\cos\frac{\pi}{16} = \sqrt{\frac{1+\frac{\sqrt{2+\sqrt{2}}}{2}}{2}} = \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}$ etc. **M1 A1 M1**

So

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \times \frac{\sqrt{2+\sqrt{2}}}{2} \times \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdots$$

as required. ***A1 (9)**

(Alternatively, by induction

$$\sin x = 2^n \cos\frac{x}{2}\cos\frac{x}{4}\cdots\cos\frac{x}{2^n}\sin\frac{x}{2^n}$$

M1 A1 E1 (as for (i))

As $\frac{y}{\sin y} \to 1$ as $y \to 0$,

$$\frac{\sin x}{x} = \cos\frac{x}{2}\cos\frac{x}{4}\cdots\cos\frac{x}{2^n}\cdots$$

M1A1

and then, substituting $x = \frac{\pi}{2}$ **M1** result follows as before **A1M1A1**.)

5. (i)

$$\int_{-a}^{a}\frac{1}{1+e^x}\,dx=\int_{-a}^{a}\frac{e^{-x}}{e^{-x}+1}\,dx=[-\ln(e^{-x}+1)]_{-a}^{a}=\ln\left(\frac{e^a+1}{e^{-a}+1}\right)=\ln e^a=a$$

M1 **A1** ***A1(3)**

Alternative 1.

$$\int_{-a}^{a}\frac{1}{1+e^x}\,dx=\int_{-a}^{a}\frac{1+e^x}{1+e^x}-\frac{e^x}{1+e^x}\,dx=[x-\ln(1+)]_{-a}^{a}=2a-\ln\left(\frac{e^a+1}{e^{-a}+1}\right)=2a-\ln e^a=a$$

M1 **A1** ***A1(3)**

Alternative 2.

Substitute $u=e^x$,

$$\int_{-a}^{a}\frac{1}{1+e^x}\,dx=\int_{e^{-a}}^{e^a}\frac{1}{1+u}\frac{1}{u}\,du=\int_{e^{-a}}^{e^a}\frac{1}{u}-\frac{1}{1+u}\,du=[\ln u-\ln(1+u)]_{e^{-a}}^{e^a}=2a-\ln\left(\frac{e^a+1}{e^{-a}+1}\right)=$$

M1 **A1**

$$2a-\ln e^a=a$$

***A1(3)**

Alternative 3.

Substitute $u=1+e^x$,

$$\int_{-a}^{a}\frac{1}{1+e^x}\,dx=\int_{1+e^{-a}}^{1+e^a}\frac{1}{u}\frac{1}{u}\,du=\int_{e^{-a}}^{e^a}\frac{1}{u}-\frac{1}{1+u}\,du=[\ln u-\ln(1+u)]_{e^{-a}}^{e^a}=2a-\ln\left(\frac{e^a+1}{e^{-a}+1}\right)=$$

M1 **A1**

$$2a-\ln e^a=a$$

***A1(3)**

Alternative 4.

$$\int_{-a}^{a}\frac{1}{1+e^x}\,dx=\int_{0}^{a}\frac{1}{1+e^x}\,dx+\int_{-a}^{0}\frac{1}{1+e^x}\,dx=\int_{0}^{a}\frac{1}{1+e^x}\,dx+\int_{a}^{0}\frac{1}{1+e^{-x}}.-dx$$

$$=\int_{0}^{a}\frac{1}{1+e^x}\,dx+\int_{0}^{a}\frac{1}{1+e^{-x}}\,dx=\int_{0}^{a}\frac{1}{1+e^x}+\frac{1}{1+e^{-x}}\,dx=\int_{0}^{a}\frac{1+e^{-x}+1+e^{-x}}{(1+e^x)(1+e^{-x})}\,dx$$

M1

$$=\int_{0}^{a}\frac{2+e^{-x}+e^{-x}}{2+e^{-x}+e^{-x}}\,dx=[x]_0^a=a$$

A1 ***A1 (3)**

(ii)

Suppose

$$\int g(x)\, dx \ = G(x) + c$$

Then if

$$\int_0^a g(x)\ dx = 0 \quad \forall a \geq 0$$

$$G(a) - G(0) = 0 \quad \forall a$$

so $G(a) = constant \ \ \forall a$ and hence $\frac{dG}{dx} = g(x) = 0 \ \ \forall x \geq 0$ as required.

Alternatively, by the FTC, $g(a) = 0 \ \ \forall a \geq 0$ **E1**

$$\int_{-a}^{a} \frac{1}{1+f(x)}\ dx = a \ \ \Leftrightarrow \ \ \int_{-a}^{0} \frac{1}{1+f(x)}\ dx + \int_{0}^{a} \frac{1}{1+f(x)}\ dx = a$$

M1

$$\Leftrightarrow \ \int_{a}^{0} \frac{1}{1+f(-x)} . - dx + \int_{0}^{a} \frac{1}{1+f(x)}\ dx = a$$

M1 A1

$$\Leftrightarrow \ \int_{0}^{a} \frac{1}{1+f(-x)} + \frac{1}{1+f(x)} - 1\ \ dx = 0$$

M1 A1

so, by stated result at start of part,

$$\Leftrightarrow \ \ \frac{1}{1+f(-x)} + \frac{1}{1+f(x)} - 1 \ = 0 \quad \forall x$$

E1 E1

$$\Leftrightarrow \ 1 + f(x) + 1 + f(-x) - \big(\,1 + f(-x)\big)\big(1 + f(x)\big) = 0$$

$$\Leftrightarrow \ f(x)\, f(-x) = 1$$

***B1 (9)**

(iii)

$$\int_{-a}^{a} \frac{h(x)}{1+f(x)}\ dx = \int_{-a}^{0} \frac{h(x)}{1+f(x)}\ dx + \int_{0}^{a} \frac{h(x)}{1+f(x)}\ dx = \int_{a}^{0} \frac{h(-x)}{1+f(-x)} . -dx + \int_{0}^{a} \frac{h(x)}{1+f(x)}\ dx$$

M1

$$= \int_0^a \frac{h(x)}{1+f(-x)} + \frac{h(x)}{1+f(x)}\, dx = \int_0^a h(x)\, dx$$

by the result of (ii). **M1 *A1 (3)**

(iv)

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^{-x}\cos x}{\cosh x}\, dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^{-x}\cos x}{\frac{e^x+e^{-x}}{2}}\, dx = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x}{1+e^{2x}}\, dx$$

M1 A1

$\cos x$ satisfies the conditions for $h(x)$ in part (iii) and e^{2x} satisfies the conditions for $f(x)$ in part (ii). **E1**

Therefore,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^{-x}\cos x}{\cosh x}\, dx = 2\int_0^{\frac{\pi}{2}} \cos x\, dx = 2\,[\sin x]_0^{\frac{\pi}{2}} = 2$$

M1 **A1 (5)**

6. (i)

$$\cos(\theta+\alpha)-\cos\theta=\cos\theta\cos\alpha-\sin\theta\sin\alpha-\cos\theta\ \approx\ \cos\theta\left(1-\frac{\alpha^2}{2}\right)-\sin\theta\,\alpha\ -\cos\theta$$

$=-\alpha\sin\theta\ -\frac{\alpha^2}{2}\cos\theta$ as required. **M1 *A1**

If $\sin\theta\neq 0$

$$\lim_{\alpha\to0}\frac{\sin(\theta+\alpha)-\sin\theta}{\cos(\theta+\alpha)-\cos\theta}=\lim_{\alpha\to0}\frac{\alpha\ \cos\theta\ -\frac{\alpha^2}{2}\sin\theta}{-\alpha\sin\theta\ -\frac{\alpha^2}{2}\cos\theta}=\lim_{\alpha\to0}\frac{\cos\theta\ -\frac{\alpha}{2}\sin\theta}{-\sin\theta\ -\frac{\alpha}{2}\cos\theta}=-\cot\theta$$

M1 A1 **A1**

(Alternative by l'Hopital, $\lim_{\alpha\to0}\frac{\sin(\theta+\alpha)-\sin\theta}{\cos(\theta+\alpha)-\cos\theta}=\lim_{\alpha\to0}\frac{\cos(\theta+\alpha)}{-\sin(\theta+\alpha)}=\lim_{\alpha\to0}-\cot(\theta+\alpha)=-\cot\theta$

M1 A1 **A1**)

If $\sin\theta=0$

$$\lim_{\alpha\to0}\frac{\sin(\theta+\alpha)-\sin\theta}{\cos(\theta+\alpha)-\cos\theta}=\lim_{\alpha\to0}\frac{\cos\theta}{-\frac{\alpha}{2}\cos\theta}=\lim_{\alpha\to0}\frac{-2}{\alpha}$$

M1

$\to-\infty$ as $\alpha\to+0$ and $\to\infty$ as $\alpha\to-0$ **A1 (7)**

(ii) (a) If Q_0 is the initial point of contact of C_1 and C_2, and if X is the point on C_2 which was initially at Q_0, then if $QOQ_0=\theta$, arc QQ_0 on C_1 is of length $(n-1)a\theta$ **E1** and this will equal the arc length QX on C_2. So if T is the centre of C_2, $QTX=(n-1)\theta$, and TP makes an angle $\theta+(n-1)\theta=n\theta$ with the x axis. **E1**

Thus the x-coordinate of P is $x(\theta)=na\cos\theta\ +a\cos(n\theta)=a(n\cos\theta\ +\cos n\theta)$ as required.

Similarly, $y(\theta)=a(n\sin\theta+\sin n\theta)$. **M1 *A1 (4)**

(b) $OP=(n-1)a$ if and only if $(n\cos\theta\ +\cos n\theta)^2+(n\sin\theta+\sin n\theta)^2=(n-1)^2$

That is if $n^2+2n\cos(n-1)\theta+1=n^2-2n+1$ which is $\cos(n-1)\theta=-1$

M1

so, when $(n-1)\theta$ is an odd multiple of π **M1**

Therefore $\theta=\frac{2r+1}{n-1}\pi$ for $r=0,1,\cdots$ **A1 (3)**

(Alternatively, $OP=(n-1)a$ only if $n\cos\theta\ +\cos n\theta=(n-1)\cos\theta$ i.e. $\cos n\theta=-\cos\theta$, and $n\sin\theta+\sin n\theta=(n-1)\sin\theta$ i.e. $\sin n\theta=-\sin\theta$ **M1**

Thus $\cos(n-1)\theta=-\cos\theta\ \cos\theta\ +-\sin\theta\ \sin\theta=-1$ so $(n-1)\theta$ is an odd multiple of π **M1**

Result as before **A1**)

(c)

$$\lim_{\alpha\to 0}\frac{y(\theta_0+\alpha)-y(\theta_0)}{x(\theta_0+\alpha)-x(\theta_0)}=\lim_{\alpha\to 0}\frac{a(n\sin(\theta_0+\alpha)+\sin n(\theta_0+\alpha))-a(n\sin\theta_0+\sin n\theta_0)}{a(n\cos(\theta_0+\alpha)\ +\cos n(\theta_0+\alpha))-a(n\cos\theta_0\ +\cos n\theta_0)}$$

M1

$$=\lim_{\alpha\to 0}\frac{n\left(\alpha\ \cos\theta_0\ -\frac{\alpha^2}{2}\ \sin\theta_0\right)+\left(n\alpha\ \cos n\theta_0\ -\frac{n^2\alpha^2}{2}\ \sin n\theta_0\right)}{n\left(-\alpha\sin\theta_0\ -\frac{\alpha^2}{2}\ \cos\theta_0\right)+\left(-n\alpha\sin n\theta_0\ -\frac{n^2\alpha^2}{2}\ \cos n\theta_0\right)}$$

M1 A1

$$=\lim_{\alpha\to 0}\frac{\cos\theta_0+\cos n\theta_0-\frac{\alpha}{2}(\sin\theta_0+n\sin n\theta_0)}{-(\sin\theta_0+\sin n\theta_0)-\frac{\alpha}{2}(\cos\theta_0+n\cos n\theta_0)}$$

$$=\frac{\sin\theta_0\ +n\sin n\theta_0}{\cos\theta_0\ +n\cos n\theta_0}$$

as $\cos\theta_0+\cos n\theta_0\ =2\cos(n+1)\frac{\theta_0}{2}\ \cos(n-1)\frac{\theta_0}{2}$ and $(n-1)\frac{\theta_0}{2}=\frac{\pi}{2}$ so $\cos(n-1)\frac{\theta_0}{2}=0$

and similarly, $\sin\theta_0\ +\sin n\theta_0=2\sin(n+1)\frac{\theta_0}{2}\ \cos(n-1)\frac{\theta_0}{2}\ =0$

Further,

$$\sin\theta_0\ +n\ \sin n\theta_0=\sin\theta_0+n\big(\sin\big((n-1)+1\big)\,\theta_0\big)$$

$$=\sin\theta_0+n(\sin(n-1)\,\theta_0\ \cos\theta_0+\cos(n-1)\,\theta_0\ \sin\theta_0)$$

$$=(1-n)\sin\theta_0$$

and

$$\cos\theta_0\ +n\ \cos n\theta_0\ =\cos\theta_0+n(\cos(n-1)\theta_0\cos\theta_0\ -\sin(n-1)\,\theta_0\sin\theta_0)$$

$$=(1-n)\cos\theta_0$$

So

$$\lim_{\alpha\to 0}\frac{y(\theta_0+\alpha)-y(\theta_0)}{x(\theta_0+\alpha)-x(\theta_0)}=\frac{(1-n)\sin\theta_0}{(1-n)\cos\theta_0}=\ \tan\theta_0$$

M1 **A1**

The LHS is the gradient of the tangent to the curve at P and the RHS is the gradient of OP, as required. **E1 (6)**

7. (i)

$$f(\boldsymbol{r}) = \boldsymbol{n} \times \boldsymbol{r} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} bz - cy \\ cx - az \\ ay - bx \end{pmatrix}$$

The x-component of $f\big(f(\boldsymbol{r})\big)$ is the x-component of $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} bz - cy \\ cx - az \\ ay - bx \end{pmatrix}$

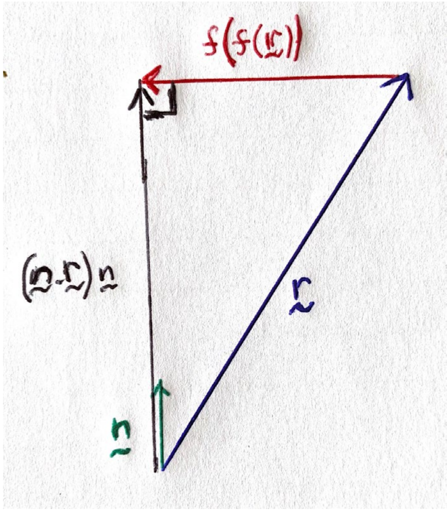
which is $b(ay - bx) - c(cx - az) = -x(b^2 + c^2) + aby + acz$ as required. **M1 *A1**

$-x(b^2 + c^2) + aby + acz = -x(a^2 + b^2 + c^2) + a^2x + aby + acz = -x + a(ax + by + cz)$

as $\boldsymbol{n}$ is a unit vector. **E1**

Similarly, the y and z -components of $f\big(f(\boldsymbol{r})\big)$ $-y + b(ax + by + cz)$ and $-z + c(ax + by + cz)$

respectively and thus $f\big(f(\boldsymbol{r})\big) = -\boldsymbol{r} + (\boldsymbol{n}.\boldsymbol{r})\boldsymbol{n}$ **M1 *A1**



G1 G1 G1 (8)

(ii)

$$g(\boldsymbol{n}) = \boldsymbol{n} + \sin\theta\ f(\boldsymbol{n}) + (1 - \cos\theta)\ f\big(f(\boldsymbol{n})\big)$$

$$= \boldsymbol{n} + \sin\theta\ \boldsymbol{n} \times \boldsymbol{n} + (1 - \cos\theta)\big((\boldsymbol{n}.\boldsymbol{n})\boldsymbol{n} - \boldsymbol{n}\big)$$

$$= \boldsymbol{n}$$

M1A1

$$g(\boldsymbol{r}) = \boldsymbol{r} + \sin\theta\ f(\boldsymbol{r}) + (1 - \cos\theta)\ f\big(f(\boldsymbol{r})\big)$$

$$= \boldsymbol{r} + \sin\theta\ \boldsymbol{n} \times \boldsymbol{r} + (1 - \cos\theta)\big((\boldsymbol{n}.\boldsymbol{r})\boldsymbol{n} - \boldsymbol{r}\big)$$

$$= \boldsymbol{r}\cos\theta + \sin\theta\ \boldsymbol{n} \times \boldsymbol{r}$$

A1

If $\boldsymbol{r}$ is perpendicular to $\boldsymbol{n}$, then $\boldsymbol{r}$, $\boldsymbol{n}$, and $\boldsymbol{n} \times \boldsymbol{r}$ form a mutually perpendicular vector triad.

g maps $\boldsymbol{r}$ to $\boldsymbol{r}\cos\theta + \sin\theta\ \boldsymbol{n} \times \boldsymbol{r}$ which represents an anticlockwise rotation by θ about an axis in the direction $\boldsymbol{n}$ as **B1** both vectors are of equal magnitude **E1** and are at angle of θ to each other **E1** and are both perpendicular to $\boldsymbol{n}$. **E1 (7)**

(iii)

$$h(\boldsymbol{s}) = -\boldsymbol{s} - 2f(\boldsymbol{s}) = -\boldsymbol{s} - 2\big((\boldsymbol{n}.\boldsymbol{s})\boldsymbol{n} - \boldsymbol{s}\big) = \boldsymbol{s} - 2(\boldsymbol{n}.\boldsymbol{s})\boldsymbol{n}$$

So, h represents a reflection **M1** in the plane through the origin perpendicular to $\boldsymbol{n}$ **A1**

Justification. If $\boldsymbol{r}$ is as in (ii).

$$h(\boldsymbol{n}) = \boldsymbol{n} - 2(\boldsymbol{n}.\boldsymbol{n})\boldsymbol{n} = -\boldsymbol{n}$$

$$h(\boldsymbol{r}) = \boldsymbol{r} - 2(\boldsymbol{n}.\boldsymbol{r})\boldsymbol{n} = \boldsymbol{r}$$

$$h(\boldsymbol{n} \times \boldsymbol{r}) = \boldsymbol{n} \times \boldsymbol{r} - 2(\boldsymbol{n}.\boldsymbol{n} \times \boldsymbol{r})\boldsymbol{n} = \boldsymbol{n} \times \boldsymbol{r}$$

B1

So any vector in the plane through the origin perpendicular to $\boldsymbol{n}$ is invariant under h, **E1** and any vector in the direction of $\boldsymbol{n}$ is reversed. **E1 (5)**

8. (i)

By de Moivre,

$$\cos(k\theta) + i\ \sin(k\theta) = (\cos\theta + i\ \sin\theta)^k$$

$$= \left[\cos^k\theta - \binom{k}{2}\cos^{k-2}\theta\ \sin^2\theta + \binom{k}{4}\cos^{k-4}\theta\ \sin^4\theta - \cdots\right]$$
$$+ i\left[\binom{k}{1}\cos^{k-1}\theta\ \sin\theta - \binom{k}{3}\cos^{k-3}\theta\ \sin^3\theta + \binom{k}{5}\cos^{k-5}\theta\ \sin^5\theta - \cdots\right]$$

M1 A1 A1

Equating imaginary parts,

$$\sin(k\theta) = \binom{k}{1}\cos^{k-1}\theta\ \sin\theta - \binom{k}{3}\cos^{k-3}\theta\ \sin^3\theta + \binom{k}{5}\cos^{k-5}\theta\ \sin^5\theta - \cdots$$

$$= \sin\theta\ \cos^{k-1}\theta\ \left(k - \binom{k}{3}\tan^2\theta + \binom{k}{5}\tan^4\theta - \cdots\right)$$

M1

$$= \sin\theta\ \cos^{k-1}\theta\ \left(k - \binom{k}{3}(\sec^2\theta - 1) + \binom{k}{5}(\sec^2\theta - 1)^2 - \cdots\right)$$

as required. ***A1**

Similarly, equating real parts,

$$\cos(k\theta) = \cos^k\theta - \binom{k}{2}\cos^{k-2}\theta\ \sin^2\theta + \binom{k}{4}\cos^{k-4}\theta\ \sin^4\theta - \cdots$$

$$= \cos^k\theta\ \left(1 - \binom{k}{2}(\sec^2\theta - 1) + \binom{k}{4}(\sec^2\theta - 1)^2 - \cdots\right)$$

B1 (6)

(ii)

$$\sin(k\theta) = 0\ \Rightarrow \sin\theta\ \cos^{k-1}\theta\ \left(k - \binom{k}{3}(\sec^2\theta - 1) + \binom{k}{5}(\sec^2\theta - 1)^2 - \cdots\right) = 0$$

Thus, if k were odd,

$$\sin\theta\ \frac{1}{a^{k-1}}\left(k - \binom{k}{3}(a^2-1) + \binom{k}{5}(a^2-1)^2 - \cdots + (-1)^{\frac{k-1}{2}}(a^2-1)^{\frac{k-1}{2}}\right) = 0$$

M1

and we are given that $\sin\theta \neq 0$

As a is odd, $(a^2 - 1)$ is even. Thus

$$\left(k - \binom{k}{3}(a^2-1) + \binom{k}{5}(a^2-1)^2 - \cdots + (-1)^{\frac{k-1}{2}}(a^2-1)^{\frac{k-1}{2}}\right)$$

is the sum of one odd number (the first) and the remainder even, and hence is odd. **A1**

We are given that $\sin\theta \neq 0$ and because a is odd, $\frac{1}{a^{k-1}} \neq 0$, and the bracketed expression is odd and thus not zero. Hence, we have a contradiction and thus k cannot be odd, and must therefore be even, as required. **E1**

If $\sin(k\theta) = 0$, and k is even, as $\sin(k\theta) = 2\sin\frac{k\theta}{2}\cos\frac{k\theta}{2}$ where $\frac{k}{2}$ is an integer, we know $\sin\frac{k\theta}{2} \neq 0$ so it would have to be that $\cos\frac{k\theta}{2} = 0$. ***B1 (4)**

Let $\frac{k}{2} = n$.

By the second result of (i), $\cos(n\theta) = \cos^n\theta\left(1 - \binom{n}{2}(\sec^2\theta - 1) + \binom{n}{4}(\sec^2\theta - 1)^2 - \cdots\right)$

$$= \frac{1}{a^n}\left(1 - \binom{n}{2}(a^2 - 1) + \binom{n}{4}(a^2 - 1)^2 - \cdots\right)$$

M1

As before, the bracketed expression is odd, being the sum of one odd number (the first which is 1) and the remainder even, and thus not zero, so $\cos(n\theta) \neq 0$ which is a contradiction. **A1**

Thus, there is no least integer k for which $\sin(k\theta) = 0$, **dM1** and hence that $k\theta = 180p$, i.e. that $\theta = \frac{180p}{k}$. Hence θ is irrational. **E1 (4)**

(iii) Suppose there is a positive odd integer k such that $\sin(k\varphi) = 0$ and $\sin(m\varphi) \neq 0$ for all integers m with $0 < m < k$.

Then $\sin(k\varphi) = \sin\varphi\cos^{k-1}\varphi\left(k - \binom{k}{3}\tan^2\varphi + \binom{k}{5}\tan^4\varphi - \cdots\right)$

$$= \sin\varphi\cos^{k-1}\varphi\left(k - \binom{k}{3}b^2 + \binom{k}{5}b^4 - \cdots\right)$$

M1

As before in (ii), the bracketed expression is odd and thus not zero, $\sin\varphi \neq 0$ and as

$\cot\varphi = \frac{1}{b} \neq 0$, $\cos\varphi \neq 0$. Hence a contradiction. **E1**

So, it would be necessary to have k even.

If $\sin(k\varphi) = 0$, and k is even, as $\sin(k\varphi) = 2\sin\frac{k\varphi}{2}\cos\frac{k\varphi}{2}$ where $\frac{k}{2}$ is an integer, we know $\sin\frac{k\varphi}{2} \neq 0$ so it would have to be that $\cos\frac{k\varphi}{2} = 0$. **E1** Let $\frac{k}{2} = n$.

$$\cos(n\varphi) = \cos^n\varphi\left(1 - \binom{n}{2}b^2 + \binom{n}{4}b^4 - \cdots\right)$$

Once again, the bracketed expression is odd and thus not zero and $\cos\varphi \neq 0$ so we have a contradiction. **E1**

Once again, there is no value k for which $\sin(k\varphi) = 0$, **M1** i.e, that $\varphi = \frac{180p}{k}$ so φ is irrational. **E1** **(6)**

9.

Conservation of linear momentum for the collision between *A* and *B* gives

$$mv_1 + kmv_2 = mu$$

M1

i.e.

$$v_1 + kv_2 = u \qquad (1)$$

Newton's experimental law of impact gives

$$v_2 - v_1 = eu \qquad (2)$$

M1

$(1) - k(2)$ gives $v_1(1+k) = u(1-ke)$ and hence $v_1 = \frac{u(1-ke)}{(1+k)}$ as required. ***A1**

$(1) + (2)$ gives $v_2(k+1) = u(1+e)$ and hence $v_2 = \frac{u(1+e)}{(1+k)}$ as required. ***A1 (4)**

Time for *B* to reach wall is $\frac{D}{\beta u}$ and the time to then return to point $\frac{1}{2}D$ from wall is $\frac{\frac{1}{2}D}{e\beta u}$

Time for *A* to reach point $\frac{1}{2}D$ from wall is $\frac{\frac{1}{2}D}{\alpha u}$

Thus

$$\frac{\frac{1}{2}D}{\alpha u} = \frac{D}{\beta u} + \frac{\frac{1}{2}D}{e\beta u}$$

M1 A1

which simplifies to

$$\frac{1}{2\alpha} = \frac{1}{\beta} + \frac{1}{2e\beta} = \frac{1}{\beta}\left(1 + \frac{1}{2e}\right)$$

Hence

$$\alpha = \beta\left(\frac{e}{1+2e}\right)$$

A1

Thus

$$(1-ke) = (1+e)\left(\frac{e}{1+2e}\right)$$

$$ke = 1 - (1+e)\left(\frac{e}{1+2e}\right) = \frac{1+2e-e-e^2}{1+2e}$$

M1

and so

$$k = \frac{1 + e - e^2}{e(1 + 2e)}$$

as required. ***A1 (5)**

(ii) The first collision (between *A* and *B*) is as in part (i).

The second collision (between *B* and *C*) is as in part (i) as the ratio of masses is the same but u is replaced by βu .

Thus, after two collisions, *A* has speed αu , *B* has speed $\alpha\beta u$, and *C* has speed $\beta^2 u$. **M1 A1**

The condition that *B* and *C* collide half the distance from the wall is as in (i) $(D = 3d)$

So

$$k = \frac{1 + e - e^2}{e(1 + 2e)}$$

E1

Equating the times of A and B to reach the point of simultaneous collision, we have

$$\frac{\frac{5}{2}d}{\alpha u} = \frac{d}{\beta u} + \frac{\frac{3}{2}d}{\alpha\beta u}$$

M1 A1

Therefore

$$\frac{5}{\alpha} = \frac{2}{\beta} + \frac{3}{\alpha\beta}$$

$$5\beta = 2\alpha + 3$$

A1

So, substituting for α and β ,

$$\frac{5(1 + e)}{(1 + k)} = \frac{2(1 - ke)}{(1 + k)} + 3$$

Thus,

$$5 + 5e = 2 - 2ke + 3 + 3k$$

$$5e = k(3 - 2e)$$

and so

$$k = \frac{5e}{3 - 2e}$$

A1

Equating these two expressions for k

$$\frac{1+e-e^2}{e(1+2e)} = \frac{5e}{3-2e}$$

M1

$$(3-2e)(1+e-e^2) = 5e^2(1+2e)$$

$$2e^3 - 5e^2 + e + 3 = 10e^3 + 5e^2$$

$$8e^3 + 10e^2 - e - 3 = 0$$

A1

Factorising we have,

$$(2e-1)(4e^2+7e+3) = 0$$

further

$$(2e-1)(e+1)(4e+3) = 0$$

M1

$e > 0$ so $e = \frac{1}{2}$ as required. ***A1 (11)**

10. (i)

$$BP = 2a\cos\theta$$

Thus, the extension of BP is $2a\cos\theta - a = a(2\cos\theta - 1)$

M1

and the tension in BP is $s_1 W\frac{a(2\cos\theta-1)}{a} = s_1 W(2\cos\theta - 1)$

Resolving in the direction BP, $W\sin\theta = s_1 W(2\cos\theta - 1)$

M1 A1

(Alternative

Resolving vertically $T_{BP}\sin\theta + T_{CP}\cos\theta = W$

Resolving horizontally $T_{BP}\cos\theta = T_{CP}\sin\theta$

Solving simultaneously $T_{BP} = W\sin\theta$

So $W\sin\theta = s_1 W(2\cos\theta - 1)$

M1 A1)

and hence

$$s_1 = \frac{\sin\theta}{(2\cos\theta - 1)}$$

as required. ***A1**

By symmetry,

$$s_2 = \frac{\cos\theta}{(2\sin\theta - 1)}$$

B1 (5)

[Both divisions are valid as both extensions are positive and so $\cos\theta > \frac{1}{2}$ and $\sin\theta > \frac{1}{2}$] @

(ii)

The GPE of the particle is $-W \times BP\sin\theta = -2Wa\sin\theta\cos\theta$

M1 A1

The EPE of BP is

$$\frac{s_1 W\big(a(2\cos\theta - 1)\big)^2}{2a}$$

M1

and the EPE of *CP* is

$$\frac{s_2 W\big(a(2\sin\theta - 1)\big)^2}{2a}$$

Thus, the total potential energy of the system is

$$\frac{-Wa}{2}\ (4\sin\theta\cos\theta - s_1(2\cos\theta - 1)^2 - s_2(2\sin\theta - 1)^2$$

A1

$$= \frac{-Wa}{2}\left(4\sin\theta\cos\theta - \frac{\sin\theta}{(2\cos\theta - 1)}(2\cos\theta - 1)^2 - \frac{\cos\theta}{(2\sin\theta - 1)}(2\sin\theta - 1)^2\right)$$

$$= \frac{-Wa}{2}(\sin\theta + \cos\theta\,)$$

So

$$p = \frac{1}{2}(\sin\theta + \cos\theta\,)$$

A1 (5)

$$(\sin\theta + \cos\theta\,) = \sqrt{2}\,\cos(\theta - 45^o)$$

M1 A1

As $\cos\theta > \frac{1}{2}$ and $\sin\theta > \frac{1}{2}$, $30^o < \theta < 60^o$

The expression is a maximum when $\theta = 45^o$ when $\frac{1}{2}(\sin\theta + \cos\theta\,) = \frac{\sqrt{2}}{2}$ ***B1** which is attainable and a minimum when $\theta = 30^o$ or 60^o (from @) **M1 E1** when $\frac{1}{2}(\sin\theta + \cos\theta\,) = \frac{1}{4}\big(1 + \sqrt{3}\,\big)$ **M1** ***A1 (7)** which cannot be attained.

(Alternative 1. $(\sin\theta + \cos\theta\,) = \sqrt{2}\sin(\theta + 45^o)$ which, similarly, is an attainable maximum when $\theta = 45^o$ and an unattainable minimum when $\theta = 30^o$ or 60^o

Alternative 2. Instead of using harmonic form

$\frac{dp}{d\theta} = \frac{1}{2}(\cos\theta - \sin\theta\,) = 0$ for stationary value **M1 A1**, giving $\tan\theta = 0$, $\theta = 45^o$ and when $\frac{1}{2}(\sin\theta + \cos\theta\,) = \frac{\sqrt{2}}{2}$ ***B1** which is attainable and a minimum)

So $\frac{\sqrt{2}}{2} \geq p > \frac{1}{4}\big(1 + \sqrt{3}\,\big)$

We require to show that $0.75 > p \geq 0.65$.

$$64 < 75 \ \Rightarrow \frac{4}{25} < \frac{3}{16} \ \Rightarrow \frac{2}{5} < \frac{\sqrt{3}}{4} \Rightarrow 0.65 < \frac{1}{4}\,(1 + \sqrt{3})$$

M1

$$\frac{9}{16} > \frac{1}{2} = \frac{2}{4} \Rightarrow 0.75 = \frac{3}{4} > \frac{\sqrt{2}}{2}$$

M1

Thus, $0.75 > \frac{\sqrt{2}}{2} \geq p > \frac{1}{4}(1+\sqrt{3}) > 0.65$ which shows that $p = 0.7$ correct to one significant figure. ***A1 (3)**

11. (i) (a) As the coin is fair, the distribution is binomial and symmetric,

so $P(X = r) = P(X = N - r) = P(X = 2n - r)$

Therefore,

$$P(X \le n-1) = \sum_{i=0}^{n-1} P(X = i) = \sum_{i=0}^{n-1} P(X = 2n - i) = \sum_{i=n+1}^{2n} P(X = i) = P(X \ge n+1)$$

E1

$$1 = P(X \le n-1) + P(X = n) + P(X \ge n+1) = 2P(X \le n-1) + P(X = n)$$

Hence,

$$P(X \le n-1) = \frac{1}{2}\left(1 - P(X = n)\right)$$

E1 (2)

(b)

$\mu = Np = 2n \times \frac{1}{2} = n$ (or by symmetry) **B1**

$$\delta = E(|X - \mu|) = \sum_{r=0}^{n-1} (n-r)\binom{2n}{r}\left(\frac{1}{2}\right)^{2n} + \sum_{r=n+1}^{2n} (r-n)\binom{2n}{r}\left(\frac{1}{2}\right)^{2n}$$

M1

$$= \sum_{r=0}^{n-1} (n-r)\binom{2n}{r}\left(\frac{1}{2}\right)^{2n} + \sum_{r=n+1}^{2n} (r-n)\binom{2n}{2n-r}\left(\frac{1}{2}\right)^{2n}$$

$$= \sum_{r=0}^{n-1} (n-r)\binom{2n}{r}\left(\frac{1}{2}\right)^{2n} + \sum_{s=0}^{n-1} (n-s)\binom{2n}{s}\left(\frac{1}{2}\right)^{2n}$$

M1

$$= 2\sum_{r=0}^{n-1} (n-r)\binom{2n}{r}\left(\frac{1}{2}\right)^{2n} = \sum_{r=0}^{n-1} (n-r)\binom{2n}{r}\frac{1}{2^{2n-1}}$$

as required. ***A1 (4)**

(c)

$$r\binom{2n}{r} = r\frac{(2n!)}{r!\,(2n-r)!} = \frac{2n \times (2n-1)!}{(r-1)!\left((2n-1)-(r-1)\right)!} = 2n\binom{2n-1}{r-1}$$

M1 ***A1(2)**

$$\delta = \sum_{r=0}^{n-1} (n-r)\binom{2n}{r}\frac{1}{2^{2n-1}} = \sum_{r=0}^{n-1} n\binom{2n}{r}\frac{1}{2^{2n-1}} - \sum_{r=0}^{n-1} r\binom{2n}{r}\frac{1}{2^{2n-1}}$$

M1

$$= \frac{1}{2^{2n-1}}\left(n\sum_{r=0}^{n-1}\binom{2n}{r} - \sum_{r=1}^{n-1} r\binom{2n}{r}\frac{1}{2^{2n-1}}\right)$$

$$= \frac{1}{2^{2n-1}}\left(n\frac{1}{2}\left(2^{2n} - \binom{2n}{n}\right) - \sum_{r=1}^{n-1} 2n\binom{2n-1}{r-1}\frac{1}{2^{2n-1}}\right)$$

M1 **M1**

$$= \frac{n}{2^{2n-1}}\left(2^{2n-1} - \frac{1}{2}\binom{2n}{n} - 2\sum_{r=0}^{n-2}\binom{2n-1}{r}\right)$$

$$= \frac{n}{2^{2n-1}}\left(2^{2n-1} - \frac{1}{2}\binom{2n}{n} - \sum_{r=0}^{n-2}\binom{2n-1}{r} - \sum_{r=n+1}^{2n-1}\binom{2n-1}{r}\right)$$

M1

$$= \frac{n}{2^{2n-1}}\left\{2^{2n-1} - \frac{1}{2}\binom{2n}{n} - \left(2^{2n-1} - \binom{2n-1}{n-1} - \binom{2n-1}{n}\right)\right\}$$

$$= \frac{n}{2^{2n-1}}\left\{-\frac{1}{2}\binom{2n}{n} + 2\binom{2n-1}{n}\right\}$$

M1

But

$$\binom{2n-1}{n} = \frac{(2n-1)!}{n!\,(n-1)!} = \frac{2n}{2n}\frac{(2n-1)!}{n!\,(n-1)!} = \frac{1}{2}\frac{(2n)!}{n!\,n!} = \frac{1}{2}\binom{2n}{n}$$

M1

Thus

$$\delta = \frac{n}{2^{2n-1}}\frac{1}{2}\binom{2n}{n} = \frac{n}{2^{2n}}\binom{2n}{n}$$

as required. ***A1 (7)**

(Alternative

$$\delta = \sum_{r=0}^{n}(n-r)\binom{2n}{r}\frac{1}{2^{2n-1}} = n\sum_{r=0}^{n}\binom{2n}{r}\frac{1}{2^{2n-1}} - 2n\sum_{r=1}^{n}\binom{2n-1}{r-1}\frac{1}{2^{2n-1}}$$

M1 A1

$$= 2n\sum_{r=0}^{n}\binom{2n}{r}\frac{1}{2^{2n}} - 2n\sum_{s=0}^{n-1}\binom{2n-1}{s}\frac{1}{2^{2n-1}}$$

M1

$$= 2nP(X \leq n) - 2nP(Y \leq n-1)$$

(where Y is a binomial variable $\left(2n-1,\frac{1}{2}\right)$) **M1**

$$= n\big(1+P(X=n)\big) - 2n\times\frac{1}{2} = n\binom{2n}{n}\frac{1}{2^{2n}}$$

M1 **M1** ***A1**)

(ii) $\mu = Np = (2n+1)\times\frac{1}{2} = \frac{2n+1}{2}$ (or by symmetry)

$$\delta = E(|X-\mu|) = \sum_{r=0}^{2n+1}\left|r-\frac{2n+1}{2}\right|\binom{2n+1}{r}\left(\frac{1}{2}\right)^{2n+1}$$

$$= \frac{1}{2^{2n}}\sum_{r=0}^{n}\left(\frac{2n+1}{2}-r\right)\binom{2n+1}{r}$$

M1 A1

$$= \frac{1}{2^{2n}}\left[\frac{2n+1}{2}\sum_{r=0}^{n}\binom{2n+1}{r} - \sum_{r=0}^{n} r\binom{2n+1}{r}\right]$$

$$= \frac{1}{2^{2n}}\left[\frac{2n+1}{2}\times 2^{2n} - \sum_{r=1}^{n} r\binom{2n+1}{r}\right]$$

$$= \frac{1}{2^{2n}}\left[\frac{2n+1}{2}\times 2^{2n} - \sum_{r=1}^{n}(2n+1)\binom{2n}{r-1}\right]$$

using the first result of (i) c) **M1**

$$= \frac{1}{2^{2n}}\left[\frac{2n+1}{2}\times 2^{2n} - (2n+1)\sum_{r=0}^{n-1}\binom{2n}{r}\right]$$

$$= \frac{(2n+1)}{2^{2n}}\left[\frac{2^{2n}}{2} - \left(\frac{2^{2n}-\binom{2n}{n}}{2}\right)\right]$$

M1

$$= \frac{(2n+1)}{2^{2n+1}}\binom{2n}{n}$$

A1 (5)

which can alternatively be written as

$$= \frac{(2n+1)}{2^{2n+1}}\frac{(2n)!}{n!\,n!} = \frac{(2n+1)!}{2^{2n+1}n!\,n!} = \frac{(2n+1)!\,(n+1)}{2^{2n+1}(n+1)!\,n!} = \frac{(n+1)}{2^{2n+1}}\binom{2n+1}{n}$$

(Alternative

$$\delta = \sum_{r=0}^{n} \left(\frac{2n+1}{2} - r\right) \binom{2n+1}{r} \frac{1}{2^{2n}}$$

M1 A1

$$= (2n+1) \sum_{r=0}^{n} \binom{2n+1}{r} \frac{1}{2^{2n+1}} - (2n+1) \sum_{r=1}^{n} \binom{2n}{r-1} \frac{1}{2^{2n}}$$

M1

$$= (2n+1)\frac{1}{2} - (2n+1) \sum_{s=0}^{n-1} \binom{2n}{s} \frac{1}{2^{2n}}$$

$$= (2n+1)\frac{1}{2} - (2n+1)\left(\frac{1}{2} - \frac{1}{2}\binom{2n}{n}\frac{1}{2^{2n}}\right)$$

M1

$$= \frac{(2n+1)}{2^{2n+1}} \binom{2n}{n}$$

A1)

12. (i)

If $AOB = \theta$, then the probability distribution function for θ $\quad f(\theta) = \frac{1}{2\pi}$ on $[0,2\pi]$

$$AB = 2a\sin\frac{\theta}{2}$$

M1 A1

$$E(AB) = \int_0^{2\pi} 2a\sin\frac{\theta}{2}\,\frac{1}{2\pi}\,d\theta$$

M1 A1

$$= \frac{2a}{\pi}\left[-\cos\frac{\theta}{2}\right]_0^{2\pi}$$

$$= \frac{4a}{\pi}$$

A1 (5)

(Alternatives replace θ with 2φ , $f(\varphi) = \frac{1}{\pi}$ on $[0,\pi]$,

or minor segment $AOB = 2\varphi$, $f(\varphi) = \frac{2}{\pi}$ on $\left[0,\frac{\pi}{2}\right]$)

(ii)

$$P(R \le x) = \frac{\pi x^2}{\pi a^2} = \frac{x^2}{a^2}$$

M1

Therefore

$$f_R(x) = \frac{2x}{a^2}$$

for $0 \le x \le a$ **A1 (2)**

If the ends of the chord are X and Y , then OXY is an isosceles triangle so

$$XY = 2\sqrt{a^2 - R^2\sin^2 t}$$

M1 A1 (2)

$$L(t) = \int_0^a 2\sqrt{a^2 - x^2\sin^2 t}\,\frac{2x}{a^2}\,dx$$

M1 A1

$$= \left[-\frac{4}{3a^2\sin^2 t}\left(a^2 - x^2\sin^2 t\right)^{\frac{3}{2}}\right]_0^a$$

$$= -\frac{4}{3a^2\sin^2 t}\left(a^3\cos^3 t - a^3\right)$$

$$= \frac{4a(1-\cos^3 t)}{3\sin^2 t}$$

as required. **dM1 *A1 (4)**

$$\frac{(1-\cos^3 t)}{\sin^2 t} = \frac{(1-\cos^3 t)}{(1-\cos^2 t)} = \frac{1+\cos t+\cos^2 t}{1+\cos t} = \frac{1}{1+\cos t} + \frac{\cos t\,(1+\cos t)}{1+\cos t}$$

M1

$$= \frac{1}{2\cos^2\frac{t}{2}} + \cos t = \cos t + \frac{1}{2}\sec^2\frac{t}{2}$$

M1 *A1 (3)

(Alternative

$$\frac{(1-\cos^3 t)}{\sin^2 t} = \frac{1-\cos t+\cos t\,(1-\cos^2 t)}{\sin^2 t} = \frac{2\sin^2\frac{t}{2}}{4\sin^2\frac{t}{2}\cos^2\frac{t}{2}} + \cos t = \frac{1}{2}\sec^2\frac{t}{2} + \cos t$$

M1 **M1** ***A1**)

giving

$$L(t) = \frac{4a}{3}\left(\cos t + \frac{1}{2}\sec^2\frac{t}{2}\right)$$

(iii)

$$E\big(L(T)\big) = \int_0^{\frac{\pi}{2}} \frac{4a}{3}\left(\cos t + \frac{1}{2}\sec^2\frac{t}{2}\right)\frac{2}{\pi}\,dt$$

M1 A1

$$= \frac{8a}{3\pi}\left[\sin t + \tan\left(\frac{t}{2}\right)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{16a}{3\pi}$$

M1 A1 (4)

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