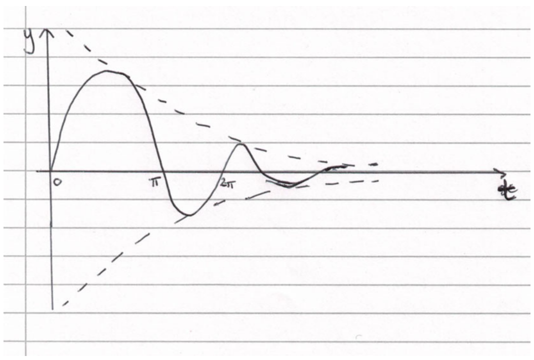
STEP MATHEMATICS 3

2019

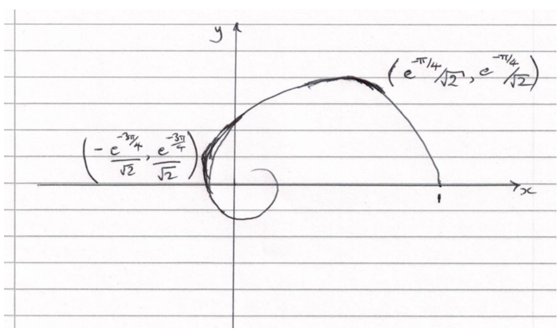
Hints and Solutions

Question 1

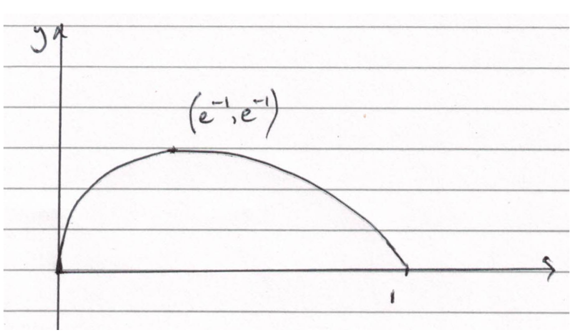
(i) Making y the subject of the first equation, it can be substituted in the second to give a second order differential equation solely in x. Using an auxiliary quadratic equation, this can be solved for x, and then y can be found from the first equation in terms of the same arbitrary constants, which can be evaluated using the initial conditions to give $x = e^{-t}\cos t$ and $y = e^{-t}\sin t$. The sketch of y as a function of t is



Differentiating y and then x with respect to t and in each case equating to zero gives the values of t for which y is greatest and for which x is least, giving the points $\left(\frac{e^{-\frac{\pi}{4}}}{\sqrt{2}}, \frac{e^{-\frac{\pi}{4}}}{\sqrt{2}}\right)$ and $\left(-\frac{e^{-\frac{3\pi}{4}}}{\sqrt{2}}, \frac{e^{-\frac{3\pi}{4}}}{\sqrt{2}}\right)$ so the sketch is



(ii) In this case, the first equation becomes a separable first order differential equation for x and substituting its solution into the second equation yields a first order differential equation for y which can be solved using an integrating factor. The arbitrary constants can be evaluated using the initial conditions to give $x = e^{-t}$ and $y = te^{-t}$. In order to sketch the path, x and y should both be differentiated with respect to t which indicate that the gradient at $(1,0)$ is -1 , that there is a maximum for y at (e^{-1}, e^{-1}) and that as the curve approaches the origin, it approaches the y axis tangentially, so the sketch is



Question 2

(i) Substituting $y = 0$ (and optionally $x = 0$) in the expression for $f(x + y)$ yields two possibilities, the required $f(0) = 1$ or a second which can be dismissed because $f'(0) \neq 0$. Using (*) and the expression for $f(x + y)$ applied for $y = h$ yields a factor $f(x)$ and the limiting expression is as required. Separating variables and applying the initial condition already found yields $f(x) = e^{kx}$.
Part (ii) proceeds similarly finding $g(0) = 0$, with a second option dismissed this time owing to the inequality for the modulus of g. The limiting follows that of (i) with a factor this time of $\left(1 - \left(g(x)\right)^2\right)$ giving $g'(x) = k\left(1 - \left(g(x)\right)^2\right)$. Separating variables and then either using partial fractions or using the standard hyperbolic result yields the solution in one of the forms

$g(x) = \frac{e^{2kx}-1}{e^{2kx}+1} = \tanh(kx)$ once the initial condition has been imposed.

Question 3

Writing $A\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x\\y\end{pmatrix}$ and expanding as two linear equations, in each of which the terms in x are on one side of the equation and y on the other, they can then be multiplied together and then rearranged to give the first desired result in (i). The further result requires consideration of the cases $x = 0$ and $y = 0$ which lead to values for either a and c, or b and d, which both similarly imply the required result. An alternative approach to this is to rewrite the original matrix equation in the form $B\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$ and as this does not have a unique solution being true for a line of invariant points, the determinant of B must be zero yielding both of the first two results. The final result of (i) can be obtained by considering $\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}x\\mx+k\end{pmatrix}=\begin{pmatrix}x\\mx+k\end{pmatrix}$ with $k \neq 0\ \ \forall x$ and $\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}k\\y\end{pmatrix}=\begin{pmatrix}k\\y\end{pmatrix}$ with $k \neq 0\ \ \forall y$, to evaluate a, b, c and d, and hence conclude that $A = I$. In (ii), a similar approach to that used at the start of the question as the condition if and only if a point is invariant leads to the line $(a-1)x + by = 0$ being the same as $cx + (d-1)y = 0$ if $(a-1)(d-1) = bc$ and $b \neq 0$. On the other hand, if $(a-1)(d-1) = bc$ and $b = 0$ then the cases of $a = 1$ or $\neq 1$ can be followed through to give lines $cx + (d-1)y = 0$ or the y axis as invariant. Part (iii) can be approached from considering $\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}x\\mx+k\end{pmatrix}=\begin{pmatrix}x'\\mx'+k\end{pmatrix}$ with $k \neq 0$. As this is true for all x, considering the case $x = 0$ yields an equation in b, d and m, and similarly $x = 1$ (of course using say x'' instead of x') yields one in a, c and m. Combining these yields $m(a-1)(d-1) = mbc$ and so the case $m = 0$ has to be considered, which returning to the initial line of working for this part and considering for two values of x gives $d = 1$ and $c = 0$ and hence the desired result.

Question 4

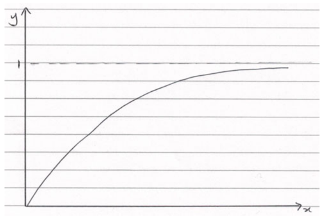
(i) For $n = 1$, it is easy to show that $x - a_1$ is reflexive. For $= 2$, Vieta's equations yield that $a_2 = 0$ and a_1 can take any value giving $x^2 - a_1 x$. For $= 3$, Vieta's equations yield $a_2 = -a_3$ from the sum of roots, and as a consequence that $a_2 = 0$ or $a_3 = 1$ from the sum of products of pairs of roots, leading, after using the third equation to the possibilities $x^3 - a_1 x^2$ and $x^3 + x^2 - x - 1$.

(ii) From the sum of roots Vieta relation a_1 can be eliminated, and the result can then be squared which with a little careful manipulation of the sum of product of roots two at a time Vieta relation yields the desired result. For $n > 3$ completion of the square yields a square on LHS and 1 minus a sum of squares on RHS. That the coefficients are integers can only yield ${a_n}^2 = 1$ for $a_n \neq 0$ and the other coefficients for $r = 3, \dots, n - 1$ are zero, which establishes a contradiction when considering the product of all roots.

(iii) As it has been deduced in (ii) that for any reflexive polynomial of degree greater than 3 that $a_n = 0$, such a polynomial can be factorised by x and then it can be carefully argued that the resulting polynomial is reflexive and so there is an inductive argument. So essentially the solutions are those found in (i) along with those solutions multiplied by arbitrary positive integer powers of x, that is $(x - a_1)x^r$ or $(x + 1)^2(x - 1)x^r$ with $r = 0, 1, 2, \dots$

Question 5

Differentiating f(x) and considering behaviour at the origin and for large x gives the following graph.



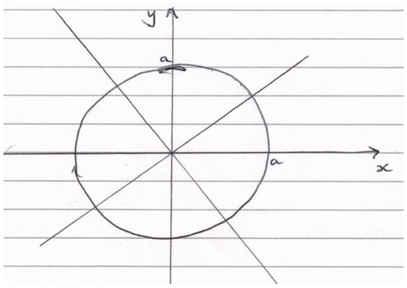
Using the substitution $y = \frac{cx}{\sqrt{x^2+p}}$ in the integral I, it can be seen to simplify to the given result if we choose $= 1$. The first evaluation uses that result with $c = \sqrt{2}$, and $= \sqrt{3}$, giving a result having used the noted standard integral of $\frac{\pi}{3\sqrt{3}}$. Making the substitution, $= \frac{1}{x}$, the second evaluation can be shown to be the same as the first. Returning to making the same general substitution (i.e. $y = \frac{bx}{\sqrt{x^2+p}}$ say) for part (iii),the resulting integral can be seen to be simplified by choice of b and p ($2b^2 = 1$ and $p = -1$) to simplify to the standard integral noted in the question and hence a result of $\frac{\pi}{4}$.

Question 6

The stem is achieved by using the direction given in the question, expansion and rearrangement to obtain the desired result, with a radius r and centre a. To obtain the locus of Q, substituting for z in terms of w, then multiplying through by ww^* and dividing by $aa^* - r^2$ followed by some simplification yields $\left|w - \frac{a^*}{aa^*-r^2}\right|^2 = \frac{r^2}{(aa^*-r^2)^2}$ which represents a circle centre $\frac{a^*}{aa^*-r^2}$ and radius $\left|\frac{r}{aa^*-r^2}\right|$. The second result of (i) is obtained by equating the radii and simplifying. Equating the centres and substituting for the denominator as each of the values derived from the result just found gives $a = \pm a^*$ and so the two possibilities for a can be concluded. In the case that is real, working back to the value that produced it, the radius is smaller than $|a|$ so the circle is centred on the real axis and does not contain the origin, whereas in the other case, the opposite is true so the origin is contained. For part (ii), the given result of part (i) still applies by a repetition of the previous working but with the new substitution, but equating the radii now gives that any a is possible with $|a| = \sqrt{r^2+1}$ (or a is zero) so the previous statement regarding a does not apply.

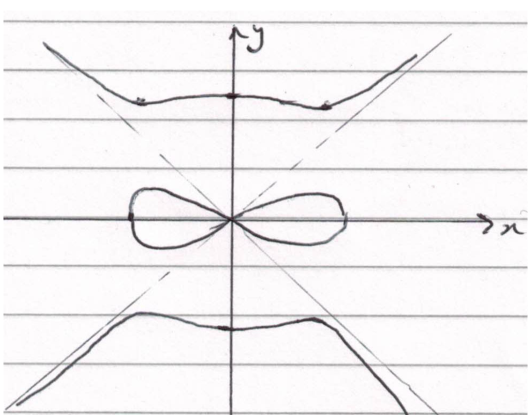
Question 7

(i) Treating both sides of the given equation as biquadratics and completing the squares, or subtracting and factorising the differences of squares, yields the line pair $y = \pm x$ and the circle $x^2 + y^2 = a^2$.



Following the instructions in the question for part (ii) (a), the positive discriminant condition for the biquadratic in x yields a biquadratic inequality for y which readily factorises to give the required result, bearing in mind that y positive ensures that two of the four linear factors are thus positive regardless. In part (b), ‘close to the origin’ indicates that only the terms of lowest degree need consideration, giving $y \approx \pm\frac{2x}{\sqrt{5}}$ (though the negative is discounted), and ‘very far from the origin’, the highest degree terms and $y \approx \pm x$ respectively. For (c) differentiating and setting $\frac{dy}{dx}$ equal to zero gives a simple cubic equation in x, which gives $\left(0, \sqrt{5}\right)$, $\left(\sqrt{2}\,,1\right)$, $\left(\sqrt{2}\,,2\right)$ but from (b) not $(0, 0)$ as points where the tangents are parallel to the x axis. Likewise for ‘parallel to the y axis’ with $\frac{dx}{dy}$, but the working of (a) and (b) restricts the possibilities to just the single one $(2\,, 0)$.

Using all the information gleaned, the sketch for (iii) is merely that for (ii)(c) employing symmetry in both axes, though, in both cases, care should be taken to ensure that the relative gradients near and from the origin reflect the results of (ii) (b).



Question 8

Using the convention of labelling ABCD anticlockwise and labelling the midpoint of AB as M, it is straightforward to find the vector MV as a scalar multiple of a unit vector and from it to deduce the required vector perpendicular to AVB as $\begin{pmatrix} 0 \\ -\sin\alpha \\ \cos\alpha \end{pmatrix}$. Doing the same for BVC yields $\begin{pmatrix} \sin\beta \\ 0 \\ \cos\beta \end{pmatrix}$ and the scalar product of these generates the result for (i). Labelling the centre of the base ACD as W, simple trigonometry enables lengths MW, BM and BW to be expressed in terms of VW, , β and θ , then the first result of (ii) is using Pythagoras. Using this result and standard trigonometric identities, it is possible to proceed to $\tan^2\varphi$, to $\sec^2\varphi$, and so to $\cos^2\varphi$; doing the comparable steps with α and β in the expression obtained and using the result of (i) where the product appears yields the required expression and considering $(\cos\alpha - \cos\beta)^2 \geq 0$ leads to the inequality. For the deduction, as θ is acute, the factor $1 - \cos\theta$ is positive and so can be cancelled, and also both $\frac{2}{(1+\cos\theta)} > 1$ and $\cos\theta > \cos\theta\cos\theta$. That $\cos^2\varphi \geq \cos^2\theta$, given the positivity of both cosines means the same applies to the unsquared quantities and hence the final result.

Question 9

The expression for the position of the particle is $= (a\sin\theta - s)\boldsymbol{i} + a\cos\theta\,\boldsymbol{j}$, obtained by adding the displacement of the particle relative to the hemisphere to that of the hemisphere, and differentiating this with respect to time gives the first required result in (i). Conserving the horizontal linear momentum of the system (i.e. the particle and the hemisphere together) and rearranging for $\dot{s}$ eliminating the masses in favour of the given variable k obtains the second result. The deduction is achieved by substituting the second result in the first. Part (ii) is obtained by conserving energy for the system, using the results for the speed of the hemisphere, the speed of the particle derived from its velocity obtained in (i), the gravitational potential energy of the particle and eliminating the masses using k. Part (iii) commences by finding $\ddot{\boldsymbol{r}}$ by differentiating $\dot{\boldsymbol{r}}$ from (i), then appreciating that the particle loses contact when $\ddot{\boldsymbol{r}} = -g\boldsymbol{j}$, equating the two expressions and taking the scalar product of the equation with the suggested vector. Substituting the required result in the result of (ii) leads to the equation $(k-1)\cos^3\alpha + 3\cos\alpha - 2 = 0$, and the deduction can be made by taking the term of degree 3 to the other side of the equation, and appreciating that in doing so it can be seen to be positive.

Question 10

A good diagram showing Q moving off along the line of centres after the collision and conserving linear momentum perpendicular to that line obtains the first result of (i). The expression for w can be found by conserving momentum perpendicular to the original direction of motion of P, or alternatively in the direction of the line of centres and then substituting for u using the first result; $w = v\frac{\sin\theta}{\sin\alpha}$. The first result of (ii) can be found by applying Newton's Law of Impact and then substituting for w and u using the results of (i). Using this result, expanding using compound angle formulae, and dividing by $\cos\theta\cos^2\alpha$ leads to $\tan\theta = \frac{(1+e)\tan\alpha}{1+2\tan^2\alpha-e}$ having applied trigonometric identities. To obtain the maximum of $\tan\theta$, it is worth simplifying the working by letting $t = \tan\alpha$ and differentiating with respect to t. The maximum value is $\frac{\sqrt{2}(1+e)}{4\sqrt{1-e}}$ (which occurs when $= \sqrt{\frac{1-e}{2}}$) which can be justified as it is the only stationary value, consideration of when $t = 0$, $t \to \infty$ and that $\tan\theta > 0$ for all t.

Question 11

To satisfy (i), it is necessary to find the probability that a number, say r, take sand which can be calculated by summing the product of the Poisson probability that a number of customers arrive and the binomial probability that r of that number take the offer; factorising out all the terms not involved in the summation index leaves a sum that can be recognised as an exponential function, and the result then follows. The mass taken by n customers can be expressed as a GP with first term kS and common ratio $(1-k)$ so the case $k=0$ needs considering separately, and then the expectation can be found by summing the product of these masses and the probabilities using the result of (i); each term naturally splits into two parts giving two exponential sums whose difference leads to the given result. Using the working from (i) and (ii), if r customers take the sand, the amount the assistant takes is $S(1-k)^r$ and so the probability the assistant takes the golden grain in that case can be shown to be $k(1-k)^r$. Summing over r the product of Poisson probabilities and that just found gives the probability the assistant takes the golden grain as $e^{-k\lambda p}$. In the case $k=0$, no sand is taken by anyone at all, so the answer is zero as per the formula, and as $\rightarrow 1$, the only way the assistant can get the grain is if no customer takes the sand so the probability approaches $e^{-\lambda p}$. To maximise, differentiation with respect to k yields a stationary value when $k=\frac{1}{\lambda p}$ which the given condition ensures is less than 1, and justification that this is a maximum can be shown by considering the gradient either side of this value or by using the second derivative.

Question 12

The result required in the stem can be derived by considering that for each subset, each integer can be an element of it or not leading to the number of possibilities stated. An alternative is to consider the sum of the number of subsets there are with r elements written as binomial coefficients and appreciating that this sum is $(1+1)^n$. $P(1 \in A_1) = \frac{1}{2}$ for (i) as 1 is equally likely to be or not in the set A_1. A similar approach for part (ii) yields $P(t \in A_1 \cap A_2) = \frac{1}{4}$ for any particular integer t and from that, the complement extended to all n integers gives the required result. The other two solutions are similarly $P(\, A_1 \cap A_2 \cap A_3 = \emptyset\,) = \left(\frac{7}{8}\right)^n$ and $P(\, A_1 \cap A_2 \cap \ldots \cap A_m = \emptyset\,) = \left(1 - \frac{1}{2^m}\right)^n$.

For (iii), considering that for any particular integer t, if $A_1 \subseteq A_2$ then $t \in A_1 \cap A_2$, $t \in A_1' \cap A_2$ or $t \in A_1' \cap A_2'$, gives $P(A_1 \subseteq A_2) = \left(\frac{3}{4}\right)^n$. The same approach yields the other two results as $P(A_1 \subseteq A_2 \subseteq A_3) = \left(\frac{4}{8}\right)^n = \left(\frac{1}{2}\right)^n$ and $P(A_1 \subseteq A_2 \subseteq \cdots \subseteq A_m) = \left(\frac{m+1}{2^m}\right)^n$.