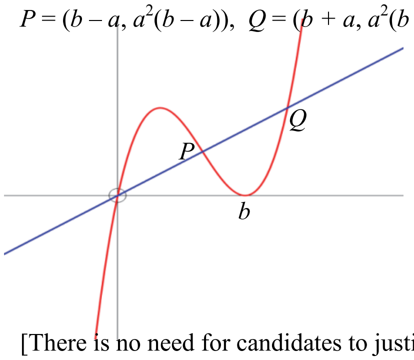
STEP MATHEMATICS 1

2018

Mark Scheme

Q1	$a^2x = x(b-x)^2$	**M1 equating the two equations (with/without the factor of x)**	
	$\Rightarrow\ a = b-x$ or $-a = b-x$	**M1 for solving method, this way or via a quadratic equation … which should be $x^2 - (b^2 - a^2)$**	
	$x = b-a$ or $b+a$	**A1 both**	
	$P = (b-a,\ a^2(b-a)),\ \ Q = (b+a,\ a^2(b+a))$	**A1 both y-coordinates**	



B1 for a fully correct graph; N.B. (b, 0) need not be noted **5**

[There is no need for candidates to justify that this is the correct arrangement: a second, more interesting, sketch arises when $0 < b < a$ but the question does not require it.]

$y = x^3 - 2bx^2 + b^2x \Rightarrow \dfrac{dy}{dx} = 3x^2 - 4bx + b^2$	**M1 for differentiating a cubic**	
or $\dfrac{dy}{dx} = (b-x)^2 - 2x(b-x)$	**using the *Product Rule* of differentiation on $y = x(b-x)^2$**	
$= 3(b^2 - 2ab + a^2) - 4b(b-a) + b^2$	**M1 for substituting $x = b - a$**	
$= 3a^2 - 2ab$ or $a(3a-2b)$ at P	**A1 (AG) for correct gradient in any form**	
Eqn. of tgt. at P is $y - a^2(b-a) = a(3a-2b)(x - [b-a])$	**M1 method for tgt. eqn. via $y - y_c = m(x - x_c)$ or $y = mx + c$ with P's coords. substd.**	
$y = a(3a-2b)x + a^2(b-a) - (3a^2 - 2ab)(b-a)$		
$y = a(3a-2b)x - (b-a)[4a^2 - 2ab]$		
$y = a(3a-2b)x + 2a(b-a)^2$	**A1 (AG) legitimately obtained & written in this form**	**5**

$S = \displaystyle\int_0^{b-a}\left(x^3 - 2bx^2 + b^2x\right)dx - \tfrac{1}{2}a^2(b-a)^2$	**M1 method for finding area by ∫n. – Δ area**	
$= \left[\dfrac{1}{4}x^4 - \dfrac{2}{3}bx^3 + \dfrac{1}{2}b^2x^2\right]_0^{b-a} - \tfrac{1}{2}a^2(b-a)^2$	**B1 for correct ∫n. of a 3 (or 4) term cubic (even if Δ omitted)**	
$= \tfrac{1}{4}(b-a)^4 - \tfrac{2}{3}b(b-a)^3 + \tfrac{1}{2}b^2(b-a)^2 - \tfrac{1}{2}a^2(b-a)^2$	**M1 for substn. of correct limits in any integrated terms**	
$= \tfrac{1}{12}(b-a)^2\{3(b-a)^2 - 8b(b-a) + 6(b^2 - a^2)\}$	**M1 for correctly factoring out at least two linear terms**	
$= \tfrac{1}{12}(b-a)^3(3b - 3a - 8b + 6b + 6a)$	**(must have a difference of two areas or equivalent)**	
$= \tfrac{1}{12}(b-a)^3(b+3a)$	**A1 (AG) legitimately obtained**	**5**

Area $\Delta OPR = \tfrac{1}{2}$ (y-coord. of R)×(x-coord. of P)	**M1 correct method for required area**	
$T = \tfrac{1}{2}.2a(b-a)^2.(b-a) = a(b-a)^3$	**A1 correct, factorised form for T seen at some stage**	
$\dfrac{S}{T} = \tfrac{1}{12}.\dfrac{b+3a}{a}$ or $S - \tfrac{1}{3}T = \ldots$ or $3S - T = \ldots$	**M1 for genuine attempt to consider any of these algebraically**	
	A1 (AG) correct result legitimately obtained	
$\dfrac{b+3a}{a} > \dfrac{a+3a}{a}$ ∵ $b > a$	**E1 for proper justification of result**	
or $3S - T = \tfrac{1}{4}(b-a)^4 > 0$ ∵ $b \neq a$	**(E0 for unexplained 'backwards' logic)**	**5**

Q2	$c = b^x$	**B1**	
		M1 Taking logs to base a and rearranging	**2**
(i)	$\log_{10} \pi^2 < 1$	**M1 Taking logs to base 10 of given inequality** **RHS might still be in terms of a log** **A1 Simplifying to an expression involving only one log** **(might be awarded later)**	
	$$\frac{1}{\log_2 \pi} + \frac{1}{\log_5 \pi} = \frac{\log_{10} 2}{\log_{10} \pi} + \frac{\log_{10} 5}{\log_{10} \pi}$$	**M1 Writing both denominators in the same base** **(might not be base 10)**	
	$$= \frac{1}{\log_{10} \pi}$$	**M1 A1 Simplifying to an expression involving only one log**	
	Linking to given inequality to complete the proof that LHS > 2 **AG**	**E1 Penalise answers which assume the result here**	**6**
(ii)	$\ln \pi > 1 + \frac{1}{5} \ln 2$	**M1 Using the change-of-base-formula to turn into "ln"** **A1 Producing a correctly simplified version** **(may be given implicitly later)**	
	$\ln 2 > \frac{2}{3}$	**M1 A1 Taking natural logs of given inequality**	
	Combining both facts to get $\ln \pi > \frac{17}{15}$ **AG**	**E1 Penalise answers which assume the result here**	**5**
(iii)	$\ln \pi < 1 + \frac{1}{2} \ln 10$	**M1 Taking a log of given inequality**	
	$$= \frac{\log_{10} 10}{2 \log_{10} e}$$	**M1 Converting to base 10 using the change-of-base-formula**	
	$\log_{10} e > \frac{1}{3} \log_{10} 20$	**M1 Taking log to base 10 of given inequality**	
	$= \frac{1}{3}(1 + \log_{10} 2)$	**M1 Linking to $\log_{10} 2$**	
	$> \frac{13}{30}$	**A1 Correct use of given result**	
	Putting it all together to get $\ln \pi < \frac{15}{13}$ **AG**	**E2 Penalise if inequality directions misused**	**7**

Q3 (i) $\tan\alpha = \frac{y}{x+a}$ and $\tan\beta = \frac{y}{2a-x}$. **B1 B1**

If $\beta = 2\alpha$ then

$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$ **M1 for using formula**

$= \frac{\frac{2y}{x+a}}{1-\frac{y^2}{(x+a)^2}}$ **A1 correct unsimplified**

$= \frac{y}{2a-x}$ **M1 equating with tan β**

i.e. $\frac{y}{2a-x} = \frac{2y(x+a)}{(x+a)^2-y^2}$ **A1 simplified equation**

$y((x+a)^2 - y^2) = 2y(x+a)(2a-x)$ **M1 for getting rid of fractions**

$(x+a)^2 - y^2 = 2(x+a)(2a-x)$ since $y > 0$ **E1 for justifying this step (this could happen earlier)**

$x^2 + a^2 + 2ax - y^2 = 4a^2 - 2x^2 + 2ax$ so

$3x^2 - 3a^2 = y^2$ **A1 (AG)**

Alt.1:

$y = \text{PR}\sin\alpha = \text{PS}\sin 2\alpha$ so $\text{PR} = 2\text{PS}\cos\alpha$ **M1 M1 for useful expression for cos α**

$x + a = PR\cos\alpha = 2PS\cos^2\alpha$

$2a - x = \text{PS}\cos 2\alpha = 2\text{PS}\cos^2\alpha - \text{PS}$

so $3x - 3a = 2\text{PS}(1-\cos^2\alpha) = 2\text{PS}\sin^2\alpha$ **M1 A1 A1 for expressing x^2 and y^2 in terms of α and a length**

so $3(x^2 - a^2) = 4\text{PS}^2\sin^2\alpha\cos^2\alpha = y^2$. **M1 A1 for expression for $3(x^2 - a^2)$**

M1 A1 (AG) for checking equality

Alt.2:

Let angle bisector of S meet PR at T.

PST and PRS are similar **B1**

so PT/PS = PS/PR **M1 A1**

$PT = PR\frac{x-a/2}{x+a}$, and so **M1 A1**

$PR^2\left(x - \frac{a}{2}\right) = PS^2(x+a)$ **M1**

Pythagoras gives **M1 A1 unsimplified cubic**

$((x-2a)^2 + y^2)(x+a) = ((x+a)^2 + y^2)\left(x - \frac{a}{2}\right)$

Simplifying: $\frac{3a}{2}y^2 = \frac{9a}{2}(x^2 - a^2)$ **A1 (AG)**

For methods (not involving similar triangles) which reach a higher-order polynomial in a, x and y, give **M1 M1 A1**.

As progress towards this, give **M1 M1** *for Sine Rule + Pythagoras.*

Alt.3:

$y = \tan\alpha\,.(x+a)$ and $y = \tan\beta\,.(2a-x)$ **B1 B1**

$\tan\alpha\,.(x+a) = \frac{2\tan\alpha}{1-\tan^2\alpha}(2a-x)$ **M1 for double tangent A1**

$\tan\alpha \neq 0$ so $x + a = \frac{2a-x}{1-\tan^2\alpha}$ **E1**

giving $x = \frac{3+\tan^2\alpha}{3-\tan^2\alpha}a$ **M1 A1 writing x in terms of a and tan α**

and $y = \cdots$ **M1 A1 for expression for y and checking $y^2 = 3(x^2 - a^2)$** **9**

(ii) If $3(x^2 - a^2) = y^2$ then

$(x+a)^2 - y^2 = 2(x+a)(2a-x)$ **M1 rearranging into something useful**

$x \neq 2a, -a$ (the latter because $y > 0$) **E1 justifying this**

meaning both sides non-zero so

$\frac{y}{2a-x} = \frac{2y(x+a)}{(x+a)^2-y^2}$ **M1 dividing through**

$= \frac{\frac{2y}{x+a}}{1-\frac{y^2}{(x+a)^2}}$ **M1 A1 for something in terms of tan α**

So $\tan\beta = \tan 2\alpha$ **A1**

Some candidates might just say "everything in (i) is reversible", without checking. I suggest such a claim would get the three M marks above but not the A or E marks. If for some reason a candidate does this part but not (i), they should also get the two B1 marks and the first M1 from part (i) for using these facts here.

Other methods exist which give instead $\cos 2\alpha = \pm\cos\beta$.

Alt.:

$\tan(\beta-\alpha) = \dfrac{\frac{y}{2a-x}-\frac{y}{x+a}}{1+\frac{y^2}{(x+a)(2a-x)}}$	**M1**	
$= \dfrac{y(2x-a)}{(x+a)(2a-x)+y^2}$	**M1 for single fraction**	
$= \dfrac{y(2x-a)}{(x+a)(2a-x)+3x^2-3a^2}$	**M1 substituting y**	
$= \dfrac{y(2x-a)}{(x+a)(2x-a)}$	**A1**	
Since $x \neq a/2$ (as otherwise $y^2 < 0$)	**E1 (be generous if there is an attempt to justify)**	
we get $\tan(\beta-\alpha) = \tan\alpha$	**A1**	**6**

This means $\beta = 2\alpha + k\pi$ for some integer k.	**B1**	
$0 < \alpha < \pi$ so $y > 0$ and $0 < \beta < \pi$	**B1 for bounding β (the bound you get depends on whether**	
OR $y > 0$ and $x < 2a$ so $0 < \beta < \pi/2$	**you use the information given in this part or given earlier)**	
so $-\pi < 2\alpha - \beta < 2\pi$	**M1 for using this to bound k**	
so $k = 0, -1$	**A1 only two values of k – don't worry about a sign error**	
giving $\beta = 2\alpha$ or $\beta = 2\alpha - \pi$.	**A1 cao (don't need to check both are possible)**	**5**

Alt. part (ii) (all 11 marks):

Construct the point $S' = (2x - 2a, 0)$,	
making PSS′ isosceles.	**M2***
Now $\text{PS}^2 = y^2 + (2a - x)^2$	
$= 3(x^2 - a^2) + (2a - x)^2$	
$= (2x - a)^2 = \text{RS}'^2$	**M2* A2***
Thus we have $\text{PS} = \text{PS}' = \text{RS}'$	
If S′ lies between R and S, this gives $\text{RPS}' = \alpha$	
and $\text{PS}'\text{R} = \pi - 2\alpha$ so $\beta = 2\alpha$.	**M1 A1**
If R lies between S′ and S, this gives	**B1 for considering both cases**
$\text{PRS}' = (\pi - \beta)/2$ so $\beta = 2\alpha - \pi$.	**M1 A1**

Candidates who attempt this are likely to do all the calculations separately for the two cases. If so, give 1 mark out of each 2 above for each part where the corresponding working appears.*

Q4

$$f'(x) = \frac{x\ln x.2\left(1-(\ln x)^2\right).-2\ln x.\frac{1}{x} - \left(1-(\ln x)^2\right)^2.\left(x.\frac{1}{x}+\ln x\right)}{\left(x\ln x\right)^2}$$

M1 Use of product or quotient rule (or alt. substn.)
A1 1st term (numerator) correct
A1 2nd term (numerator) & denominator correct

Showing both f(x) and f ′(x) = 0 when $(\ln x)^2 = 1$ — **E1** — **4**

(i) $u = \ln t$ — **M1 Any sensible substitution**

$I = \int \frac{(1-u^2)^2}{u}\,du$ — **M1 A1 Full substitution used; correct** $= \int\left(\frac{1}{u} - 2u + u^3\right)du$

$= \ln|u| - u^2 + \frac{1}{4}u^4 \quad (+\,c)$ — **A1 Penalise absence of modulus signs here (but allow for next 2 marks)**

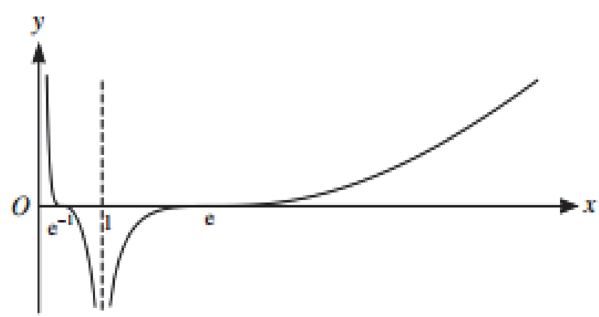
$= \ln|\ln x| - (\ln x)^2 + \frac{1}{4}(\ln x)^4 + \frac{3}{4}$ for $0 < x < 1$ — **A1**

$= \ln|\ln x| - (\ln x)^2 + \frac{1}{4}(\ln x)^4 + \frac{3}{4}$ for $x > 1$ — **A1** — **6**

$F(x^{-1}) = \ln|-\ln x| - (-\ln x)^2 + \frac{1}{4}(-\ln x)^4 + \frac{3}{4}$
$= F(x)$

M1 For using $\ln(x^{-1}) = -\ln x$
E1 For candidates who notice that F(x) takes the same functional form, this will be quite easy. Otherwise, two cases are required. — **2**

(ii)



G1 Asymptote $x = 0$
G1 Asymptote $x = 1$
G1 Negative gradient for $0 < x < 1$
G1 Positive gradient for $x > 1$
G1 Stationary points at $x = e^{-1}$ and $x = e$
G1 Points of inflexion at $x = e^{-1}$ and $x = e$
G1 Zeroes at $x = e^{-1}$ and $x = e$
G1 Generally correct shape — **8**

Q5 **(i)** $k(x-1)(x-2)(x-3)(x-4)+1$ — **B1 no need to state that $k \neq 0$** — **1**

(ii)

$P(x) = k(x-1)(x-2)(x-3)\ldots(x-N)+1$ — **B1**

$P(N+1) = k(N)(N-1)(N-2)\ldots(1)+1$ — **M1 $P(N+1)$ for an N^{th}-degree polynomial**

$\quad = k(N!)+1 = 1 \text{ iff } k = 0$

Alt. $P(x) = 1$ is a polynomial of degree N so has N roots, 1 to N inclusive; but if $P(N+1) = 1$ also then it has $N+1$ … a contradiction — **A1**

$P(N+1) = 2 \text{ iff } k = \dfrac{1}{N!}$ — **A1 (no k, no mark)**

$P(N+r) = \dfrac{1}{N!}(N+r-1)(N+r-2)(N+r-3)\ldots(r)+1$

$\quad = \dfrac{(N+r-1)!}{N!(r-1)!}+1 \text{ or } \dbinom{N+r-1}{N}+1$ — **B1 any form**

Let $m = N+r$ (so that $m > N$)

Require $P(m) = \dbinom{m-1}{N}+1 = m$ or $\dbinom{m-1}{N} = m-1$ — **M1 or equivalent statement**

$\dbinom{m-1}{N} = \dfrac{(m-1)(m-2)(m-3)\ldots(m-N)}{N(N-1)\ldots\times 2} = m-1$ — **M1 for general approach**

$\Rightarrow m = N+2 \quad \text{i.e. } r = 2$ — **A1**

<u>Notes:</u> Question only requires candidates to *find* a suitable r so noting $r = 2$ (**M1**) and checking that it works (**M1 A1**) can score all of these final 3 marks — **8**

(iii) $S(x) = (x-a)(x-b)(x-c)(x-d)+2001$ — **B1 stated (a, b, c, d distinct integers)** — **1**

(a)

$S(e) = (e-a)(e-b)(e-c)(e-d)+2001 = 2018$ — **M1**

$\Rightarrow (e-a)(e-b)(e-c)(e-d) = 17$ — **A1**

$\Rightarrow$ 17 has (at least) 4 distinct integer factors — **M1 looking at factorisations of 17**

However, 17 has only four factors; ± 1, ± 17 and, as both 17s cannot be used, no such integer e exists — **A1 must be fully explained** — **4**

(b)

$S(x) = (x-a)(x-b)(x-c)(x-d)+2001$ (integers $a < b < c < d$)

$S(0) = abcd + 2001 = 2017 \Rightarrow abcd = 16$ — **B1**

and we require 16 to be written as the product of four distinct integers a, b, c, d with $a < b < c < d$

Thus $a, b, c, d \in \{\pm 1, \pm 2, \pm 4, \pm 8, \pm 16\}$; allow $\{\pm 1, \pm 2, \pm 4, \pm 8\}$ **M1**

I If $a = -16$, then $b, c, d \in \{\pm 1\}$ and this cannot be done distinctly

II If $a = -8$, then $b, c, d \in \{\pm 1, \pm 2\}$ with exactly one of them $-_{ve}$

$\Rightarrow (a, b, c, d) = \mathbf{(-8, -1, 1, 2)}$

III If $a = -4$, then $b, c, d \in \{\pm 1, \pm 2, \pm 4\}$ with exactly one of them $-_{ve}$

$\Rightarrow (a, b, c, d) = \mathbf{(-4, -2, 1, 2)}$ or $\mathbf{(-4, -1, 1, 4)}$

IV If $a = -2$, then $b, c, d \in \{\pm 1, \pm 2, \pm 4, \pm 8\}$ with exactly one of them $-_{ve}$

$\Rightarrow (a, b, c, d) = \mathbf{(-2, -1, 2, 4)}$ or $\mathbf{(-2, -1, 1, 8)}$

V $a \neq -1$ since then $abcd < 0$ and if $a > 0$ then $abcd \geq 64$

There are thus 5 ways in which a, b, c, d can be chosen s.t. $S(0) = 2017$

M1 for a (partially) systematic case analysis
A1 for any three correct solutions
A1 for all five and no extras
E1 for correct justification no solutions in cases I, V

<u>Important note:</u> Candidates need to identify clearly the *number* of cases (so the actual solutions are not required) and may still gain the marks despite numerical errors if the method for finding them is clearly explained. However, I very much doubt this will happen.

Alt. 1 The cases could be argued by sign first and then value, as follows.

a, b, c, d cannot be all $+_{\text{ve}}$ or all $-_{\text{ve}}$ since then $abcd \geq 64$

so we must have two $+_{\text{ve}}$ and two $-_{\text{ve}}$.

Note that $|a| \neq 16$ since all three others must then have $|..| = 1$.

So the options are:

I $(a, b) = (-8, -4)$ impossible since $abcd$ already too big

II $(a, b) = (-8, -2)$ impossible since then both c, d must equal 1

III $(a, b) = (-8, -1) \Rightarrow (c, d) = (1, 2)$

IV $(a, b) = (-4, -2) \Rightarrow (c, d) = (1, 2)$

V $(a, b) = (-4, -1) \Rightarrow (c, d) = (1, 4)$

VI $(a, b) = (-2, -1) \Rightarrow (c, d) = (1, 8)$ or $(2, 4)$

and there are thus 5 ways in which a, b, c, d can be chosen s.t. S(0) = 2017

M1 for a (partially) systematic case analysis
A1 for any three correct solutions
A1 for all five and no extras
E1 for correct justification no solutions in cases I, II

Alt. 2 Instead, one might reason thus:

As a product of four factors, in magnitude order,

$16 = 1.1.1.16$ or $1.1.2.8$ or $1.1.4.4$ or $1.2.2.4$ or $2.2.2.2$

We reject the first and last of these since we can have at most two of equal magnitude (two $+_{\text{ve}}$ and two $-_{\text{ve}}$). This leaves us with

I 1.1.2.8 gives $(a, b, c, d) = \{-1, 1, -2, 8\}$ or $\{-1, 1, 2, -8\}$
i.e. $(a, b, c, d) = (-2, -1, 1, 8)$ or $(-8, -1, 1, 2)$

II 1.1.4.4 gives $(a, b, c, d) = \{-1, 1, -4, 4\}$
i.e. $(a, b, c, d) = (-4, -1, 1, 4)$

III 1.2.2.4 gives $(a, b, c, d) = \{-2, 2, 1, -4\}$ or $\{-2, 2, -1, 4\}$
i.e. $(a, b, c, d) = (-4, -2, 1, 2)$ or $(-2, -1, 2, 4)$

M1 for a (partially) systematic case analysis
A1 for any three correct solutions
A1 for all five and no extras
E1 for initial justification which 4-term factorisations of 16 work

6

Q6 $2\sin\theta\left(\sin\theta+\sin 3\theta+\sin 5\theta+\ldots+\sin(2n-1)\theta\right)$

$\equiv 2\sin\theta\sin\theta+2\sin\theta\sin 3\theta+2\sin\theta\sin 5\theta+\ldots+2\sin\theta\sin(2n-1)\theta$

$\equiv \left(\cos 0-\cos 2\theta\right)+\left(\cos 2\theta-\cos 4\theta\right)+\left(\cos 4\theta-\cos 6\theta\right)+\ldots$ **M1 complete method using given identity**

$\ldots+\left(\cos(2n-2)\theta-\cos 2n\theta\right)$

$\equiv 1-\cos 2n\theta$ since all intermediate terms cancel **A1 (AG) legitimately obtained** **2**

(i) The midpoint of the k^{th} strip is at $x=\dfrac{(k-\frac{1}{2})\pi}{n}$ or $\dfrac{(2k-1)\pi}{2n}$ **B1**

Ht. of strip is $\sin\dfrac{(2k-1)\pi}{2n}$ and its area is $\dfrac{\pi}{n}\sin\dfrac{(2k-1)\pi}{2n}$ **B1**

A_n = sum of all strips

$=\dfrac{\pi}{n}\left(\sin\theta+\sin 3\theta+\sin 5\theta+\ldots+\sin(2n-1)\theta\right),\ \theta=\dfrac{\pi}{2n}$ **M1**

$=\dfrac{\pi}{n}\left(\dfrac{1-\cos 2n\theta}{2\sin\theta}\right)$ **M1 for using the initial result**

$=\dfrac{\pi}{n}\left(\dfrac{1-\cos\pi}{2\sin\theta}\right)=\dfrac{\pi}{n}\left(\dfrac{2}{2\sin\left(\dfrac{\pi}{2n}\right)}\right)$

$\Rightarrow A_n\sin\dfrac{\pi}{2n}=\dfrac{\pi}{n}$ **A1 (AG) fully established** **5**

(ii) $B_n=\frac{1}{2}\left(\dfrac{\pi}{n}\right)\left\{\sin 0+2\left(\sin\left(\dfrac{\pi}{n}\right)+\sin\left(\dfrac{2\pi}{n}\right)+\sin\left(\dfrac{3\pi}{n}\right)+\ldots+\sin\left(\dfrac{(n-1)\pi}{n}\right)\right)+\sin\pi\right\}$

M1 use of Trapezium Rule formula in context

$=\left(\dfrac{\pi}{n}\right)\left(\sin\theta+\sin 2\theta+\sin 3\theta+\ldots+\sin(n-1)\theta\right),\ \theta=\dfrac{\pi}{n}$

$B_n\sin\dfrac{\pi}{2n}=\left(\dfrac{\pi}{n}\right)\sin\tfrac{1}{2}\theta\left(\sin\theta+\sin 2\theta+\sin 3\theta+\ldots+\sin(n-1)\theta\right)$

$=\left(\dfrac{\pi}{2n}\right)\left(2\sin\tfrac{1}{2}\theta\sin\theta+2\sin\tfrac{1}{2}\theta\sin 2\theta+2\sin\tfrac{1}{2}\theta\sin 3\theta+\ldots+2\sin\tfrac{1}{2}\theta\sin(n-1)\theta\right)$

M1 use of initial result

$=\left(\dfrac{\pi}{2n}\right)\left\{\left(\cos\tfrac{1}{2}\theta-\cos\tfrac{3}{2}\theta\right)+\left(\cos\tfrac{3}{2}\theta-\cos\tfrac{5}{2}\theta\right)+\ldots+\left(\cos(n-\tfrac{3}{2})\theta-\cos(n-\tfrac{1}{2})\theta\right)\right\}$

$=\left(\dfrac{\pi}{2n}\right)\left\{\cos\left(\dfrac{\pi}{2n}\right)-\cos(n-\tfrac{1}{2})\left(\dfrac{\pi}{n}\right)\right\}$ **A1 all intermediate terms cancelled**

Now, $\cos(n-\tfrac{1}{2})\left(\dfrac{\pi}{n}\right)=\cos\left(\pi-\dfrac{\pi}{2n}\right)=-\cos\left(\dfrac{\pi}{2n}\right)$ **M1 dealing with the final term in { }**

so $B_n\sin\dfrac{\pi}{2n}=\left(\dfrac{\pi}{2n}\right)\left\{2\cos\left(\dfrac{\pi}{2n}\right)\right\}=\dfrac{\pi}{n}\cos\left(\dfrac{\pi}{2n}\right)$ **A1** **5**

(iii) $A_n + B_n$

$$= \frac{\pi}{n\sin\left(\frac{\pi}{2n}\right)} + \frac{\pi\cos\left(\frac{\pi}{2n}\right)}{n\sin\left(\frac{\pi}{2n}\right)} = \frac{\pi}{n\sin\left(\frac{\pi}{2n}\right)}\left(1+\cos\left(\frac{\pi}{2n}\right)\right)$$ **M1**

$$= \frac{\pi}{n\sin\left(\frac{\pi}{2n}\right)}\left(1+2\cos^2\left(\frac{\pi}{4n}\right)-1\right)$$ **M1 use of double-angle formula**

$$= \frac{2\pi\cos^2\left(\frac{\pi}{4n}\right)}{n\sin\left(\frac{\pi}{2n}\right)}$$ **A1 or equivalent later tidying up**

$$B_{2n} = \frac{\pi\cos\left(\frac{\pi}{4n}\right)}{2n\sin\left(\frac{\pi}{4n}\right)} = \frac{\pi\cos\left(\frac{\pi}{4n}\right)\cos\left(\frac{\pi}{4n}\right)}{n.2\sin\left(\frac{\pi}{4n}\right)\cos\left(\frac{\pi}{4n}\right)} = \frac{\pi\cos^2\left(\frac{\pi}{4n}\right)}{n\sin\left(\frac{\pi}{2n}\right)}$$ **M1 B_n result with $n \to 2n$**

$= \frac{1}{2}(A_n + B_n)$ as required **A1 (AG) fully established**

Alt. $A_n = \frac{\pi}{n}\operatorname{cosec}\frac{\pi}{2n}$ and $B_n = \frac{\pi}{n}\cot\frac{\pi}{2n}$, so the result now boils down to trig. identity work:

$A_n + B_n = \frac{\pi}{n}\left(\frac{1+\cos\theta}{\sin\theta}\right)$, $\theta = \frac{\pi}{2n}$ **M1**

$= \frac{1+2c^2-1}{2sc} = \frac{c}{s}$ where $c = \cos\frac{1}{2}\theta$ and $s = \sin\frac{1}{2}\theta$ **M1 A1 using half-angle results**

$= 2\frac{\pi}{2n}\cot\frac{\pi}{4n} = 2B_{2n}$ **A1 B1** **5**

$$A_n B_{2n} = \frac{\pi}{n\sin\left(\frac{\pi}{2n}\right)} \times \frac{\pi\cos^2\left(\frac{\pi}{4n}\right)}{n\sin\left(\frac{\pi}{2n}\right)}$$ using the work above

$$= \left[\frac{\pi\cos\left(\frac{\pi}{4n}\right)}{n\sin\left(\frac{\pi}{2n}\right)}\right]^2$$ **B1 sensible expression for LHS**

$$A_{2n} = \frac{\pi}{2n\sin\left(\frac{\pi}{4n}\right)} = \frac{\pi\cos\left(\frac{\pi}{4n}\right)}{n.2\sin\left(\frac{\pi}{4n}\right)\cos\left(\frac{\pi}{4n}\right)} = \frac{\pi\cos\left(\frac{\pi}{4n}\right)}{n\sin\left(\frac{\pi}{2n}\right)}$$ **M1 attempt at RHS**

Since all terms are positive, we can take positive square roots to get the required result,

$$\sqrt{A_n B_{2n}} = A_{2n}$$ **A1 (AG) fully established**

Alt. Done via trig. identity work: $A_n B_{2n} = \frac{\pi}{n}.\frac{1}{2sc}.\frac{\pi}{2n}.\frac{c}{s} = \frac{\pi^2}{4n^2s^2} = \left(\frac{\pi}{2n}.\frac{1}{s}\right)^2 = (A_{2n})^2$ etc. **3**

Q7 (i) If $x = \frac{pz+q}{z+1}$ and $x^3 - 3pqx + pq(p+q) = 0$ then

$\left(\frac{pz+q}{z+1}\right)^3 - 3pq\left(\frac{pz+q}{z+1}\right) + pq(p+q) = 0$ so	**M1 substitution**
$(pz+q)^3 - 3pq(pz+q)(z+1)^2$ $+ pq(p+q)(z+1)^3 = 0$	**M1 multiplying by $(z+1)^3$**
i.e. $(p^3 - 3p^2q + p^2q + pq^2)z^3$	**M1 for expanding & collecting like terms**
$+(3p^2q - 3pq^2 - 6p^2q + 3p^2q + 3pq^2)z^2$	**A1 (AG) for checking middle terms vanish**
$+(3pq^2 - 3p^2q - 6pq^2 + 3p^2q + 3pq^2)z$	
$+(q^3 - 3pq^2 + p^2q + pq^2) = 0$	**A1 for correct first/last terms (do not need to divide by**
i.e. $(p-q)^2(pz^3+q) = 0$ and $p \neq q$ so $pz^3 + q = 0$	**$(p-q)^2$ to get this mark, but it will help later!)**

Alt.: *write $z = \frac{x-q}{p-x}$ (**M1**) and substitute in $az^3 + b = 0$ (**M1**). Expand and collect terms (**M1**) and check $\frac{b}{a} = \frac{q}{p}$ for no quadratic term (**A1**). Initial cubic legitimately obtained (**A1**).* **5**

(ii)

We need $pq = c$ and $pq(p+q) = d$ so $p + q = \frac{d}{c}$	**M1 conditions on p, q**
These are roots of a quadratic $y^2 - \frac{d}{c}y + c = 0$	**M1 A1**
This has distinct real roots iff $\left(\frac{d}{c}\right)^2 - 4c > 0$	**M1 for evaluating discriminant**
Since $c^2 > 0$, iff $d^2 > 4c^3$.	**A1 (AG)**

Candidates who evaluate $d^2 - 4c^3$ in terms of p and q and show the inequality holds just get first M1 (this is the converse). By writing $p = c/q$ or vice versa it is possible to get a quadratic for one of them, but unless they justify that p, q distinct when the discriminant is positive, don't give the final A1. Another alternative is to calculate $(p-q)^2 = \frac{d^2}{c^2} + 4c$ and use this to deduce values for p and q (this is equivalent to the normal solution). **5**

(iii)

We need $p + q = 1$ and $pq = -2$ so $p = 2, q = -1$	**M1 A1 (using quadratic or by inspection)**
So this reduces to $2z^3 - 1 = 0$ and $z = 2^{-1/3}$	**M1 A1 ft for value of z**
and $x = \frac{2z-1}{z+1} = \frac{2^{2/3}-1}{2^{-1/3}+1}$	**M1 A1 ft calculating x**
OR … so $p = -1, q = 2$	
So this reduces to $2 - z^3 = 0$ and $z = 2^{1/3}$	**(only one of these needed)**
and $x = \frac{2-z}{z+1} = \frac{2-2^{1/3}}{2^{1/3}+1}$ (equivalent to above)	

6

(iv)

$x = p$ is a root;	**M1 spotting (any) root**
factoring gives $(x-p)(x^2 + p - 2p^2) = 0$	**A1**
and $(x-p)(x-p)(x+2p) = 0$ so $x = p, -2p$	**A1**
Thus the equation reduces to the above with $p = \frac{d}{2c}$ so has roots $x = \frac{d}{2c}, \frac{-d}{c}$.	**A1 ft**

*Equivalent values of p, such as $\sqrt[3]{d/2}$ are fine here, but **NOT** $p = \sqrt{c}$ as this isn't necessarily the correct sign.* **4**

Q8

(i) $\frac{\mathrm{d}}{\mathrm{d}x}\left(s(x)^3 + c(x)^3\right) = 3s(x)^2 s'(x) + 3c(x)^2 c'(x)$ **using the *Chain Rule* of differentiation M1**

$= 3s(x)^2 . c(x)^2 + 3c(x)^2 . - s(x)^2 = 0 \Rightarrow s(x)^3 + c(x)^3 = \text{constant}$ **A1**

Since $s(0)^3 + c(0)^3 = 0^3 + 1^3 = 1$, $s(x)^3 + c(x)^3 = 1$ for all x **A1** **3**

(ii) $\frac{\mathrm{d}}{\mathrm{d}x}\left(s(x)c(x)\right) = s(x)c'(x) + s'(x)c(x)$ **using the *Product Rule* of differentiation**

$= s(x)^2 . - s(x)^2 + c(x)^2 . c(x)$ **with derivatives substd. M1**

$= c(x)^3 - s(x)^3 = c(x)^3 - \left[1 - c(x)^3\right]$ using **(i)**'s result **A1**

$= 2c(x)^3 - 1$ **must show that given answer is obtained from (i)** **2**

$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{s(x)}{c(x)}\right) = \frac{c(x)s'(x) - s(x)c'(x)}{c(x)^2}$ **using the *Quotient Rule* of differentiation**

$= \frac{c(x)c(x)^2 - s(x). - s(x)^2}{c(x)^2}$ **with derivatives substd. M1**

$= \frac{c(x)^3 + s(x)^3}{c(x)^2} = \frac{1}{c(x)^2}$ using **(i)**'s result **m.s.t.g.a.i.o.f.(i) A1** **2**

(iii) $\int s(x)^2 \, \mathrm{d}x = -c(x) + K$ **ignore missing *K*'s throughout B1** **1**

$\int s(x)^5 \, \mathrm{d}x = \int s(x)^3 s(x)^2 \, \mathrm{d}x$ **correct splitting and …**

$= \int \left[1 - c(x)^3\right] s(x)^2 \, \mathrm{d}x$ using **(i)**'s result **… use of (i) or ∫n. by parts* M1**

$= \int s(x)^2 \, \mathrm{d}x - \int c(x)^3 . - c'(x) \, \mathrm{d}x$ **or use of parts twice M1**

$= -c(x) + \tfrac{1}{4} c(x)^4 + K$ **using "reverse *Chain Rule*" integration A1**

NB* $I = \int s^5 \, \mathrm{d}x = \int s^3 s^2 \, \mathrm{d}x = s^3 . {-c} + \int 3s^2 c^3 \, \mathrm{d}x = -c\, s^3 + 3\int s^2 (1 - s^3) \, \mathrm{d}x$

$= -c\, s^3 + 3\int s^2 \, \mathrm{d}x - 3I \Rightarrow 4I = -c\, s^3 - 3c \Rightarrow I = -\tfrac{3}{4} c(x) - \tfrac{1}{4} c(x) s(x)^3 + K$ **3**

(iv) $u = s(x) \Rightarrow \mathrm{d}u = s'(x)\, \mathrm{d}x = c(x)^2 \, \mathrm{d}x$ & $1 - u^3 = 1 - s(x)^3 = c(x)^3$ **full substn. prepn. B1**

$\int \frac{1}{\left(1 - u^3\right)^{\frac{2}{3}}} \, \mathrm{d}u = \int \frac{1}{c(x)^2} . c(x)^2 \, \mathrm{d}x = \int 1 \, \mathrm{d}x = x + K = s^{-1}(u) + K$ **M1 A1** **3**

(v) $\int \frac{1}{\left(1 - u^3\right)^{\frac{4}{3}}} \, \mathrm{d}u = \int \frac{1}{c(x)^4} . c(x)^2 \, \mathrm{d}x$ **full substn. M1**

$= \int \frac{1}{c(x)^2} \, \mathrm{d}x = \frac{s(x)}{c(x)} + K$ using **(ii)**'s result **A1 in *s/c* (ft sign)** $= \frac{u}{\left(1 - u^3\right)^{\frac{1}{3}}} + K$ **A1 in *u*** **3**

$\int\left(1-u^3\right)^{\frac{1}{3}}\,\mathrm{d}u=\int c(x).c(x)^2\ \mathrm{d}x=\int c(x)^3\ \mathrm{d}x$	**full substn.**	**M1**	
$=\int\left(\frac{1}{2}+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left[s(x)c(x)\right]\right)\mathrm{d}x$	using **(ii)**'s result	**M1**	
$=\frac{1}{2}x+\frac{1}{2}s(x)c(x)+K=\frac{1}{2}s^{-1}(u)+\frac{1}{2}u\left(1-u^3\right)^{\frac{1}{3}}+K$		**A1**	**3**

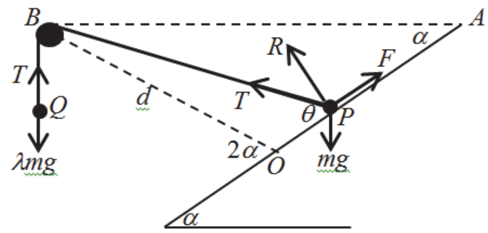
Q9	$mg\,x\sin\alpha \geq mg\,d\sin\beta$	**M1 Attempt to use conservation of energy**	
	$\Rightarrow\ x\sin\alpha \geq d\sin\beta$	**A1 Correct**	**2**
	Acceleration down the slope is $g\sin\alpha$	**B1**	
	$v^2 = 2g\,x\sin\alpha$	**M1A1 Use of appropriate kinematic formula; correct**	
	$t_1 = \dfrac{x}{\frac{1}{2}\sqrt{2gx\sin\alpha}}$ or $\sqrt{\dfrac{2x}{g\sin\alpha}}$	**A1**	
	Acceleration up the slope is $-g\sin\beta$	**B1 Clear use of correct sign convention required**	
	$d = v\,t_2 - \frac{1}{2}g\sin\beta\ t_2^2$	**M1 A1 Use of appropriate kinematic formulae.** **NB: this and the previous M1A1 can also be gained from conservation of energy considerations**	
	$t_2 = \dfrac{v \pm \sqrt{v^2 - 2dg\sin\beta}}{g\sin\beta}$	**M1 A1 Use of the quadratic formula; correct**	
	Justifying taking the negative sign	**E1**	
		M1 Algebraic working towards correct form	
	$\left(\dfrac{g\sin\alpha}{2}\right)^{\frac{1}{2}}T = (1+k)\sqrt{x} - \sqrt{k^2x - kd}$	**A1 Given Answer convincingly obtained**	**12**
	$\dfrac{\mathrm{d}}{\mathrm{d}x}(RHS) = \dfrac{1+k}{2\sqrt{x}} - \dfrac{k^2}{2\sqrt{k^2x-kd}}$	**M1 A1 Finding the derivative; correct**	
	$= 0$ when $\dfrac{(1+k)^2}{x} = \dfrac{k^4}{k^2x - kd}$	**M1 Setting derivative to zero and attempting to solve**	
	$(1+k)^2\left(k^2x - kd\right) = k^4x$	**M1 Sensible separating and squaring**	
	$x = \dfrac{(1+k)^2}{k(1+2k)}d$	**M1 A1 Isolating *x*; correct**	**6**

Q10

(i)	$2D - nR = (2M + nm)a \Rightarrow a = \dfrac{2D - nR}{2M + nm}$	**M1 A1 Use of N2L for the train; *a* correct**	
	$D - T = Ma$	**B1 N2L for the front engine**	
	$T = \dfrac{D(2M + nm) - M(2D - nR)}{2M + nm}$	**M1 Combining results**	
	$= \dfrac{n(mD + MR)}{2M + nm}$	**A1 Getting Given Answer legitimately**	**5**
(ii)	For the r^{th} carriage, with $1 \le r \le k$	**M1 Considering a general carriage between the two engines**	
	$T_{r-1} - T_r - R = ma$	**A1**	
	$T_{r-1} - T_r = R + ma > 0 \Rightarrow$ tensions decreasing	**E1**	
	Noting the same applies after the 2^{nd} engine	**E1**	
	If U is the tension of the connection to 2^{nd} engine …	**M1 Considering the tension just after 2^{nd} engine (can be done in several ways)**	
	$U - (n - k)R = (n - k)ma$	**A1**	
	Then $T - U = \dfrac{n(mD + MR)}{2M + nm} - (n - k)(R + ma)$	**M2 Finding an expression for $T - U$**	
	From first line, $2D - nR = (2M + nm)a$ $\Rightarrow 2mD - mnR = (2M + nm)ma$ $\Rightarrow 2mD + 2MR = (ma + R)(2M + nm)$ Substituting in: $T - U = \frac{1}{2}n(R + ma) - (n - k)(R + ma)$	**M1 Reasonable strategy for dealing with the algebra; such as eliminating *D*. Don't reward those going round in circles or reverse logic.**	
	$= (k - \frac{1}{2}n)(R + ma)$	**A1 Getting to a correct factorised expression**	
	So $T > U$ if $k > \frac{1}{2}n$	**E1 Given Answer suitably justified**	**11**
(iii)	For the 2^{nd} engine, $T_k + D - U = Ma$	**M1 A1 N2L for the 2^{nd} engine**	
	$\Rightarrow T_k = U + Ma - D = U - T$	**M1 Eliminating *D* using N2L on 1^{st} engine ($T = D - Ma$)**	
	From above, $T > U$ if $k > \frac{1}{2}n$ so $T_k < 0$	**E1 Correct justification using (ii)'s result**	**4**

Q11 (i)	$\angle BAO = \alpha$ (due to || lines)	**E1 Accept this just labelled on a diagram**	
	$\angle ABO = \alpha$ (due to Isos.Δ) so $\angle BOA = 180° - 2\alpha$	**E1 Or accept Ext.$\angle$ of Δ**	
	So when P is at O, $\theta = 2\alpha$		
	and when P is at A, $\theta = \alpha$ and the result follows	**E1 (since qn. says *P* is between *O* and *A*)**	**3**

(ii) Labelled diagram:



B1

$R + T\sin\theta = mg\cos\alpha$	**M1 A1 Resolving perpr. to plane for *P***	
$T = \lambda mg$	**B1 Resolving vertically for freely-hanging mass**	
$R > 0$ if P is in contact with the plane,		
so $\cos\alpha \ge \lambda\sin\theta$	**E1**	
If this is true for all values then it holds at the		
largest possible θ … so $\cos\alpha \ge \lambda\sin 2\alpha$	**E1 Condone no mention that $\alpha < 45°$**	
i.e. $\cos\alpha \ge \lambda \, . \, 2\sin\alpha\cos\alpha \Rightarrow 1 \ge 2\lambda\sin\alpha$	**M1 Use of trig. identity and "cancelling" to get Given Answer**	
	Condone oversight of checking for division by zero	**7**

(iii) $mg\sin\alpha + T\cos\theta = F$ — **M1 A1 Resolving parallel to plane; correct**

$F = mg\tan\beta(\cos\alpha - \lambda\sin\theta)$ — **B1 Correct use of $F \le \mu R$ (condone =)**

$$\sin\alpha + \lambda\cos\theta = \frac{\sin\beta}{\cos\beta}(\cos\alpha - \lambda\sin\theta)$$

$$\Rightarrow \sin\alpha\cos\beta + \lambda\cos\theta\cos\beta = \sin\beta\cos\alpha - \lambda\sin\beta\sin\theta$$

$$\Rightarrow \lambda(\cos\theta\cos\beta + \sin\beta\sin\theta) = \sin\beta\cos\alpha - \sin\alpha\cos\beta$$

$$\Rightarrow \lambda = \frac{\sin\beta\cos\alpha - \sin\alpha\cos\beta}{\cos\theta\cos\beta + \sin\beta\sin\theta}$$ **M1 Isolating λ**

$$= \frac{\sin(\beta - \alpha)}{\cos(\beta - \theta)}$$ **M1 A1 Use of compound angle formula; answer correct**

Since $\theta < 2\alpha$ and $\beta \ge 2\alpha$ then $\beta - \theta > 0$	**E1 Explaining how given condition is used**	
The minimum of $\sec(\beta - \theta)$ is achieved when		
θ is a maximum; i.e. $\theta = 2\alpha$. For the system to		
be in equilibrium for all P between O and A then	**E1 Explaining how considering $\theta = 2\alpha$ leads to the**	
λ must be less than this	**necessary condition**	
If $\alpha \le \beta \le 2\alpha$ then the minimum of $\sec(\beta - \theta)$		
is achieved when $\theta = \beta$	**E1**	
at which point the condition becomes $\lambda \le \sin(\beta - \alpha)$	**B1**	**10**

Q12 (i) $P(H) = \frac{1}{3}p_1 + \frac{1}{3}p_2 + \frac{1}{3}p_3$ **B1** **1**

NB follow through marks below are for candidates who have a probability of $p_1 + p_2 + p_3$ above, and work with this as the probability of an individual head, but check that they actually have $(1 - p_1 - p_2 - p_3)$ for the probability of a tail, not $(3 - p_1 - p_2 - p_3)$; the latter is a wrong method and gets nothing.

(ii)

value	prob
2	p^2
1	$2p(1-p)$
0	$(1-p)^2$

M1 A1 ft (need P(2) and either P(1) or P(0), but if they give all three, require all correct)

gives $\mathrm{E}(N_1) = 2p^2 + 2p(1-p) + 0 = 2p$ **A1 cao**
$\mathrm{Var}(N_1) = \mathrm{E}(N_1^2) - \mathrm{E}(N_1)^2$ **M1 for this or other plausible method**
and $\mathrm{E}(N_1^2) = 4p^2 + 2p(1-p) + 0 = 2p^2 + 2p$ **A1 ft for this or other intermediate calculation for Var**
so $\mathrm{Var}(N_1) = 2p(1-p)$ **A1 (AG)**

Alt. *approach is to argue that these are twice the corresponding values for one toss (***M1 A1***)*
*where $E(X) = E(X^2) = p$ (***M1 A1***), then getting the two values (***A1 A1 AG***),*
Or *just to say this is $\mathrm{Bin}(2, p)$ (***M2***) and quote the mean (***A2***) and variance (***A2***) for that*
(which is in formula book: to get marks for this candidates MUST explicitly write $\mathrm{Bin}(2, p)$). **6**

(iii)

value	prob
2	$(p_1p_2 + p_2p_3 + p_3p_1)/3$
1	$[p_1(1-p_2) + p_2(1-p_1) + p_2(1-p_3) + p_3(1-p_2) + p_3(1-p_1) + p_1(1-p_3)]/3$
0	$[(1-p_1)(1-p_2) + (1-p_2)(1-p_3) + (1-p_3)(1-p_1)]/3$

M1 for working out probabilities by conditioning on the chosen coins
A1 for at least one prob correct
A1 for all correct (of at least two given)

gives $\mathrm{E}(N_2) = \cdots = 2p$ **A1 ft correct expression + A1 cao simplified**
$\mathrm{Var}(N_2) = \mathrm{E}(N_2^2) - \mathrm{E}(N_2)^2$ **M1 for this or other plausible method**
and $\mathrm{E}(N_2^2) = 2p + 2(p_1p_2 + p_2p_3 + p_3p_1)/3$ **A1 ft for this or other intermediate calculation for Var**
so $\mathrm{Var}(N_2) = 2p + \frac{2(p_1p_2+p_2p_3+p_3p_1)}{3} - 4p^2$ **A1 cao any equivalent expression**

Alt. *is to calculate probs for each pair of coins (***M1 A1 A1***) then $E(N)$ and $E(N^2)$ for each pair of coins (***A1 A1***),*
*then average at this point to give $E(N_2)$ and $E(N_2^2)$ (***M1 A1***), then calculate variance (***A1***). For a correct evaluation of the expectation by conditioning on which coins are chosen, but no probabilities/variance, give* **M1 A1 A1**. **8**

(iv) Look at $\mathrm{Var}(N_1) - \mathrm{Var}(N_2) =$
$2(p_1^2 + p_2^2 + p_3^2 - p_1p_2 - p_2p_3 - p_3p_1)/9$ **B1 ft for suitable simplified expression**
$= ((p_1 - p_2)^2 + (p_2 - p_3)^2 + (p_3 - p_1)^2)/9$ **M1 for attempt to complete square A1 for partial completion**
≥ 0, with equality iff **A1 for full completion to get the inequality**
$p_1 - p_2 = p_2 - p_3 = p_3 - p_1 = 0$, i.e $p_1 = p_2 = p_3$ **E1 for justifying when equality occurs**
Anyone who uses the rearrangement inequality here is likely to get full marks – but check final E1!
An example of "partial completion" is writing as $p_1(p_1 - p_2) + p_2(p_2 - p_3) + p_3(p_3 - p_1)$ from which the result would follow by w.l.o.g.-ing $p_1 \geq p_2 \geq p_3$. **5**

Q13 (i) If $k \le 2$ she can get at most 4 marks, so P(pass)=0 — **B1 for ruling out $k < 3$**

If $k = 3$ the only way to pass is 3 right answers, with probability $\frac{1}{n^3}$. — **B1 for needs all correct if k = 3** **(give this even if prob. wrong/missing)**

If $k = 4$, 3 or 4 correct will pass, and this has prob. $\frac{4(n-1)}{n^4} + \frac{1}{n^4} = \frac{4n-3}{n^4}$. — **M1 for 3/4 needed & attempt to get prob. (but see remark)** **A1* correct prob.**

If $k = 5$, 4 or 5 correct will pass, and this has prob. $\frac{5(n-1)}{n^5} + \frac{1}{n^5} = \frac{5n-4}{n^5}$. — **M1 for 4/5 needed & attempt to get prob. (but see remark)** **A1* correct prob.** **6**

$\frac{4n-3}{n^4} - \frac{1}{n^3} = \frac{3(n-1)}{n^4} > 0$ since $n > 1$. — **M1 comparing via difference/quotient A1 inequality justified**

$\frac{4n-3}{n^4} - \frac{5n-4}{n^5} = \frac{4n^2-8n-4}{n^5} = \frac{4(n-1)^2}{n^5} > 0$ — **M1 A1 for correct difference, A1 for justifying inequality**

So $k = 4$ is best. **5**

NB *It is possible to do part (i) without calculating any probabilities: if you would pass with three questions, answering another question cannot hurt you, and if you would fail after four questions, answering another question cannot help you. A candidate who explains this will get most of the marks above, but do not give the two marks* **A1*** *unless these probabilities appear later on – you do need to calculate these probabilities at some point.*

(ii) $\mathrm{P}(k = 4 \mid \text{pass}) = \frac{\mathrm{P}(k=4 \cap \text{pass})}{\mathrm{P}(\text{pass})}$ — **M1 for this in any form**

$= \frac{\frac{1}{6}\times\frac{4n-3}{n^4}}{\frac{1}{6}\times\frac{4n-3}{n^4}+\frac{1}{6}\times\frac{1}{n^3}+\frac{1}{6}\times\frac{5n-4}{n^5}}$ — **M1 A1ft for substituting probs from before or re-calculating**

$= \frac{4n^2-3n}{5n^2+2n-4}$ — **A1cao (simplified to a quotient of polys, not nec. lowest terms)**

If a candidate jumps straight to the second line, assume they know where it comes from. However, if they jump straight to the second line without the $\frac{1}{6}$*s (and without justifying that they cancel), withhold the final* **A1**. **4**

(iii) $\mathrm{P}(\text{pass}) = \mathrm{P}(3\text{ heads})\mathrm{P}(\text{pass} \mid 3\text{ heads}) + \mathrm{P}(4\text{ heads})\mathrm{P}(\text{pass} \mid 4\text{ heads}) + \mathrm{P}(5\text{ heads})\mathrm{P}(\text{pass} \mid 5\text{ heads})$ — **M1 for conditioning on the number of heads**

$= 10\frac{n^3}{(n+1)^5}\times\frac{1}{n^3} + 5\frac{n^4}{(n+1)^5}\times\frac{4n-3}{n^4} + \frac{n^5}{(n+1)^5}\times\frac{5n-4}{n^5}$ — **M1 A1 for calculating binomial probabilities**

M1 A1cao for substituting probs from before or re-calculating **5**

If a candidate works throughout with a particular value of n (typically 2) they can get at most the following marks: B1 B1 M1 A0 M1 A0 (4/6) M1 A0 M0 A0 A0 (1/5) M1 M0 A0 A0 (1/4) M1 M1 A1 M1 A0 (4/5), total 10.
