Section A: Pure Mathematics

1 Sketch the graph of the function h, where

$$\mathrm{h}(x) = \frac{\ln x}{x}, \qquad (x > 0).$$

Hence, or otherwise, find all pairs of distinct positive integers m and n which satisfy the equation

$$n^m = m^n.$$

2 The functions f and g are related (for all real x) by

$$\mathrm{g}(x) = \mathrm{f}(x) + \frac{1}{\mathrm{f}(x)}.$$

Express $\mathrm{g}'(x)$ and $\mathrm{g}''(x)$ in terms of $\mathrm{f}(x)$ and its derivatives.

If $\mathrm{f}(x) = 4 + \cos 2x + 2\sin x$, find the stationary points of g for $0 \leq x \leq 2\pi$, and determine which are maxima and which are minima.

3 Two points P and Q lie within, or on the boundary of, a square of side 1 cm, one corner of which is the point O. Show that the length of at least one of the lines OP, PQ and QO must be less than or equal to $(\sqrt{6} - \sqrt{2})$ cm.

4 Each of m distinct points on the positive y-axis is joined by a line segment to each of n distinct points on the positive x-axis. Except at the endpoints, no three of these segments meet in a single point. Derive formulae for

(i) the number of such line segments;

(ii) the number of points of intersection of the segments, ignoring intersections at the endpoints of the segments.

If $m = n \geq 3$, and the two segments with the greatest number of points of intersection, and the two segments with the least number of points of intersection, are excluded, prove that the average number of points of intersection per segment on the remaining segments is

$$\frac{n^3 - 7n + 2}{4(n+2)}.$$

5 Given that $b > a > 0$, find, by using the binomial theorem, coefficients c_m $(m = 0, 1, 2, \ldots)$ such that

$$\frac{1}{(1-ax)(1-bx)} = c_0 + c_1x + c_2x^2 + \ldots + c_mx^m + \ldots ,$$

for $b|x| < 1$.

Show that

$$c_m^2 = \frac{a^{2m+2} - 2(ab)^{m+1} + b^{2m+2}}{(a-b)^2}.$$

Hence, or otherwise, show that

$$c_0^2 + c_1^2x + c_2^2x^2 + \ldots + c_m^2x^m + \ldots = \frac{1+abx}{(1-abx)(1-a^2x)(1-b^2x)},$$

for x in a suitable interval which you should determine.

6 The complex numbers $z_1, z_2, \ldots, z_6$ are represented by six distinct points $P_1, P_2, \ldots, P_6$ in the Argand diagram. Express the following statements in terms of complex numbers:

(i) $\overrightarrow{P_1P_2} = \overrightarrow{P_5P_4}$ and $\overrightarrow{P_2P_3} = \overrightarrow{P_6P_5}$;

(ii) $\overrightarrow{P_2P_4}$ is perpendicular to $\overrightarrow{P_3P_6}$.

If (i) holds, show that $\overrightarrow{P_3P_4} = \overrightarrow{P_1P_6}$.

Suppose that the statements (i) and (ii) both hold, and that $z_1 = 0$, $z_2 = 1$, $z_3 = z$, $z_5 = i$ and $z_6 = w$. Determine conditions which $\mathrm{Re}\,(z)$ and $\mathrm{Re}\,(w)$ must satisfy in order that $P_1P_2P_3P_4P_5P_6$ should form a convex hexagon.

Find the distance between P_3 and P_6 when $\tan(P_3\widehat{P_2}P_6) = -2/3$.

7 The function f is defined by

$$\mathrm{f}(x) = ax^2 + bx + c.$$

Show that

$$\mathrm{f}'(x) = \mathrm{f}(1)\,(x + \tfrac{1}{2}) + \mathrm{f}(-1)\,(x - \tfrac{1}{2}) - 2\mathrm{f}(0)\,x.$$

If a, b and c are real and such that $|\mathrm{f}(x)| \leq 1$ for $|x| \leq 1$, show that $|\mathrm{f}'(x)| \leq 4$ for $|x| \leq 1$.

Find particular values of a, b and c such that, for the corresponding function f of the above form, $|\mathrm{f}(x)| \leq 1$ for all x with $|x| \leq 1$ and $\mathrm{f}'(x) = 4$ for some x satisfying $|x| \leq 1$.

8 $ABCD$ is a skew (non-planar) quadrilateral, and its pairs of opposite sides are equal, i.e. $AB = CD$ and $BC = AD$. Prove that the line joining the midpoints of the diagonals AC and BD is perpendicular to each diagonal.

9 Find the following integrals:

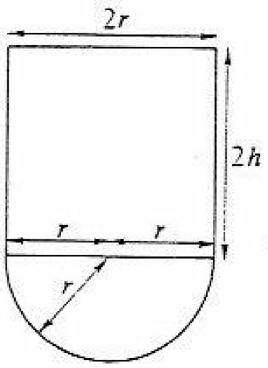
$$\text{(a)} \quad \int_1^e \frac{\ln x}{x^2}\,\mathrm{d}x;$$

$$\text{(b)} \quad \int \frac{\cos x}{\sin x\sqrt{(1+\sin x)}}\,\mathrm{d}x.$$

Section B: Mechanics

10 A sniper at the top of a tree of height h is hit by a bullet fired from the undergrowth covering the horizontal ground below. The position and elevation of the gun which fired the shot are unknown, but it is known that the bullet left the gun with speed v. Show that it must have been fired from a point within a circle centred on the base of the tree and of radius $(v/g)\sqrt{(v^2 - 2gh)}$.
[Neglect air resistance.]

11 Derive a formula for the position of the centre of mass of a uniform circular arc of radius r which subtends an angle 2θ at the centre.



A plane framework consisting of a rectangle and a semicircle, as in the above diagram, is constructed of uniform thin rods. It can stand in equilibrium if it is placed in a vertical plane with any point of the semicircle in contact with a horizontal floor. Express h in terms r.

12 A skater of mass M is skating inattentively on a smooth frozen canal. She suddenly realises that she is heading perpendicularly towards the straight canal bank at speed V. She is at a distance d from the bank and can choose one of two methods of trying to avoid it; either she can apply a force of constant magnitude F, acting at right-angles to her velocity, so that she travels in a circle; or she can apply a force of magnitude $\frac{1}{2}F(V^2 + v^2)/V^2$ directly backwards, where v is her instantaneous speed. Treating the skater as a particle, find the set of values of d for which she can avoid hitting the bank. Comment briefly on the assumption that the skater is a particle.

13 A piece of circus apparatus consists of a rigid uniform plank of mass 1000 kg, suspended in a horizontal position by two equal light vertical ropes attached to the ends. The ropes each have natural length 10 m and modulus of elasticity 490 000 N. Initially the plank is hanging in equilibrium. Nellie, an elephant of mass 4000 kg, lands in the middle of the plank while travelling vertically downwards at speed $5\,\mathrm{ms}^{-1}$. While carrying Nellie, the plank comes instantaneously to rest at a negligible height above the floor, and at this instant Nellie steps nimbly and gently off the plank onto the floor. Assuming that the plank remains horizontal, and the ropes remain vertical, throughout this motion, find to three significant figures its initial height above the floor.

During the motion after Nellie alights, do the ropes ever become slack?

[Take g to be $9.8\,\mathrm{ms}^{-2}$.]

Section C: Probability and Statistics

14 Let X be a standard normal random variable. If M is any real number, the random variable X_M is defined in terms of X by

$$X_M = \begin{cases} X & \text{if } X < M, \\ M & \text{if } X \geq M. \end{cases}$$

Show that the expectation of X_M is given by

$$\mathrm{E}(X_M) = -\phi(M) + M\big(1 - \Phi(M)\big),$$

where ϕ is the probability density function, and Φ is the cumulative distribution function, of X.

Fifty times a year, 1024 tourists disembark from a cruise liner at the port of Slaka. From there they must travel to the capital either by taxi or by bus. Officials of HOGPo are equally likely to direct a tourist to the bus station or to the taxi rank. Each bus of the bus cooperative holds 31 passengers, and the cooperative currently runs 16 buses. The bus cooperative makes a profit of 1 vloska for each passenger carried. It carries all the passengers it can, with any excess being (eventually) transported by taxi. What is the largest annual bribe the bus cooperative should consider paying to HOGPo in order to be allowed to run an extra bus?

15 In Fridge football, each team scores two points for a goal and one point for a foul committed by the opposing team. In each game, for each team, the probability that the team scores n goals is $(3 - |2 - n|)/9$ for $0 \leq n \leq 4$ and zero otherwise, while the number of fouls committed against it will with equal probability be one of the numbers from 0 to 9 inclusive. The numbers of goals and fouls of each team are mutually independent. What is the probability that in some game a particular team gains more than half its points from fouls?

In response to criticisms that the game is boring and violent, the ruling body increases the number of penalty points awarded for a foul, in the hope that this will cause large numbers of fouls to be less probable. During the season following the rule change, 150 games are played and on 12 occasions (out of 300) a team committed 9 fouls. Is this good evidence of a change in the probability distribution of the number of fouls? Justify your answer.

16 Wondergoo is applied to all new cars. It protects them completely against rust for three years, but thereafter the probability density of the time of onset of rust is proportional to $t^2/(1+t^2)^2$ for a car of age $3+t$ years ($t \geq 0$). Find the probability that a car becomes rusty before it is $3+t$ years old.

Every car is tested for rust annually on the anniversary of its manufacture. If a car is not rusty, it will certainly pass; if it is rusty, it will pass with probability $\frac{1}{2}$. Cars which do not pass are immediately taken off the road and destroyed. What is the probability that a randomly selected new car subsequently fails a test taken on the fifth anniversary of its manufacture?

Find also the probability that a car which was destroyed immediately after its fifth anniversary test was rusty when it passed its fourth anniversary test.