Cambridge Assessment
Admissions Testing

Sixth Term Examination Papers **9475**
MATHEMATICS 3 Morning
Monday 20 June 2022 Time: 3 hours

8924095334

Additional Material: Answer Booklet

INSTRUCTIONS TO CANDIDATES

Read this page carefully, but do not open this question paper until you are told that you may do so.

Read the additional instructions on the front of the answer booklet.

Write your name, centre number, candidate number, date of birth, and indicate the paper number in the spaces provided on the answer booklet.

INFORMATION FOR CANDIDATES

There are 12 questions in this paper.

Each question is marked out of 20. There is no restriction of choice.

All questions attempted will be marked.

Your final mark will be based on the **six** questions for which you gain the highest marks.

You are advised to concentrate on no more than **six** questions. Little credit will be given for fragmentary answers.

You must shade the appropriate Question Answered circle on every page of the answer booklet that you write on. Failure to do so might mean that some of your answers are not marked.

There is NO Mathematical Formulae Booklet.

Calculators are not permitted.

Wait to be told you may begin before turning this page.

BLANK PAGE

Section A: Pure Mathematics

1 Let C_1 be the curve given by the parametric equations

$$x = ct\,, \quad y = \frac{c}{t}\,,$$

where $c > 0$ and $t \neq 0$, and let C_2 be the circle

$$(x-a)^2 + (y-b)^2 = r^2\,.$$

C_1 and C_2 intersect at the four points P_i $(i = 1, 2, 3, 4)$, and the corresponding values of the parameter t at these points are t_i.

(i) Show that t_i are the roots of the equation

$$c^2t^4 - 2act^3 + (a^2 + b^2 - r^2)t^2 - 2bct + c^2 = 0\,. \qquad (*)$$

(ii) Show that

$$\sum_{i=1}^{4} t_i^2 = \frac{2}{c^2}(a^2 - b^2 + r^2)$$

and find a similar expression for $\displaystyle\sum_{i=1}^{4} \frac{1}{t_i^2}$.

(iii) Hence show that $\displaystyle\sum_{i=1}^{4} OP_i^2 = 4r^2$, where OP_i denotes the distance of the point P_i from the origin.

(iv) Suppose that the curves C_1 and C_2 touch at two distinct points.

By considering the product of the roots of $(*)$, or otherwise, show that the centre of circle C_2 must lie on either the line $y = x$ or $y = -x$.

2 **(i)** Suppose that there are three non-zero integers a, b and c for which $a^3 + 2b^3 + 4c^3 = 0$. Explain why there must exist an integer p, with $|p| < |a|$, such that $4p^3 + b^3 + 2c^3 = 0$, and show further that there must exist integers p, q and r, with $|p| < |a|$, $|q| < |b|$ and $|r| < |c|$, such that $p^3 + 2q^3 + 4r^3 = 0$. Deduce that no such integers a, b and c can exist.

(ii) Prove that there are no non-zero integers a, b and c for which $9a^3 + 10b^3 + 6c^3 = 0$.

(iii) By considering the expression $(3n \pm 1)^2$, prove that, unless an integer is a multiple of three, its square is one more than a multiple of 3. Deduce that the sum of the squares of two integers can only be a multiple of three if each of the integers is a multiple of three.

Hence prove that there are no non-zero integers a, b and c for which $a^2 + b^2 = 3c^2$.

(iv) Prove that there are no non-zero integers a, b and c for which $a^2 + b^2 + c^2 = 4abc$.

3 **(i)** The curve C_1 has equation

$$ax^2 + bxy + cy^2 = 1$$

where $abc \neq 0$ and $a > 0$.

Show that, if the curve has two stationary points, then $b^2 < 4ac$.

(ii) The curve C_2 has equation

$$ay^3 + bx^2y + cx = 1$$

where $abc \neq 0$ and $b > 0$.

Show that the x-coordinates of stationary points on this curve satisfy

$$4cb^3x^4 - 8b^3x^3 - ac^3 = 0\,.$$

Show that, if the curve has two stationary points, then $4ac^6 + 27b^3 > 0$.

(iii) Consider the simultaneous equations

$$\begin{aligned} ay^3 + bx^2y + cx &= 1 \\ 2bxy + c &= 0 \\ 3ay^2 + bx^2 &= 0 \end{aligned}$$

where $abc \neq 0$ and $b > 0$.

Show that, if these simultaneous equations have a solution, then $4ac^6 + 27b^3 = 0$.

4 *You may assume that all infinite sums and products in this question converge.*

(i) Prove by induction that for all positive integers n,

$$\sinh x = 2^n \cosh\left(\frac{x}{2}\right)\cosh\left(\frac{x}{4}\right)\cdots\cosh\left(\frac{x}{2^n}\right)\sinh\left(\frac{x}{2^n}\right)$$

and deduce that, for $x \neq 0$,

$$\frac{\sinh x}{x}\,\frac{\dfrac{x}{2^n}}{\sinh\left(\dfrac{x}{2^n}\right)} = \cosh\left(\frac{x}{2}\right)\cosh\left(\frac{x}{4}\right)\cdots\cosh\left(\frac{x}{2^n}\right).$$

(ii) You are given that the Maclaurin series for $\sinh x$ is

$$\sinh x = \sum_{r=0}^{\infty} \frac{x^{2r+1}}{(2r+1)!}.$$

Use this result to show that, as y tends to 0, $\dfrac{y}{\sinh y}$ tends to 1.

Deduce that, for $x \neq 0$,

$$\frac{\sinh x}{x} = \cosh\left(\frac{x}{2}\right)\cosh\left(\frac{x}{4}\right)\cdots\cosh\left(\frac{x}{2^n}\right)\cdots.$$

(iii) Let $x = \ln 2$. Evaluate $\cosh\left(\dfrac{x}{2}\right)$ and show that

$$\cosh\left(\frac{x}{4}\right) = \frac{1+2^{\frac{1}{2}}}{2\times 2^{\frac{1}{4}}}.$$

Use part **(ii)** to show that

$$\frac{1}{\ln 2} = \frac{1+2^{\frac{1}{2}}}{2} \times \frac{1+2^{\frac{1}{4}}}{2} \times \frac{1+2^{\frac{1}{8}}}{2}\cdots.$$

(iv) Show that

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \times \frac{\sqrt{2+\sqrt{2}}}{2} \times \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdots.$$

5 **(i)** Show that

$$\int_{-a}^{a} \frac{1}{1+\mathrm{e}^{x}}\,\mathrm{d}x = a \quad \text{for all } a \geqslant 0.$$

(ii) Explain why, if g is a continuous function and

$$\int_{0}^{a} \mathrm{g}(x)\,\mathrm{d}x = 0 \quad \text{for all } a \geqslant 0,$$

then $\mathrm{g}(x) = 0$ for all $x \geqslant 0$.

Let f be a continuous function with $\mathrm{f}(x) \geqslant 0$ for all x. Show that

$$\int_{-a}^{a} \frac{1}{1+\mathrm{f}(x)}\,\mathrm{d}x = a \quad \text{for all } a \geqslant 0$$

if and only if

$$\frac{1}{1+\mathrm{f}(x)} + \frac{1}{1+\mathrm{f}(-x)} - 1 = 0 \quad \text{for all } x \geqslant 0,$$

and hence if and only if $\mathrm{f}(x)\mathrm{f}(-x) = 1$ for all x.

(iii) Let f be a continuous function such that, for all x, $\mathrm{f}(x) \geqslant 0$ and $\mathrm{f}(x)\mathrm{f}(-x) = 1$. Show that, if h is a continuous function with $\mathrm{h}(x) = \mathrm{h}(-x)$ for all x, then

$$\int_{-a}^{a} \frac{\mathrm{h}(x)}{1+\mathrm{f}(x)}\,\mathrm{d}x = \int_{0}^{a} \mathrm{h}(x)\,\mathrm{d}x\,.$$

(iv) Hence find the exact value of

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\mathrm{e}^{-x}\cos x}{\cosh x}\,\mathrm{d}x\,.$$

6 **(i)** Show that when α is small, $\cos(\theta+\alpha)-\cos\theta \approx -\alpha\sin\theta - \frac{1}{2}\alpha^2\cos\theta$.

Find the limit as $\alpha \to 0$ of

$$\frac{\sin(\theta+\alpha)-\sin\theta}{\cos(\theta+\alpha)-\cos\theta} \qquad (*)$$

in the case $\sin\theta \neq 0$.

In the case $\sin\theta = 0$, what happens to the value of expression $(*)$ when $\alpha \to 0$?

(ii) A circle C_1 of radius a rolls without slipping in an anti-clockwise direction on a fixed circle C_2 with centre at the origin O and radius $(n-1)a$, where n is an integer greater than 2. The point P is fixed on C_1. Initially the centre of C_1 is at $(na, 0)$ and P is at $\big((n+1)a, 0\big)$.

(a) Let Q be the point of contact of C_1 and C_2 at any time in the rolling motion. Show that when OQ makes an angle θ, measured anticlockwise, with the positive x-axis, the x-coordinate of P is $x(\theta) = a\,(n\cos\theta + \cos n\theta)$, and find the corresponding expression for the y-coordinate, $y(\theta)$, of P.

(b) Find the values of θ for which the distance OP is $(n-1)a$.

(c) Let $\theta_0 = \dfrac{1}{n-1}\pi$. Find the limit as $\alpha \to 0$ of

$$\frac{y(\theta_0+\alpha)-y(\theta_0)}{x(\theta_0+\alpha)-x(\theta_0)}\,.$$

Hence show that, at the point $\big(x(\theta_0)\,,\ y(\theta_0)\big)$, the tangent to the curve traced out by P is parallel to OP.

7 Let $\mathbf{n}$ be a vector of unit length in three dimensions. For each vector $\mathbf{r}$, $\mathrm{f}(\mathbf{r})$ is defined by $\mathrm{f}(\mathbf{r}) = \mathbf{n} \times \mathbf{r}$.

(i) Given that

$$\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ and } \mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

show that the x-component of $\mathrm{f}(\mathrm{f}(\mathbf{r}))$ is $-x(b^2 + c^2) + aby + acz$. Show further that

$$\mathrm{f}(\mathrm{f}(\mathbf{r})) = (\mathbf{n}.\mathbf{r})\mathbf{n} - \mathbf{r}.$$

Explain, by means of a diagram, how $\mathrm{f}(\mathrm{f}(\mathbf{r}))$ is related to $\mathbf{n}$ and $\mathbf{r}$.

(ii) Let R be the point with position vector $\mathbf{r}$ and P be the point with position vector $\mathrm{g}(\mathbf{r})$, where g is defined by

$$\mathrm{g}(\mathbf{s}) = \mathbf{s} + \sin\theta\, \mathrm{f}(\mathbf{s}) + (1 - \cos\theta)\, \mathrm{f}(\mathrm{f}(\mathbf{s})).$$

By considering $\mathrm{g}(\mathbf{n})$ and $\mathrm{g}(\mathbf{r})$ when $\mathbf{r}$ is perpendicular to $\mathbf{n}$, state, with justification, the geometric transformation which maps R onto P.

(iii) Let R be the point with position vector $\mathbf{r}$ and Q be the point with position vector $\mathrm{h}(\mathbf{r})$, where h is defined by

$$\mathrm{h}(\mathbf{s}) = -\mathbf{s} - 2\, \mathrm{f}(\mathrm{f}(\mathbf{s})).$$

State, with justification, the geometric transformation which maps R onto Q.

8 **(i)** Use De Moivre's theorem to prove that for any positive integer $k > 1$,

$$\sin(k\theta) = \sin\theta \cos^{k-1}\theta \left(k - \binom{k}{3}(\sec^2\theta - 1) + \binom{k}{5}(\sec^2\theta - 1)^2 - \cdots \right)$$

and find a similar expression for $\cos(k\theta)$.

(ii) Let $\theta = \cos^{-1}(\frac{1}{a})$, where θ is measured in degrees, and a is an odd integer greater than 1.

Suppose that there is a positive integer k such that $\sin(k\theta) = 0$ and $\sin(m\theta) \neq 0$ for all integers m with $0 < m < k$.

Show that it would be necessary to have k even and $\cos(\frac{1}{2}k\theta) = 0$.

Deduce that θ is irrational.

(iii) Show that if $\phi = \cot^{-1}(\frac{1}{b})$, where ϕ is measured in degrees, and b is an even integer greater than 1, then ϕ is irrational.

Section B: Mechanics

9 **(i)** Two particles A and B, of masses m and km respectively, lie at rest on a smooth horizontal surface. The coefficient of restitution between the particles is e, where $0 < e < 1$. Particle A is then projected directly towards particle B with speed u.

Let v_1 and v_2 be the velocities of particles A and B, respectively, after the collision, in the direction of the initial velocity of A.

Show that $v_1 = \alpha u$ and $v_2 = \beta u$, where $\alpha = \dfrac{1-ke}{k+1}$ and $\beta = \dfrac{1+e}{k+1}$.

Particle B strikes a vertical wall which is perpendicular to its direction of motion and a distance D from the point of collision with A, and rebounds. The coefficient of restitution between particle B and the wall is also e.

Show that, if A and B collide for a second time at a point $\frac{1}{2}D$ from the wall, then $k = \dfrac{1+e-e^2}{e(2e+1)}$.

(ii) Three particles A, B and C, of masses m, km and k^2m respectively, lie at rest on a smooth horizontal surface in a straight line, with B between A and C. A vertical wall is perpendicular to this line and lies on the side of C away from A and B. The distance between B and C is equal to d and the distance between C and the wall is equal to $3d$. The coefficient of restitution between each pair of particles, and between particle C and the wall, is e, where $0 < e < 1$. Particle A is then projected directly towards particle B with speed u.

Show that, if all three particles collide simultaneously at a point $\frac{3}{2}d$ from the wall, then $e = \frac{1}{2}$.

10 Two light elastic springs each have natural length a. One end of each spring is attached to a particle P of weight W. The other ends of the springs are attached to the end-points, B and C, of a fixed horizontal bar BC of length $2a$. The moduli of elasticity of the springs PB and PC are s_1W and s_2W respectively; these values are such that the particle P hangs in equilibrium with angle BPC equal to 90°.

(i) Let angle $PBC = \theta$. Show that $s_1 = \dfrac{\sin\theta}{2\cos\theta - 1}$ and find s_2 in terms of θ.

(ii) Take the zero level of gravitational potential energy to be the horizontal bar BC and let the total potential energy of the system be $-paW$. Show that p satisfies

$$\frac{1}{2}\sqrt{2} \geqslant p > \frac{1}{4}(1+\sqrt{3})$$

and hence that $p = 0.7$, correct to one significant figure.

Section C: Probability and Statistics

11 A fair coin is tossed N times and the random variable X records the number of heads. The mean deviation, δ, of X is defined by

$$\delta = \mathrm{E}\big(|X - \mu|\big)$$

where μ is the mean of X.

(i) Let $N = 2n$ where n is a positive integer.

(a) Show that $\mathrm{P}(X \leqslant n - 1) = \frac{1}{2}\big(1 - \mathrm{P}(X = n)\big)$.

(b) Show that

$$\delta = \sum_{r=0}^{n-1} (n - r) \binom{2n}{r} \frac{1}{2^{2n-1}} \,.$$

(c) Show that for $r > 0$,

$$r\binom{2n}{r} = 2n\binom{2n-1}{r-1}.$$

Hence show that

$$\delta = \frac{n}{2^{2n}} \binom{2n}{n} \,.$$

(ii) Find a similar expression for δ in the case $N = 2n + 1$.

12 **(i)** The point A lies on the circumference of a circle of radius a and centre O. The point B is chosen at random on the circumference, so that the angle AOB has a uniform distribution on $[\,0\,,2\pi]$. Find the expected length of the chord AB.

(ii) The point C is chosen at random in the interior of a circle of radius a and centre O, so that the probability that it lies in any given region is proportional to the area of the region. The random variable R is defined as the distance between C and O.

Find the probability density function of R.

Obtain a formula in terms of a, R and t for the length of a chord through C that makes an acute angle of t with OC.

Show that as C varies (with t fixed), the expected length $\mathrm{L}(t)$ of such chords is given by

$$\mathrm{L}(t) = \frac{4a(1-\cos^3 t)}{3\sin^2 t}\,.$$

Show further that

$$\mathrm{L}(t) = \frac{4a}{3}\big(\cos t + \tfrac{1}{2}\sec^2(\tfrac{1}{2}t)\big)\,.$$

(iii) The random variable T is uniformly distributed on $[\,0\,,\frac{1}{2}\pi]$. Find the expected value of $\mathrm{L}(T)$.

BLANK PAGE