## Section A: Pure Mathematics

**1** The points $S$, $T$, $U$ and $V$ have coordinates $(s, ms)$, $(t, mt)$, $(u, nu)$ and $(v, nv)$, respectively. The lines $SV$ and $UT$ meet the line $y = 0$ at the points with coordinates $(p, 0)$ and $(q, 0)$, respectively. Show that

$$p = \frac{(m-n)sv}{ms - nv}\,,$$

and write down a similar expression for $q$.

Given that $S$ and $T$ lie on the circle $x^2 + (y-c)^2 = r^2$, find a quadratic equation satisfied by $s$ and by $t$, and hence determine $st$ and $s+t$ in terms of $m$, $c$ and $r$.

Given that $S$, $T$, $U$ and $V$ lie on the above circle, show that $p + q = 0$.

**2** **(i)** Let $y = \displaystyle\sum_{n=0}^{\infty} a_n x^n$ , where the coefficients $a_n$ are independent of $x$ and are such that this series and all others in this question converge. Show that

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}\,,$$

and write down a similar expression for $y''$.

Write out explicitly each of the three series as far as the term containing $a_3$.

**(ii)** It is given that $y$ satisfies the differential equation

$$xy'' - y' + 4x^3 y = 0\,.$$

By substituting the series of part (i) into the differential equation and comparing coefficients, show that $a_1 = 0$.

Show that, for $n \geqslant 4$,

$$a_n = -\frac{4}{n(n-2)}\,a_{n-4}\,,$$

and that, if $a_0 = 1$ and $a_2 = 0$, then $y = \cos(x^2)$.

Find the corresponding result when $a_0 = 0$ and $a_2 = 1$.

**3** The function $\mathrm{f}(t)$ is defined, for $t \neq 0$, by

$$\mathrm{f}(t) = \frac{t}{\mathrm{e}^t - 1}\,.$$

**(i)** By expanding $\mathrm{e}^t$, show that $\lim_{t\to 0} \mathrm{f}(t) = 1$. Find $\mathrm{f}'(t)$ and evaluate $\lim_{t\to 0} \mathrm{f}'(t)$.

**(ii)** Show that $\mathrm{f}(t) + \frac{1}{2}t$ is an even function. [**Note:** A function $\mathrm{g}(t)$ is said to be *even* if $\mathrm{g}(t) \equiv \mathrm{g}(-t)$.]

**(iii)** Show with the aid of a sketch that $\mathrm{e}^t(1-t) \leqslant 1$ and deduce that $\mathrm{f}'(t) \neq 0$ for $t \neq 0$.

Sketch the graph of $\mathrm{f}(t)$.

**4** For any given (suitable) function f, the *Laplace transform* of f is the function F defined by

$$\mathrm{F}(s) = \int_0^\infty \mathrm{e}^{-st}\mathrm{f}(t)\mathrm{d}t \qquad (s > 0)\,.$$

**(i)** Show that the Laplace transform of $\mathrm{e}^{-bt}\mathrm{f}(t)$, where $b > 0$, is $\mathrm{F}(s+b)$.

**(ii)** Show that the Laplace transform of $\mathrm{f}(at)$, where $a > 0$, is $a^{-1}\mathrm{F}(\frac{s}{a})$.

**(iii)** Show that the Laplace transform of $\mathrm{f}'(t)$ is $s\mathrm{F}(s) - \mathrm{f}(0)$.

**(iv)** In the case $\mathrm{f}(t) = \sin t$, show that $\mathrm{F}(s) = \dfrac{1}{s^2+1}$.

Using only these four results, find the Laplace transform of $\mathrm{e}^{-pt}\cos qt$, where $p > 0$ and $q > 0$.

**5** The numbers $x$, $y$ and $z$ satisfy

$$x + y + z = 1$$
$$x^2 + y^2 + z^2 = 2$$
$$x^3 + y^3 + z^3 = 3\,.$$

Show that

$$yz + zx + xy = -\frac{1}{2}\,.$$

Show also that $x^2y + x^2z + y^2z + y^2x + z^2x + z^2y = -1$, and hence that

$$xyz = \frac{1}{6}\,.$$

Let $S_n = x^n + y^n + z^n$. Use the above results to find numbers $a$, $b$ and $c$ such that the relation

$$S_{n+1} = aS_n + bS_{n-1} + cS_{n-2}\,,$$

holds for all $n$.

**6** Show that $\left|e^{i\beta} - e^{i\alpha}\right| = 2\sin\frac{1}{2}(\beta - \alpha)$ for $0 < \alpha < \beta < 2\pi$. Hence show that

$$\left|e^{i\alpha} - e^{i\beta}\right|\,\left|e^{i\gamma} - e^{i\delta}\right| + \left|e^{i\beta} - e^{i\gamma}\right|\,\left|e^{i\alpha} - e^{i\delta}\right| = \left|e^{i\alpha} - e^{i\gamma}\right|\,\left|e^{i\beta} - e^{i\delta}\right|,$$

where $0 < \alpha < \beta < \gamma < \delta < 2\pi$.

Interpret this result as a theorem about cyclic quadrilaterals.

**7** **(i)** The functions $\mathrm{f}_n(x)$ are defined for $n = 0, 1, 2, \ldots$ , by

$$\mathrm{f}_0(x) = \frac{1}{1+x^2} \qquad \text{and} \qquad \mathrm{f}_{n+1}(x) = \frac{\mathrm{d}\mathrm{f}_n(x)}{\mathrm{d}x}\,.$$

Prove, for $n \geqslant 1$, that

$$(1+x^2)\mathrm{f}_{n+1}(x) + 2(n+1)x\mathrm{f}_n(x) + n(n+1)\mathrm{f}_{n-1}(x) = 0\,.$$

**(ii)** The functions $\mathrm{P}_n(x)$ are defined for $n = 0, 1, 2, \ldots$ , by

$$\mathrm{P}_n(x) = (1+x^2)^{n+1}\mathrm{f}_n(x)\,.$$

Find expressions for $\mathrm{P}_0(x)$, $\mathrm{P}_1(x)$ and $\mathrm{P}_2(x)$.

Prove, for $n \geqslant 0$, that

$$\mathrm{P}_{n+1}(x) - (1+x^2)\frac{\mathrm{d}\mathrm{P}_n(x)}{\mathrm{d}x} + 2(n+1)x\mathrm{P}_n(x) = 0\,,$$

and that $\mathrm{P}_n(x)$ is a polynomial of degree $n$.

**8** Let $m$ be a positive integer and let $n$ be a non-negative integer.

**(i)** Use the result $\lim\limits_{t\to\infty} \mathrm{e}^{-mt}t^n = 0$ to show that

$$\lim_{x\to 0} x^m(\ln x)^n = 0\,.$$

By writing $x^x$ as $\mathrm{e}^{x\ln x}$ show that

$$\lim_{x\to 0} x^x = 1\,.$$

**(ii)** Let $I_n = \displaystyle\int_0^1 x^m(\ln x)^n \mathrm{d}x$ . Show that

$$I_{n+1} = -\frac{n+1}{m+1}I_n$$

and hence evaluate $I_n$.

**(iii)** Show that

$$\int_0^1 x^x \mathrm{d}x = 1 - \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{3}\right)^3 - \left(\tfrac{1}{4}\right)^4 + \cdots\,.$$

## Section B: Mechanics

**9** A particle is projected under gravity from a point $P$ and passes through a point $Q$. The angles of the trajectory with the positive horizontal direction at $P$ and at $Q$ are $\theta$ and $\phi$, respectively. The angle of elevation of $Q$ from $P$ is $\alpha$.

**(i)** Show that $\tan\theta + \tan\phi = 2\tan\alpha$.

**(ii)** It is given that there is a second trajectory from $P$ to $Q$ with the same speed of projection. The angles of this trajectory with the positive horizontal direction at $P$ and at $Q$ are $\theta'$ and $\phi'$, respectively. By considering a quadratic equation satisfied by $\tan\theta$, show that $\tan(\theta + \theta') = -\cot\alpha$. Show also that $\theta + \theta' = \pi + \phi + \phi'$.

**10** A light spring is fixed at its lower end and its axis is vertical. When a certain particle $P$ rests on the top of the spring, the compression is $d$. When, instead, $P$ is dropped onto the top of the spring from a height $h$ above it, the compression at time $t$ after $P$ hits the top of the spring is $x$. Obtain a second-order differential equation relating $x$ and $t$ for $0 \leqslant t \leqslant T$, where $T$ is the time at which $P$ first loses contact with the spring.

Find the solution of this equation in the form

$$x = A + B\cos(\omega t) + C\sin(\omega t),$$

where the constants $A$, $B$, $C$ and $\omega$ are to be given in terms of $d$, $g$ and $h$ as appropriate.

Show that

$$T = \sqrt{d/g}\left(2\pi - 2\arctan\sqrt{2h/d}\right).$$

**11** A comet in deep space picks up mass as it travels through a large stationary dust cloud. It is subject to a gravitational force of magnitude $Mf$ acting in the direction of its motion. When it entered the cloud, the comet had mass $M$ and speed $V$. After a time $t$, it has travelled a distance $x$ through the cloud, its mass is $M(1 + bx)$, where $b$ is a positive constant, and its speed is $v$.

**(i)** In the case when $f = 0$, write down an equation relating $V$, $x$, $v$ and $b$. Hence find an expression for $x$ in terms of $b$, $V$ and $t$.

**(ii)** In the case when $f$ is a non-zero constant, use Newton's second law in the form

$$\text{force} = \text{rate of change of momentum}$$

to show that

$$v = \frac{ft + V}{1 + bx}\,.$$

Hence find an expression for $x$ in terms of $b$, $V$, $f$ and $t$.

Show that it is possible, if $b$, $V$ and $f$ are suitably chosen, for the comet to move with constant speed. Show also that, if the comet does not move with constant speed, its speed tends to a constant as $t \to \infty$.

## Section C: Probability and Statistics

**12** **(i)** Albert tosses a fair coin $k$ times, where $k$ is a given positive integer. The number of heads he gets is $X_1$. He then tosses the coin $X_1$ times, getting $X_2$ heads. He then tosses the coin $X_2$ times, getting $X_3$ heads. The random variables $X_4$, $X_5$, ... are defined similarly. Write down $\mathrm{E}(X_1)$.

By considering $\mathrm{E}(X_2 \mid X_1 = x_1)$, or otherwise, show that $\mathrm{E}(X_2) = \frac{1}{4}k$.

Find $\displaystyle\sum_{i=1}^{\infty} \mathrm{E}(X_i)$.

**(ii)** Bertha has $k$ fair coins. She tosses the first coin until she gets a tail. The number of heads she gets before the first tail is $Y_1$. She then tosses the second coin until she gets a tail and the number of heads she gets with this coin before the first tail is $Y_2$. The random variables $Y_3$, $Y_4$, ... , $Y_k$ are defined similarly, and $Y = \sum_{i=1}^{k} Y_i$ .

Obtain the probability generating function of $Y$, and use it to find $\mathrm{E}(Y)$, $\mathrm{Var}\,(Y)$ and $\mathrm{P}(Y = r)$.

**13** **(i)** The point $P$ lies on the circumference of a circle of unit radius and centre $O$. The angle, $\theta$, between $OP$ and the positive $x$-axis is a random variable, uniformly distributed on the interval $0 \leqslant \theta < 2\pi$. The cartesian coordinates of $P$ with respect to $O$ are $(X, Y)$. Find the probability density function for $X$, and calculate $\mathrm{Var}\,(X)$.

Show that $X$ and $Y$ are uncorrelated and discuss briefly whether they are independent.

**(ii)** The points $P_i$ ($i = 1, 2, \ldots , n$) are chosen independently on the circumference of the circle, as in part (i), and have cartesian coordinates $(X_i, Y_i)$. The point $\overline{P}$ has coordinates $(\overline{X}, \overline{Y})$, where $\overline{X} = \dfrac{1}{n}\sum_{i=1}^{n} X_i$ and $\overline{Y} = \dfrac{1}{n}\sum_{i=1}^{n} Y_i$. Show that $\overline{X}$ and $\overline{Y}$ are uncorrelated.

Show that, for large $n$, $\mathrm{P}\left(|\overline{X}| \leqslant \sqrt{\dfrac{2}{n}}\right) \approx 0.95$ .