PHYSICS ADMISSIONS TEST
November 2022

Time allowed: 2 hours

For candidates applying to Physics, Physics and Philosophy, Engineering, or Materials Science

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Total 23 questions [100 Marks]

Answers should be written on the question sheet in the spaces provided, and you are encouraged to show your working. You should attempt as many questions as you can.

No tables, or formula sheets may be used.

Answers should be given exactly and in simplest terms unless indicated otherwise.

Indicate multiple-choice answers by circling the best answer. Partial credit may be given for correct workings in multiple choice questions.

The numbers in the margin indicate the marks expected to be assigned to each question. You are advised to divide your time according to the marks available.

You may take the gravitational field strength on the surface of Earth to be $g \approx 10\,\mathrm{m\,s^{-2}}$

Do NOT turn over until told that you may do so.

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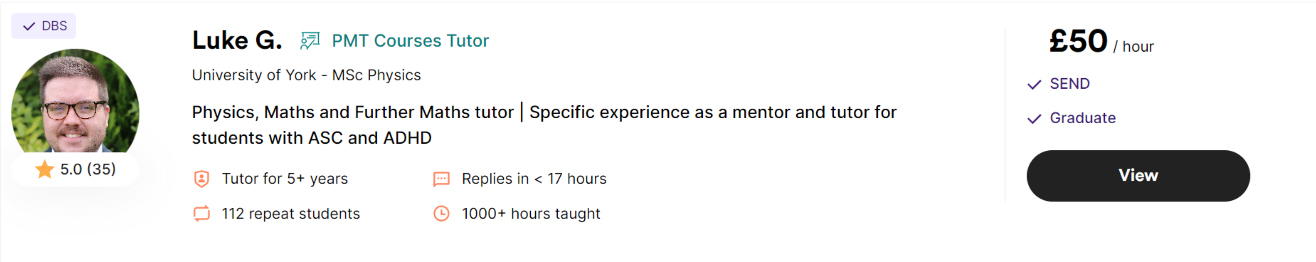
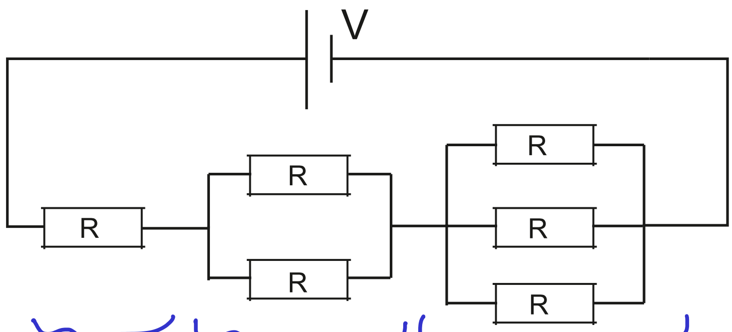
These solutions are provided by Luke, an experienced PAT tutor. You can find more information and contact him through his profile below:



1. What is the total resistance of the circuit? [2]



R

$R_T = \frac{R}{2}$

$R_T = \frac{1}{\frac{1}{R}+\frac{1}{R}+\frac{1}{R}} = \frac{R}{3}$

$R_T = R + \frac{R}{2} + \frac{R}{3} = \frac{6R+3R+2R}{6} = \frac{11R}{6}$

Parallel:

$\frac{1}{R_T} = \frac{1}{R} + \frac{1}{R} + \dots$

Series:

$R_T = R + R + \dots$

A	B	C	D	E
$\frac{11R}{6}$	$6R$	$\frac{6R}{11}$	$3R$	$\frac{R}{3}$

2. For which values of x is $\left(24 - 14x - 3x^2\right)^{-1}$ positive? [2]

A	B	C	D	E
$x < -4/3$ and $x > 6$	$x < -6$ and $x > 4/3$	$-4/3 < x < 6$	$-6 < x < 4/3$	$-\infty < x < \infty$

Find region where $24-14x-3x^2$ is positive. This will also be the region where $(24-14x-3x^2)^{-1}$ is positive (since $\frac{1}{\text{positive}}$ = positive)

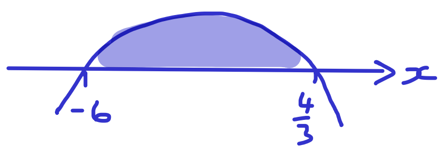
$$-\left[3x^2+14x-24\right] \quad \oplus\ 14 \quad \otimes\ -72$$

$$= -\left[3x^2+18x-4x-24\right]$$

$$= -\left[3x(x+6)-4(x+6)\right]$$

$$= -(3x-4)(x+6) = 0$$

$$\therefore \quad x = \frac{4}{3} \qquad x = -6$$



$$-6 < x < \frac{4}{3}$$

3. Molecules of oxygen in the atmosphere absorb solar radiation in bands centred at about 80 nm, 650 nm and 1000 nm. In which parts of the electromagnetic spectrum are these absorption bands? [2]

A: Visible, Infrared and Microwave

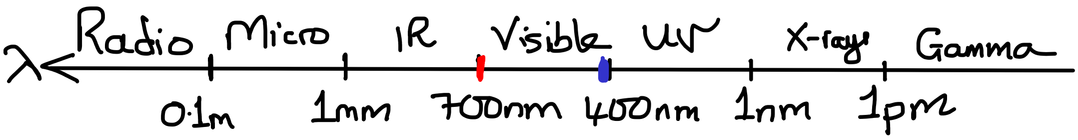
B: Visible and Infrared

C: Ultraviolet and Infrared

D: Ultraviolet, Visible and Infrared

E: X-ray, Ultraviolet and Visible

- 80nm UV
- 650nm visible (red light)
- 1000nm IR



4. Which of these polynomial functions has the largest second derivative at $x = 0$? [2]

	A	B	C	D	E
$f(x) =$	$5x^5 - x^3 + 4x$	$3x^4 + x^2 + 16$	$4x^6 + x^2 - 1$	$x^3 + 2x^2 - 5x + 10$	$10x^5 + 3x^3 - 7x + 2$

- At $x = 0$, the value of $f''(x)$ will be zero for A and E as neither contain x^2 terms.
- For B and C, the x^2 will become 2 when differentiating twice.
- For D the $2x^2$ will become 4.

5. An asteroid of mass $10^3\,\mathrm{kg}$ is moving towards a space station at $1\,\mathrm{m\,s^{-1}}$. It is proposed to stop it by firing a 1 MW laser at it. For how long must the laser be fired? You may assume that the surface of the asteroid is perfectly reflective, all photons are incident perpendicular to the surface of the asteroid, and a photon's momentum is related to its energy by $p = \frac{E}{c}$, where $c = 3 \times 10^8\,\mathrm{m\,s^{-1}}$ is the speed of light. [2]

A	B	C	D	E
$3 \times 10^{-3}\,\mathrm{s}$	$7.5 \times 10^4\,\mathrm{s}$	$1.5 \times 10^5\,\mathrm{s}$	$3 \times 10^5\,\mathrm{s}$	$3 \times 10^{11}\,\mathrm{s}$

Initial: $p = \frac{E}{c}$ $\longrightarrow$ Photons $\qquad$ $p = mv = 10^3\ \mathrm{kgms^{-1}}$ $\longleftarrow$ Asteroid

Final: $\longleftarrow$ $p = \frac{E}{c}$ (Since the photons are perfectly reflected) $\qquad$ $p = 0\ \mathrm{kgms^{-1}}$ (since the asteroid has stopped)

Cons of mmtm: ($\longrightarrow$ +)

$$\frac{E}{c} - 10^3 = -\frac{E}{c} + 0$$

$$\frac{2E}{c} = 10^3$$

Substitute $P = \frac{E}{t} \Rightarrow 10^6 = \frac{E}{t}$

$$\frac{2 \times \cancel{10^6}^{\,3}\, t}{c} = \cancel{10^3}$$

$$\therefore \quad t = \frac{3 \times 10^8}{2 \times 10^3} = 1.5 \times 10^5\,\mathrm{s}$$

Geometric series

$$S_n = \frac{a(1-r^n)}{1-r}$$

6. Which expression correctly represents the sum $\sum_{k=0}^{n} ar^{2k}$? [2]

A	B	C	D	E
$\frac{a}{1-r^2}k$	$\frac{a(1-r^{2n})}{1-r}$	$\frac{a(1-r^{2n})}{1-r^2}$	$\frac{a(1-r^{2n+2})}{1-r}$	$\frac{a(1-r^{2n+2})}{1-r^2}$

$a, ar^2, ar^4 \ldots ar^{2n}$

common ratio $= r^2$
first term $= a$

- Since $\sum$ starts at 0 finishes at n, there are $n+1$ terms

$\underbrace{0}_{\text{+ an extra term}}, \underbrace{1, 2 \ldots\ldots, n}_{n \text{ terms}}$

$$S_n = \frac{a\left(1-(r^2)^{n+1}\right)}{1-r^2} = \frac{a\left(1-r^{2n+2}\right)}{1-r^2}$$

7. In a cathode ray tube, an electron (mass 9.1×10^{-31} kg, charge -1.6×10^{-19} C) is accelerated from rest by a uniform electric field of strength $20\,\text{kV}\,\text{m}^{-1}$. How much time does it take to travel 50 cm? [2]

A	B	C	D	E
1.1×10^{-18} s	2.8×10^{-16} s	1.7×10^{-8} s	5.3×10^{-7} s	3.2×10^{-5} s

s 0.5m
u $0 ms^{-1}$
~~v~~
a a
t ?

$S = ut + \frac{1}{2}at^2$

$0.5 = \frac{1}{2}at^2$

$\therefore t = \sqrt{\frac{1}{a}}$ (1)

$E = \frac{V}{d} = \frac{F}{q}$

$\therefore F = qE$

By N2, $F = ma$

$\therefore a = \frac{qE}{m}$ (2)

Sub (2) → (1)

$t = \sqrt{\frac{m}{qE}} = \sqrt{\frac{9.1 \times 10^{-31}}{1.6 \times 10^{-19} \times 20 \times 10^{3}}} = 1.7 \times 10^{-8} s$

8. If a function $y = f(x)$ has a stationary point at (x_0, y_0), what are the co-ordinates of the corresponding stationary point of the function $y = af(bx + c)$? [2]

A	B	C	D	E
$(\frac{x_0}{b} - c, ay_0)$	$(bx_0 + c, ay_0)$	$(\frac{x_0 - c}{b}, ay_0)$	$(x_0 - \frac{c}{b}, ay_0)$	$(\frac{x_0 + c}{b}, ay_0)$

$f(x) \longrightarrow af(bx+c)$

horizontal stretch/dilation by factor $\frac{1}{b}$

horizontal translation by $-c$

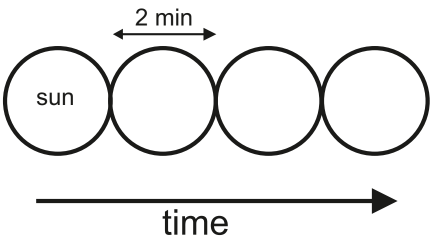
vertical stretch/dilation by factor a.

Horizontal composite transformations follow inverse order ("outside in")

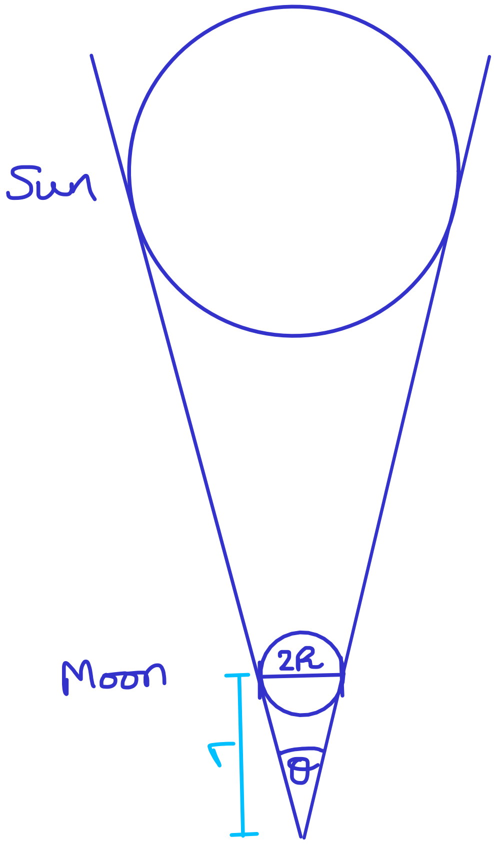
$$b x_{new} + c = x_0$$

$$\therefore x_{new} = \frac{x_0 - c}{b}$$

9. As it appears to move across the sky, the Sun moves through an angle equal to that subtended by its diameter in about two minutes, as in the diagram. In a solar eclipse, the Moon covers the Sun almost exactly in the sky. Using this, what is the approximate ratio of the Moon's $\underbrace{\text{radius}}_{R}$ to its $\underbrace{\text{orbital distance}}_{r}$ from Earth? [2]



$360° = 24$ hours $\times 60$
$360° = 1440$ minutes
$\therefore 0.5° = 2$ minutes
$\therefore \theta = 0.5° \times \frac{2\pi}{360} = \frac{\pi}{360}$ rad



- Moon covers the Sun during an eclipse $\therefore$ they have the same angular size
- Arc length : $l = r\theta$. Since r is very large compared with the size of the Moon $\rightarrow l \approx 2R$

$$\therefore \frac{2R}{r} = \theta = \frac{\pi}{360} = 0.0087$$

$$\therefore \frac{R}{r} = 0.00436\ldots$$
$$= 0.0044$$

A	B	C	D	E
0.0014	0.0022	0.0028	0.0044	0.0056

10. What is the next number in the sequence $0, \frac{3}{4}, \frac{3}{8}, \frac{9}{16}, \frac{15}{32}, \frac{33}{64}$? [2]

A	**B**	**C**	**D**	**E**
$\frac{51}{128}$	$\frac{53}{128}$	$\frac{63}{128}$	$\frac{65}{128}$	$\frac{71}{128}$

Numerator : $(3\times2)\pm3$ alternating

Denominator : 2^n for n^{th} term.

$(0\times2)+3=3$

$(3\times2)-3=3$

$(3\times2)+3=9$

$(9\times2)-3=15$

$(15\times2)+3=33$

$(33\times2)-3=63$

11. Two moons occupy circular orbits around a planet. The smaller moon has mass 1.5×10^{15} kg and orbital radius 2.3×10^{4} km. The larger moon has mass 1.1×10^{16} kg and orbital radius 9.4×10^{3} km. If the gravitational force exerted by the planet on the smaller moon is 10^{14} N, what force does the planet exert on the larger moon? [2]

$$F_G = \frac{GMm}{r^2}$$

A	B	C	D	E
2.4×10^{14} N	6.0×10^{14} N	7.3×10^{14} N	1.8×10^{15} N	4.4×10^{15} N

Small Moon:

$$10^{14} = \frac{GM(1.5 \times 10^{15})}{(2.3 \times 10^{7})^2}$$

$$\therefore GM = 3.53 \times 10^{13}$$

Large Moon:

$$F_G = \frac{GM(1.1 \times 10^{16})}{(9.4 \times 10^{6})^2}$$

$$F_G = \frac{(3.53 \times 10^{13})(1.1 \times 10^{16})}{(9.4 \times 10^{6})^2}$$

$$= 4.4 \times 10^{15} \text{ N}$$

12. What is the derivative of $y = x^6 + 6x^5 + 12x^4 + 8x^3$? [2]

A: $(3x+3)(x^2+2x)^2$

B: $(2x+2)(x^2+2x)^2$

x^3 ~~**C:**~~ $(6x+6)(x^2+2x)$

x^7 ~~**D:**~~ $(2x+2)(x^2+2x)^3$

E: $(6x+6)(x^2+2x)^2$

$$\frac{dy}{dx} = 6x^5 + 30x^4 + 48x^3 + 24x^2$$

- Highest power is x^5 so discount C and D
- Only E will give a $6x^5$ term, A gives $3x^5$, B gives $2x^5$

$$(6x+6)(x^4+4x^3+4x^2)$$

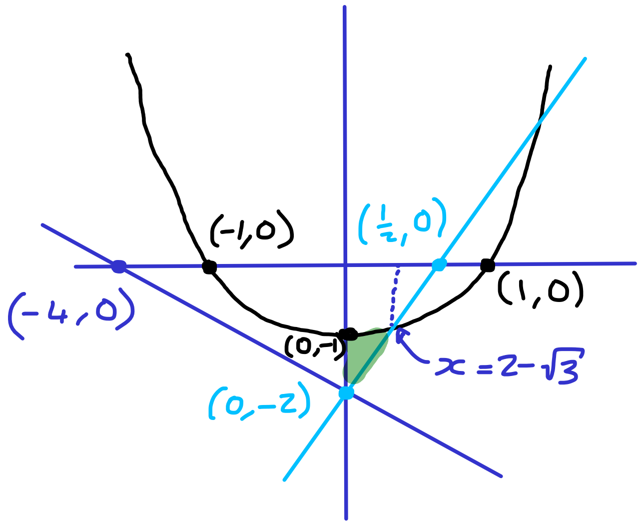
13.

(a) Draw the functions $y_1(x) = x^2 - 1$, $y_2(x) = 4x - 2$ and $y_3(x) = -\frac{x}{2} - 2$ on a common set of axes. Label where they cross the axes. [3]

(b) Work out the x-values of the intersection points of these three functions. [3]

(c) Write down a single integral which describes a finite area bounded by two of the three functions. You do *not* need to evaluate the integral. [2]

a.)



b.) • y_2 and y_3 cross at $x=0$

• y_2 and y_1 cross at

$$4x-2 = x^2-1$$
$$x^2-4x+1 = 0$$
$$(x-2)^2-3 = 0$$
$$\therefore x = 2 \pm \sqrt{3}$$

• y_1 and y_3 do not cross:

$$x^2-1 = -\frac{x}{2}-2$$
$$2x^2+x+2 = 0$$
$$b^2-4ac = 1-16 = -15 < 0$$
$$\Delta < 0 \ \therefore \text{no solutions}$$

c.) $$A = \int_0^{2-\sqrt{3}} 4x-2\,dx - \int_0^{2-\sqrt{3}} x^2-1\,dx$$
$$= \int_0^{2-\sqrt{3}} 4x-x^2-1\,dx$$

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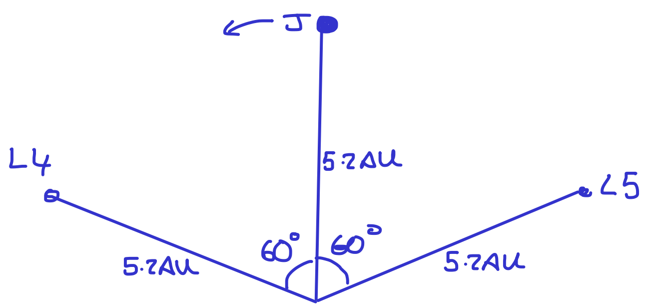
14. The Trojan asteroids share Jupiter's orbit around the Sun: approximately circular with a mean radius 5.2 AU (1 AU $= 1.5 \times 10^{11}$ m is the mean radius of the Earth's orbit around the Sun). The Trojans are clustered around two points labelled L4 and L5, where the L4 point is 60° ahead of Jupiter in its orbit and the L5 point is 60° behind Jupiter in its orbit.

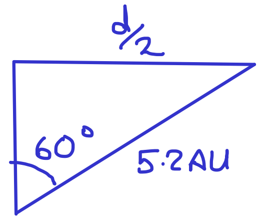
(a) Determine the mean distance between the asteroids *588 Achilles* (at the L4 point) and *617 Patroclus* (at the L5 point). [2]

(b) A spacecraft travels in a straight line between the two asteroids, accelerating at $10\,\mathrm{m\,s^{-2}}$ until the half-way point between the asteroids, and decelerating at $10\,\mathrm{m\,s^{-2}}$ from there to the end-point. Assuming that the asteroids are approximately stationary on the timescale of the journey, and neglecting any gravitational effects of Jupiter or the Sun, find the total travel time. [3]

(c) Explain why the assumption that the asteroids are approximately stationary during the journey is well-justified. [1]



a.)



$$\sin 60 = \frac{\frac{d}{2}}{5.2}$$

$\therefore d = 10.4 \sin 60$

$d = 9\,\mathrm{AU}$

$d = 1.35 \times 10^{12}\,\mathrm{m}$

b.) First half:

s $\frac{d}{2}$

u 0

~~v~~

a 10

t ?

$$\frac{d}{2} = \frac{1}{2} \times 10 \times t^2$$

$$\therefore t = \sqrt{\frac{d}{10}}$$

$$t = \sqrt{1.35 \times 10^{11}}$$

$\therefore$ Twice this between L4 & L5:

$$t_{TOT} = 2\sqrt{1.35 \times 10^{11}} = 7.35 \times 10^{5}\,\mathrm{s}$$

c.) $7.35 \times 10^{5}\,\mathrm{s} \approx 8.5$ days. Kepler's third states that $T^2 \propto r^3$.

$$\frac{T_{Earth}^{2}}{T_{Jupiter}^{2}} = \frac{r_{Earth}^{3}}{r_{Jupiter}^{3}}$$

$$\therefore T_J = \sqrt{\frac{T_E^2 r_J^3}{r_E^3}} = \sqrt{\frac{(365)^2 \times (5.2)^3}{(1)^3}} = 4328 \text{ days}$$

4328 days for one Jupiter orbit. So in 8.5 days, the asteroids will have only moved 0.2% of their full orbit $\therefore$ approx stationary.

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15. A projectile is launched at speed v and angle θ (as measured from the horizontal) outwards from the top of a high cliff.

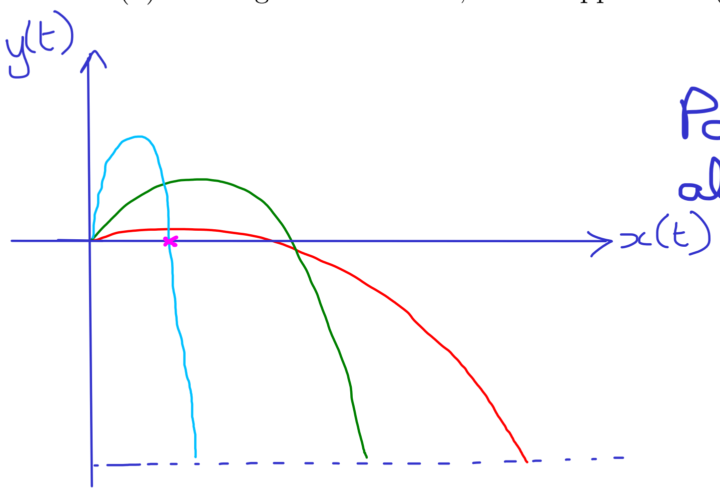
(a) Sketch the trajectory of the projectile for launch angles $\theta = 5°$, $45°$ and $85°$. Use $x(t)$ for the horizontal displacement from the launch point and $y(t)$ for the vertical displacement from the launch point. [2]

(b) Using separate axes, now sketch the absolute distance, $r(t) = \sqrt{x(t)^2 + y(t)^2}$, from its launch point as a function of time for all of the three launch angles above. [2]

(c) Obtain an expression for $r(t)$. For which angles does $r(t)$ have a stationary point? [5]

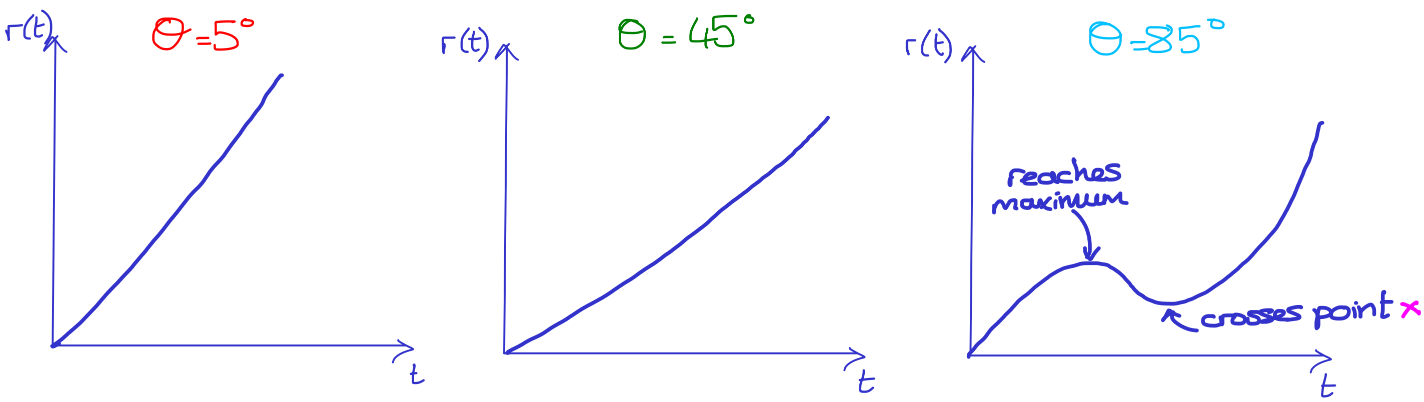
(d) For angles below these, what happens to $r(t)$ as time increases? [1]

a.)



Parabolic paths, symmetric about their maximum.

b)



- For $\theta = 5°$, the distance of the projectile from the start is continually increasing.
- For $\theta = 45°$, similar to $5°$ however the gradient will be less (the ball doesn't travel as far from the start)
- For $\theta = 85°$, the maximum height of the ball is greater than its horizontal distance when $y(t) = 0$ (i.e. the x-intercept in a.). So, the distance of the ball from the start will increase up to its maximum height, decrease until it crosses "the x-axis" and increase again as it falls below the cliff edge.

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c.) Horizontal displacement:

$$s = ut$$
$$x(t) = v\cos\theta \times t$$
$$x(t) = vt\cos\theta$$
$$\therefore \quad x^2(t) = v^2t^2\cos^2\theta$$

Vertical displacement:

$(\overset{+}{\uparrow})$

s $y(t)$
u $v\sin\theta$
$\cancel{v}$
a $-g$
t t

$$y(t) = vt\sin\theta - \frac{1}{2}gt^2$$
$$\therefore \quad y^2(t) = v^2t^2\sin^2\theta - gt^2vt\sin\theta + \frac{1}{2}g^2t^4$$

Absolute distance:

$$r(t) = \sqrt{x^2(t) + y^2(t)}$$
$$r(t) = \sqrt{v^2t^2\cos^2\theta + v^2t^2\sin^2\theta - gt^3v\sin\theta + \frac{1}{4}g^2t^4}$$
$$r(t) = \sqrt{v^2t^2 - gt^3v\sin\theta + \frac{1}{4}g^2t^4} \quad (*)$$
$$r(t) = t\sqrt{\frac{1}{4}g^2t^2 - gtv\sin\theta + v^2}$$

Differentiate $(*)$ w.r.t. t:

$$\frac{dr}{dt} = \frac{1}{2}\left(v^2t^2 - gt^3v\sin\theta + \frac{1}{4}g^2t^4\right)^{-\frac{1}{2}} \times \left(2v^2t - 3gt^2v\sin\theta + g^2t^3\right)$$

$$\frac{dr}{dt} = \frac{2v^2t - 3gt^2v\sin\theta + g^2t^3}{2\sqrt{v^2t^2 - gt^3v\sin\theta + \frac{1}{4}g^2t^4}}$$

Set $\frac{dr}{dt} = 0$ to find stationary points. $\frac{dr}{dt}$ will be zero when the numerator is zero:

$$2v^2t - 3gt^2v\sin\theta + g^2t^3 = 0$$
$$t(g^2t^2 - 3gtv\sin\theta + 2v^2) = 0$$

$t = 0$ or $g^2t^2 - 3gtv\sin\theta + 2v^2 = 0$

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$$g^2t^2 - 3gtv\sin\theta + 2v^2 = 0$$

Quadratic in t. Solutions to this will give t coords at which r(t) has stationary points. We aren't asked for these though (we want the values of θ which give stationary points). For there to be stationary point(s), the determinant to this quadratic $\Delta \geqslant 0$:

$$b^2 - 4ac \geqslant 0$$

$$(-3gv\sin\theta)^2 - 4(g^2)(2v^2) \geqslant 0$$

$$9g^2v^2\sin^2\theta - 8g^2v^2 \geqslant 0$$

$$9\sin^2\theta - 8 \geqslant 0$$

$$\sin^2\theta \geqslant \frac{8}{9}$$

$$\sin\theta \geqslant \frac{2\sqrt{2}}{3}$$

$$\underline{\underline{\theta \geqslant 70.5^\circ}}$$

d.) For angles below 70.5°, r(t) is monotonically increasing (continuously increasing with t).

16. Suppose $f(t) = 4t$ and $g(x) = \frac{3}{2}\left(3x - x^2\right)$. Consider the inequality

$$\frac{\mathrm{d}g(x)}{\mathrm{d}x} > \int_{3/2}^{x} f(t)\,\mathrm{d}t\,.$$

For which values of x is this inequality satisfied? [4]

$$\frac{dg}{dx} = \frac{9}{2} - 3x$$

$$\int_{\frac{3}{2}}^{x} 4t\,dt = 2t^2\Big|_{\frac{3}{2}}^{x}$$

$$= 2x^2 - 2\left(\frac{3}{2}\right)^2$$

$$= 2x^2 - \frac{9}{2}$$

$$\frac{dg}{dx} > \int f(t)\,dt$$

$$\frac{9}{2} - 3x > 2x^2 - \frac{9}{2}$$

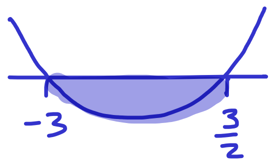
$$\therefore 2x^2 + 3x - 9 < 0$$

$$2x^2 + 6x - 3x - 9 < 0$$

$$2x(x+3) - 3(x+3) < 0$$

$$(2x-3)(x+3) < 0$$

$+$ 3
$\times$ $ac = -18$

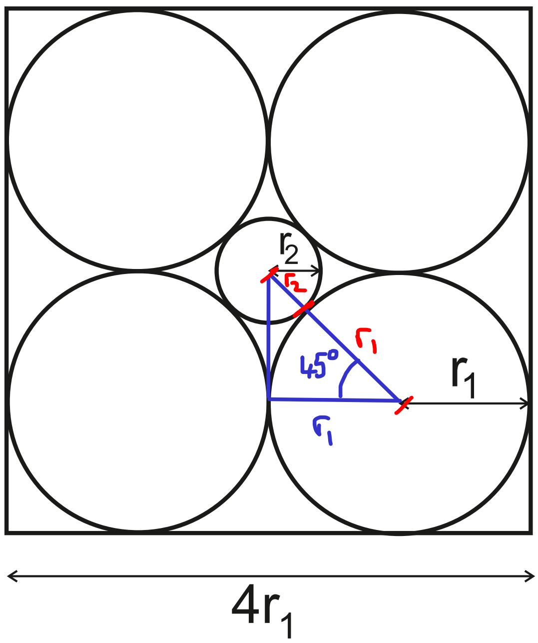


$$\therefore \underline{\underline{-3 < x < \frac{3}{2}}}$$

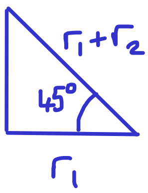
17. Four circles of radius r_1 are inscribed inside a square of side $4r_1$ as shown in the diagram below.

(a) What is the radius r_2 of the largest circle that can fit in the space at the centre of the square, bounded by the outer circles? [3]

(b) If 8 spheres of radius r_1 are now similarly arranged inside a cube of edge length $4r_1$, what is the radius r_3 of the largest sphere that can fit in the space at the centre of the cube? [3]



a.)



$$\cos(45) = \frac{r_1}{r_1 + r_2}$$

$$r_1 + r_2 = \sqrt{2}\, r_1$$

$$\therefore \quad r_2 = \underline{\underline{(\sqrt{2} - 1)\, r_1}} = 0.414\, r_1$$

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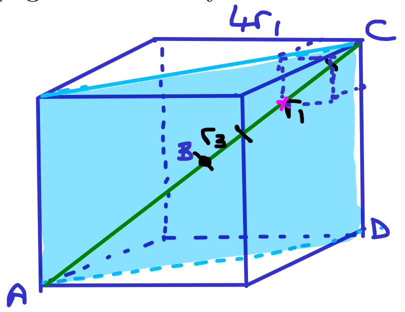
b)



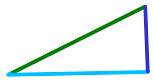
- A & C are opposite corners.
- B is the centre of the cube.

$AD = \sqrt{(4r_1)^2 + (4r_1)^2} = 4\sqrt{2}\,r_1$

$AC = \sqrt{(4\sqrt{2}r_1)^2 + (4r_1)^2}$

$= 4\sqrt{3}\,r_1$

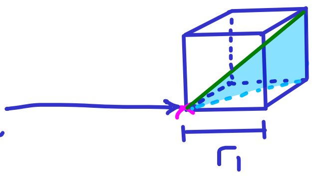
(So $BC = 2\sqrt{3}r_1$)



Spheres will meet along the leading (green) diagonal AC.

- Take a cube which contains one-eighth of one of the spheres radius r_1. The green diagonal will pass through opposite corners.

centre of one sphere



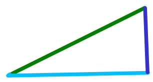
face-diagonal:

$\sqrt{r_1^2 + r_1^2} = \sqrt{2}\,r_1$

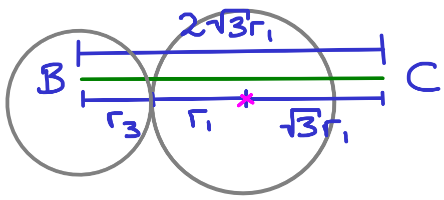
leading diagonal:

$\sqrt{(\sqrt{2}r_1)^2 + (r_1)^2}$

$= \sqrt{3}\,r_1$



- Now take half the diagonal of the full cube:



$\therefore\ r_3 + r_1 + \sqrt{3}r_1 = 2\sqrt{3}\,r_1$

$r_3 = (\sqrt{3} - 1)\,r_1$

$$\Delta = b^2 - 4ac$$

18. Consider the function

$$f(x) = -\frac{P}{x^3} + \frac{Q}{x^2} - \frac{R}{x}$$

in the region $x > 0$, where P, Q and R are all positive constants.

(a) Find an inequality satisfied by P, Q and R in order for $f(x)$ to have at least one real root. [2]

(b) Find a relationship between P, Q and R in order for $f(x)$ to have exactly one stationary point. [3]

(c) If the relationshiop of the previous part holds, so that exactly one stationary point exists, what is the nature of that stationary point and at what value of x (expressed in terms of P, Q and R) is it? *It is not necessary to work out a second derivative to answer this.* [3]

a.) To have at least one real root, $f(x) = 0$ for some x.

$$-\frac{P}{x^3} + \frac{Q}{x^2} - \frac{R}{x} = 0$$

$$-P + Qx - Rx^2 = 0$$

$$Rx^2 - Qx + P = 0$$

$$x = \frac{Q \pm \sqrt{Q^2 - 4RP}}{2R}$$

To have at least one real root (i.e. one or more) then $\Delta = 0$ or $\Delta > 0$

$$\underline{\underline{Q^2 - 4RP \geq 0}}$$

b.) At a stationary point, $\frac{df}{dx} = 0$

$$\frac{df}{dx} = \frac{d}{dx}\left\{-Px^{-3} + Qx^{-2} - Rx^{-1}\right\}$$

$$= \frac{3P}{x^4} - \frac{2Q}{x^3} + \frac{R}{x^2} = 0$$

$$3P - 2Qx + Rx^2 = 0$$

$$Rx^2 - 2Qx + 3P = 0$$

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To have exactly one solution, $\Delta = 0$

$$\Delta = (-2Q)^2 - 4(R)(3P) = 0$$

$$4Q^2 - 12RP = 0$$

$$\underline{\underline{Q^2 - 3RP = 0}}$$

$81 = Q^2$ $R = 0.5$
$Q = 9$ $P = 1$
$Q = 1.224$

c.) $Rx^2 - 2Qx + 3P = 0$

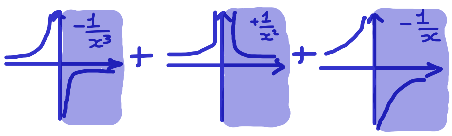
$$\therefore x = \frac{2Q \pm \cancel{\sqrt{4Q^2 - 12RP}}}{2R}$$

$\Delta = 0$ for one turning point as in part b.)

$$x = \frac{2Q}{2R} = \underline{\underline{\frac{Q}{R}}}$$

- To determine the nature of the stationary point, consider the shape of the individual terms in $f(x)$

$$f(x) = \frac{-P}{x^3} + \frac{Q}{x^2} - \frac{R}{x}$$

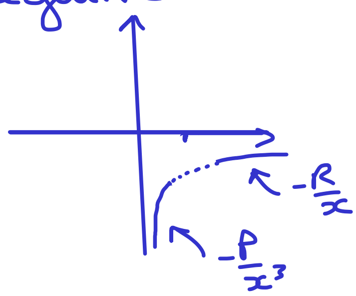
$f(x) =$ 

In the question we're told $x > 0$, so only the shaded regions are summed to form $f(x)$.

- For large x, the dominant term is $-\frac{R}{x}$ (i.e. $f(x)$ will be negative for large x). For small x, the dominant term is $-\frac{P}{x^3}$, which is also negative.

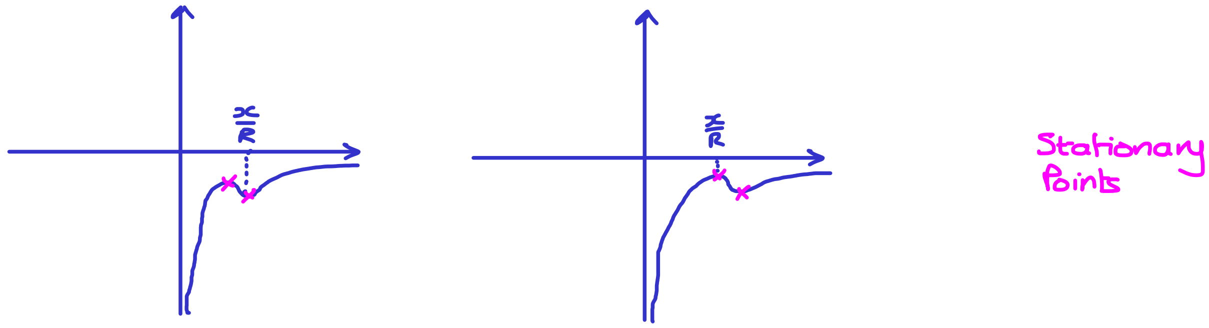
- Hence the ends of the curve will always be negative. The shape of the region around $\frac{Q}{R}$ is dependent on the values of P, Q and R. For one stationary point, $Q^2 = 3RP$ from b. From a., for there to be at least one real root $Q^2 \geqslant 4RP$.

⋮

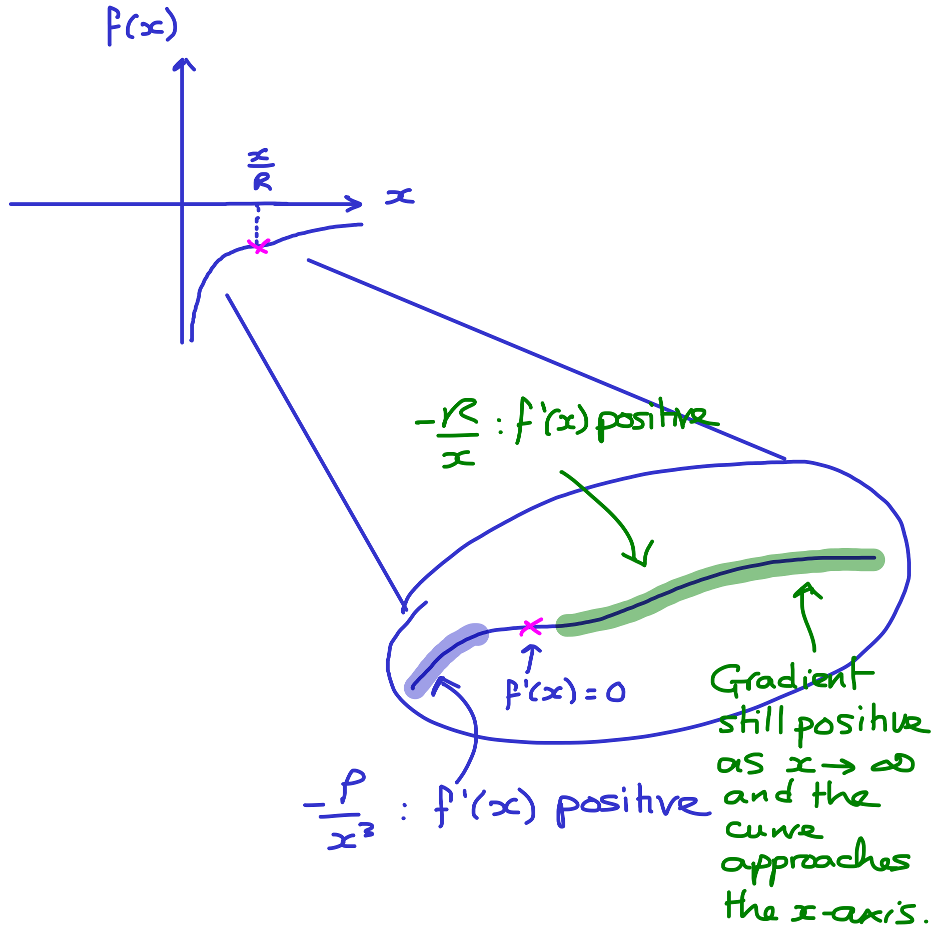


- Hence the curve has no roots (it does not cross the x-axis) when $Q^2 = 3RP$.

- For there to be a maximum or minimum at $x = \frac{Q}{R}$, there would also be another stationary point on either the left or right of $\frac{Q}{R}$ to "join the curve up" to the $-\frac{R}{x}$ to the right, or $-\frac{P}{x^3}$ to the left.



- So it is not possible for there to be a single maximum or minimum on $f(x)$. Hence the stationary point at $x = \frac{Q}{R}$ must be a point of inflection.



19. Following Bohr, we assume that a hydrogen-like atom may be modelled as a single electron (mass m and charge $-e$) in a circular orbit around a much more massive nucleus (charge $+Ze$).

(a) By balancing forces, find the speed v of the electron in terms of its orbital radius. [3]

(b) Show that the total energy of the electron is equal to the negative of its kinetic energy. You may assume that its potential energy U is given by (where r is the radius of the orbit) [3]

$$U = -\frac{Ze^2}{4\pi\epsilon_0 r}$$

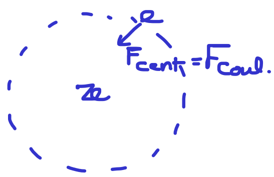
(c) Assuming that for the electron the product $mvr = n\hbar$, where n is an integer and $\hbar$ (pronounced h-bar) is a constant, find an expression for the electron energy in terms of n (and which does not depend on either v or r). [2]

(d) If $E(n=1) = -13.6\,\text{eV}$ for hydrogen, what is $E(n=3)$ for once-ionised helium (He^+)? [2]

a.) Centripetal provided by Coulomb force. Equate magnitudes:

$$\frac{mv^2}{r} = \frac{Ze^2}{4\pi\varepsilon_0 r^2}$$

$$v = \sqrt{\frac{Ze^2}{4\pi\varepsilon_0 m r}}$$



b.) $E_{TOT} = K + U$

$$= \frac{1}{2}mv^2 - \left(\frac{Ze^2}{4\pi\varepsilon_0 r}\right)\frac{\times 2}{\times 2}$$

$$= \frac{Ze^2}{8\pi\varepsilon_0 r} - \frac{2Ze^2}{8\pi\varepsilon_0 r}$$

$$= \frac{-Ze^2}{8\pi\varepsilon_0 r} = -K$$

$$K = \frac{1}{2}mv^2 = \frac{m}{2}\left(\frac{Ze^2}{4\pi\varepsilon_0 m r}\right) = \frac{Ze^2}{8\pi\varepsilon_0 r}$$

c.) $mvr = n\hbar$

$$v = \frac{n\hbar}{mr}$$

From a).

$$\therefore v^2 = \frac{n^2\hbar^2}{m^2r^2} = \frac{Ze^2}{4\pi\varepsilon_0 m r}$$

$$\therefore r = \frac{4\pi\varepsilon_0 n^2\hbar^2}{mZe^2}$$

Sub into $E_{TOT} = \dfrac{-Ze^2}{8\pi\varepsilon_0 r}$

$$E_{TOT} = -\frac{Ze^2}{8\pi\varepsilon_0} \times \frac{mZe^2}{4\pi\varepsilon_0 n^2\hbar^2}$$

$$E_{TOT} = -\frac{mZ^2e^4}{32\pi^2\varepsilon_0^2 n^2\hbar^2} \quad (*)$$

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d.) $E(n=1) = -13.6\,eV$ for H $(Z=1)$

Sub $n=1, Z=1$ into $(*)$:

$$-13.6\,eV = \frac{-me^4}{32\pi\varepsilon_0^2\hbar^2}$$

$E(n=3)$ for He^+ $(Z=2)$

Sub $n=3, Z=2$ into $(*)$

$$E(n=3) = \frac{(2)^2}{(3)^2} \times \underbrace{\frac{-me^4}{32\pi\varepsilon_0^2\hbar^2}}_{-13.6\,eV}$$

$$= \frac{4}{9} \times -13.6\,eV$$

$$= \underline{\underline{-6.04\,eV}}$$

20. Two unbiased dice are rolled. The numbers obtained are multiplied.

(a) What is the probability that the product is even? [1]

(b) Which product has a probability of $\frac{1}{12}$ to occur? [1]

(c) What is the probability that the product is greater than 28? [1]

(d) Which product(s) has(ve) the highest probability to occur? [2]

(e) If the product is known to be even, what is the probability that it is also divisible by 4? [2]

Outcome Space $m \times n$:

m\n	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	8	10	12
3	3	6	9	12	15	18
4	4	8	12	16	20	24
5	5	10	15	20	25	30
6	6	12	18	24	30	36

36 possible outcomes

a.) $P(\text{even}) = \frac{27}{36} = \frac{3}{4}$

m\n	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	8	10	12
3	3	6	9	12	15	18
4	4	8	12	16	20	24
5	5	10	15	20	25	30
6	6	12	18	24	30	36

b.) $\frac{1}{12} = \frac{3}{36}$

Product $= \underline{4}$

m\n	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	8	10	12
3	3	6	9	12	15	18
4	4	8	12	16	20	24
5	5	10	15	20	25	30
6	6	12	18	24	30	36

c.)

$P(mn > 28) = \frac{3}{36} = \frac{1}{12}$

m\n	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	8	10	12
3	3	6	9	12	15	18
4	4	8	12	16	20	24
5	5	10	15	20	25	30
6	6	12	18	24	30	36

d.) 6 and 12

with probabilities

$P(mn=6) = P(mn=12)$

$= \frac{4}{36} = \frac{1}{9}$

m\n	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	8	10	12
3	3	6	9	12	15	18
4	4	8	12	16	20	24
5	5	10	15	20	25	30
6	6	12	18	24	30	36

e.) Of the 27 even outcomes, 15 are divisible by 4

$P(\text{Divisible by } 4 \mid \text{Even}) = \frac{15}{27} = \frac{5}{9}$

m\n	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	8	10	12
3	3	6	9	12	15	18
4	4	8	12	16	20	24
5	5	10	15	20	25	30
6	6	12	18	24	30	36

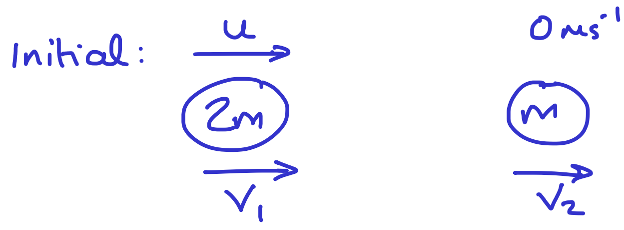
21. A ball of mass $2m$ slides along a frictionless track with speed u. Starting from a long distance away, it collides elastically with a stationary ball of mass m.

(a) Calculate the final speeds of both balls (you may neglect any rotation of the balls). [5]

(b) If both balls were now positively electrically charged, describe qualitatively either how the results would change or why you would leave the results unaltered. [2]

a)

Initial:



Cons of mmtm:

$$(\rightarrow +)\ 2mu = 2mv_1 + mv_2$$

$$2u = 2v_1 + v_2$$

$$\therefore v_2 = 2u - 2v_1 \quad (1)$$

Elastic ∴ K.E. conserved too:

$$\frac{1}{2}(2m)u^2 = \frac{1}{2}(2m)v_1^2 + \frac{1}{2}mv_2^2$$

$$2u^2 = 2v_1^2 + v_2^2 \quad (2)$$

Substitute (1) into (2) to get a quadratic in v_1.

$$2u^2 = 2v_1^2 + (2u - 2v_1)^2$$

$$2u^2 = 2v_1^2 + 4u^2 - 8uv_1 + 4v_1^2$$

$$6v_1^2 - 8uv_1 + 2u^2 = 0$$

$$3v_1^2 - 4uv_1 + u^2 = 0$$

(+) $-4u$

(×) $ac = 3u^2$

$$3v_1^2 - 3uv_1 - uv_1 + u^2 = 0$$

$$3v_1(v_1 - u) - u(v_1 - u) = 0$$

$$(3v_1 - u)(v_1 - u) = 0$$

$$\therefore \underline{\underline{v_1 = \frac{u}{3}\ \text{ms}^{-1}}},\quad v_1 = u \rightarrow \text{causes } v_2 = 0.\ \text{Discard.}$$

$$\therefore v_2 = 2u - 2\left(\frac{u}{3}\right) = \frac{6u}{3} - \frac{2u}{3} = \underline{\underline{\frac{4u}{3}}}\ \text{ms}^{-1}$$

b) Momentum and energy would still be conserved ∴ the velocities would not change. However, they would not make contact with each other due to the Coulombic repulsion. The distance of closest approach will be:

KE lost = Electric PE gained

$$\frac{1}{2}mu^2 = \frac{q^2}{4\pi\varepsilon_0 d^2} \Longrightarrow d = \frac{q}{u\sqrt{2\pi\varepsilon_0 m}}$$

22. Consider the following set of equations:

$$\begin{aligned} 2x + y &= z, \quad ① \\ x^2 &= y, \quad ② \\ z + 2y &= 2x^3. \quad ③ \end{aligned}$$

Find the possible values of x which satisfy these equations. [5]

Eliminate z, $① \rightarrow ③$:

$2x + 3y = 2x^3$ ④

Eliminate y, $② \rightarrow ④$

$2x + 3x^2 = 2x^3$

$2x^3 - 3x^2 - 2x = 0$

$x(2x^2 - 3x - 2) = 0$

$x = 0$

$2x^2 - 3x - 2 = 0$

$(+) \; -3$

$(\times) \; -4$

$2x^2 - 4x + x - 2 = 0$

$2x(x-2) + 1(x-2) = 0$

$(2x+1)(x-2) = 0$

$\therefore x = -\frac{1}{2}, \; x = 2$

$x = -\frac{1}{2}, 0, 2$

23. The number of atoms N_x in a sample of a radioactive substance x decays with time according to the equation,

$$N_x(t) = N_x(0)e^{-\lambda_x t},$$

where $N_x(0)$ is the number of atoms at time $t = 0$ and λ_x is a constant for substance x.

The half-life of a substance is defined as the time taken for N_x to reach half of its initial value. Substance a has a half-life of 1 hour. 36% of its decays emit an alpha particle and 64% of its decays emit a beta particle. Substance b has a half-life of 15 minutes. 56% of its decays emit an alpha particle and 44% of its decays emit a beta particle.

If the total particle emission rate of substance x (where $x = a, b$) is $\lambda_x N_x(t)$ and $N_a(0) = N_b(0)$, what time in minutes passes before the beta particle emission rates from the two samples are equal? [5]

$t^a_{1/2} = 1\text{hr}$

$N_a(t) = N_a(0)e^{-\lambda_a t}$

$t^b_{1/2} = 0.25\text{ hrs}$

$N_b(t) = N_b(0)e^{-\lambda_b t}$

When $t = t_{1/2}$, $N(t) = \frac{N(0)}{2}$

$\frac{\cancel{N_a(0)}}{2} = \cancel{N_a(0)}e^{-\lambda_a t_{1/2}}$

$\frac{1}{2} = e^{-\lambda_a(1)}$

$\therefore \lambda_a = -\ln(\frac{1}{2})$

$\lambda_a = \ln 2$

$\frac{\cancel{N_b(0)}}{2} = \cancel{N_b(0)}e^{-\lambda_b t_{1/2}}$

$\frac{1}{2} = e^{-\lambda_b(0.25)}$

$\therefore \lambda_b = -4\ln(\frac{1}{2})$

$\lambda_b = 4\ln 2$

Particle emission rate (activity) is defined as $\lambda N(t)$. The total particle emission rate for substance a is $\lambda_a N_a(t)$. Since 64% of these decays produce β particles, the rate at which substance a produces β particles is $0.64\lambda_a N_a(t)$.

Similarly, the rate at which substance b produces β particles is $0.44\lambda_b N_b(t)$.

- Equate these rates and solve for t to find the time at which the β emission rates are equal:

$$0.64\lambda_a N_a(t) = 0.44\lambda_b N_b(t)$$

$$0.64\cancel{\ln(2)}N_a(t) = 0.44 \times 4\cancel{\ln(2)}\, N_b(t)$$

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Substitute $N_a(t) = N_a(0)e^{-\ln(2)t}$ and $N_b(t) = N_b(0)e^{-4\ln(2)t}$:

$$0.64\,N_a(0)\,e^{-\ln(2)t} = 1.76\,N_b(0)\,e^{-4\ln(2)t}$$

Using $N_a(0) = N_b(0)$:

$$0.64\,e^{-\ln(2)t} = 1.76\,e^{-4\ln(2)t}$$

$$\frac{4}{11}e^{-\ln(2)t} = e^{-4\ln(2)t}$$

$$\ln\left\{\frac{4}{11}e^{-\ln(2)t}\right\} = \ln\left\{e^{-4\ln(2)t}\right\}$$

$$\ln\left(\frac{4}{11}\right) + \ln\left(e^{-\ln(2)t}\right) = -4\ln(2)t$$

$$\ln\left(\frac{4}{11}\right) - \ln(2)t = -4\ln(2)t$$

$$\ln\left(\frac{4}{11}\right) = -3\ln(2)t$$

$$\ln\left(\frac{4}{11}\right) = \ln\left(\frac{1}{8}\right)t$$

$$t = \frac{\ln\left(\frac{4}{11}\right)}{\ln\left(\frac{1}{8}\right)} = 0.486\ldots\text{hrs}$$

$$= \underline{\underline{29\text{mins}}}$$

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